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ON NUMERICAL SOLUTION OF WEIGHT MINIMIZATION OF ELASTIC BODIES WEAKLY SUPPORTING TENSION

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Summary. Shape optimization of a two-dimensional elastic body is considered, provided the material is weakly supporting tension. The problem generalizes that of a masonry dam subjected to its weight and to the hydrostatic pressure. A part of the boundary has to be found so as to minimize a given cost functional. The numerical realization using a penalty method and finite element technique is presented. Some typical results are shown.

Keywords: shape optimization, weight minimization, penalty method, masonry-like materials, finite elements

AMS classification: 65N30, 65K10, 73C99

INTRODUCTION

The problems of weight minimization of elastic bodies weakly supporting tension have been studied by Hlaváček and Křížek [1], [2].

This paper presents the numerical realization of two different weight minimization problems (problems 1 and 2). The first was proposed in [1], where also the convergence of both the theoretical and the approximate solution was proved. On the basis of the analysis of the first problem, the second shape optimization problem is proposed and studied.

Finite element method using piecewise linear triangular elements is applied for the solution of elastostatic problems of both models under consideration. Then a penalty approach enables us to remove both the mechanical and the geometrical constraints and thus to simplify the resulting nonlinear programming problem.

1. Formulation of problem 1

Let us consider a class of admissible domains $\Omega(v)$, where

$$\Omega(v) = \{x = (x_1, x_2) \mid 0 < x_1 < v(x_2),\ 0 < x_2 < L\}$$

and the function $v(x_2)$ — the design variable — belongs to the set of admissible functions

$$U_{\text{ad}} = \Big\{ v \in C^{(1),1}([0,L]) \mid \alpha \leqslant v(x_2) \leqslant \beta,$$
$$\Big|\frac{\mathrm{d}v}{\mathrm{d}x_2}\Big| \leqslant C_1 \text{ in } [0,L],\ \Big|\frac{d^2v}{\mathrm{d}x_2^2}\Big| \leqslant C_2 \text{ a.e. in } [0,L] \Big\}$$

with given positive constants α, β, C_1, C_2, L, $(\alpha < \beta)$. Here $C^{(1),1}([0,L])$ denotes the space of functions with Lipschitz-continuous derivatives.

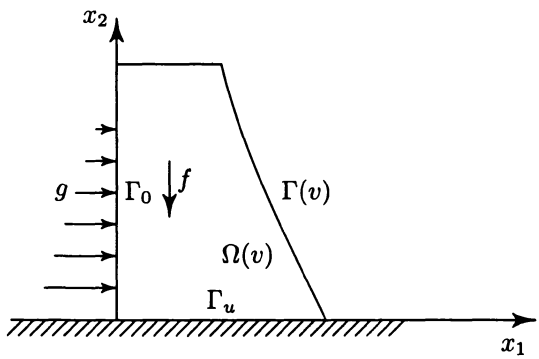

Figure 1. Problem 1—system under study

We shall consider the following boundary value problem (plane strain)

$$
\begin{aligned}
&\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0,\\
&\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),\\
&\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \quad \text{in } \Omega(v),\\
&u_i = 0 \quad \text{on } \Gamma_u,\\
&\sigma_{ij} n_j = g_i \quad \text{on } \Gamma_0,\ i,j = 1,2,
\end{aligned}
\tag{1.1}
$$

where σ_{ij}, ε_{ij} are the components of the stress and strain tensors, respectively, u_i are the components of the displacements and n_i are the components of the unit outward normal with respect to Γ_0.

A (weak) solution of the elastostatic problem (1.1) is described as follows:

$$\text{(1.2)} \qquad \begin{cases} \text{Find } u(v) \in V(v) \text{ such that} \\ a\big(v; u(v), w\big) = F(v; w) \quad \forall w \in V(v), \end{cases}$$

where

$$\begin{aligned} V(v) &= \big\{ w \in [H^1(\Omega)]^2 \mid w = 0 \quad \text{on } \Gamma_u(v) \big\}, \\ a\big(v; u(v), w\big) &= \int_{\Omega(v)} E_{ijkl} e_{ij}(w) e_{kl}\big(u(v)\big)\,\mathrm{d}x, \\ F(v; w) &= \int_{\Omega(v)} f_i w_i\,\mathrm{d}x + \int_{\Gamma_0} g_i w_i\,\mathrm{d}\Gamma \end{aligned}$$

with $f_i \in [L^2(\Omega_\delta)]^2$ and $g_i \in [L^2(\Gamma_0)]^2$ ($\Omega_\delta = (0,\delta) \times (0,L)$, $\delta > \beta$). Here $H^1(\Omega)$ denotes the standard Sobolev space $W^{1,2}(\Omega)$.

Let us consider the penalized cost functional

$$\text{(1.3)} \qquad j_\varepsilon(v,\sigma) = j(v) + \frac{1}{\varepsilon} \sum_{i=1}^{3} s_i(v,\sigma), \quad \varepsilon > 0,$$

where

$$\begin{aligned} j(v) &= \int_0^L v(x_2)\,\mathrm{d}x_2, \\ s_i(v,\sigma) &= \int_{\Omega(v)} (\sigma_{ii} - k)^+\,\mathrm{d}x, \quad i = 1,2 \text{ (no sum)}, \\ s_3(v,\sigma) &= \int_{\Omega(v)} \big(\det(\sigma - \kappa)\big)^-\,\mathrm{d}x, \quad (\kappa_{ij} = k \cdot \delta_{ij}), \end{aligned}$$

where $(\cdot)^+$ and $(\cdot)^-$ denote the positive and negative parts, respectively. The functions $j(v)$ and $s_1(v,\sigma)$, $s_2(v,\sigma)$, $s_3(v,\sigma)$ represent the optimized area and material constraints, respectively, k is a given positive constant representing the maximum admissible tensile stress.

We define the Optimal Design Problem:

$$\text{(1.4)} \qquad \begin{cases} \text{Find } v^* \in U_{\mathrm{ad}} \text{ such that} \\ j_\varepsilon\big(v^*, \sigma(v^*)\big) \leqslant j_\varepsilon\big(v, \sigma(v)\big) \quad \forall v \in U_{\mathrm{ad}}, \end{cases}$$

where $\sigma(v)$ denotes the stress field which follows from the solution $u(v)$ of the state problem (1.2).

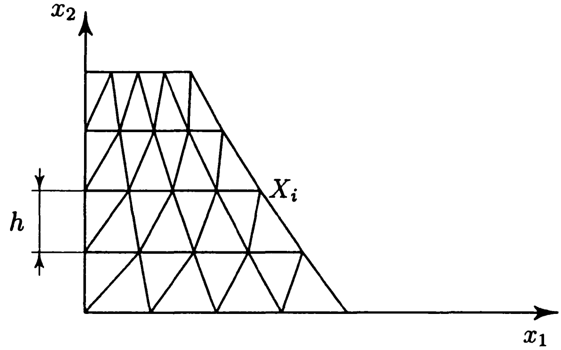

Figure 2. Scheme of the computational model

2. Numerical realization of problem 1

After approximating the state problem (1.2) and the penalized cost functional (1.3) by finite elements we get the equivalent mathematical programming problem for fixed $h > 0$ in matrix notation.

Let us denote

$$\mathsf{X} = (X_0, X_1, \ldots, X_N)^T \equiv \big(V_h(0), V_h(h), \ldots, V_h(Nh)\big)^T \in \mathbb{R}^{N+1},$$

the vector of the optimization parameters.

For a fixed h the approximate solution $u_h(v_h)$ is given by the vector of nodal values $\mathsf{q} = \mathsf{q}(\mathsf{x})$, which is the solution of the linear system of equations

$$\mathsf{K}(\mathsf{x})\mathsf{q} = \mathsf{f}(\mathsf{x}), \tag{2.1}$$

where $\mathsf{K}(\mathsf{x})$ is the stiffness matrix and $\mathsf{f}(\mathsf{x})$ is the force vector. The approximated form of (1.4) is equivalent to the mathematical programming problem

$$\begin{cases} \text{Find } \mathsf{X}^* \in U \text{ such that} \\ J(\mathsf{X}^*, \mathsf{q}(\mathsf{X}^*)) \leqslant J(\mathsf{X}, \mathsf{q}(\mathsf{X})) \quad \forall \mathsf{X} \in U, \end{cases} \tag{2.2}$$

where $\mathsf{q}(\mathsf{x})$ solves (2.1) and U denotes the set of the admissible design vectors given by

$$\begin{aligned} U = \big\{\mathsf{X} \in \mathbb{R}^{N+1} \mid \alpha \leqslant X_j \leqslant \beta, \quad & j = 0, 1, \ldots, N; \\ -C_1 h \leqslant X_{j+1} - X_j \leqslant C_1 h, \quad & j = 0, 1, \ldots, N-1; \\ -C_2 h^2 \leqslant X_{j+2} - 2X_{j+1} + X_j \leqslant C_2 h^2, \quad & j = 0, 1, \ldots, N-2\big\}. \end{aligned}$$

2.1. Sensitivity analysis. According to [1] the gradient of the cost functional (1.3) with respect to X is given by

$$\nabla_x J(\mathsf{X}, \mathsf{q}(\mathsf{X})) = \nabla_x J(\mathsf{X}, \mathsf{q}) + \left(\nabla_x \mathsf{f}(\mathsf{X}) - \nabla_x \mathsf{K}(\mathsf{X}) \cdot \mathsf{q}\right)^T \mathsf{p}, \tag{2.3}$$

where p is the solution of the adjoint state problem

$$\mathsf{K}(\mathsf{X})\mathsf{p} = \nabla_q J(\mathsf{X}, \mathsf{q}). \tag{2.4}$$

The following sensitivity analysis is derived in detail in [3]. The components of the stiffness matrix and the force vector of the e-th finite element are given by

$$\mathsf{K}^{(e)} = \frac{1}{2A^{(e)}} \mathsf{B}^{(e)T} \mathsf{C}^{(e)} \mathsf{B}^{(e)},$$

$$\mathsf{f}^{(e)} = \mathsf{f}_v^{(e)} + \mathsf{f}_s^{(e)} = \int_{A^{(e)}} \mathsf{N}^{(e)T} \mathsf{F}^{(e)} \, dx + \int_l \mathsf{L}^T \mathsf{L} \mathsf{P}^{(e)} \, dl.$$

Here N and B contain the values of the elements basis functions and their derivatives, respectively, A is the element area, F and P are the vectors of the volume forces and surface tractions, respectively, L contains the basis functions of the $1D$ element and l is the length of the element side loaded with surface tractions.

It is obvious that we obtain the global stiffness matrix and the force vector by summing over all elements:

$$\mathsf{K} = \sum_{e=1}^{NE} \mathsf{K}^{(e)}, \qquad \mathsf{f} = \sum_{e=1}^{NE} \mathsf{f}^{(e)},$$

where NE is the number of all finite elements.

The derivatives of the e-th element stiffness matrix and the force vector with respect to the optimization parameter X_j are given by

$$\begin{aligned} \frac{\partial \mathsf{K}^{(e)}(\mathsf{X})}{\partial X_j} &= 2\mathsf{B}^{(e)T}(\mathsf{X}) \mathsf{C}^{(e)} \frac{\partial \mathsf{B}^{(e)}(\mathsf{X})}{\partial X_j} A^{(e)} \\ &\quad + \mathsf{B}^{(e)T}(\mathsf{X}) \cdot \mathsf{C}^{(e)} \mathsf{B}^{(e)}(\mathsf{X}) \frac{\partial A^{(e)}}{\partial X_j}, \\ \frac{\partial \mathsf{f}^{(e)}(\mathsf{X})}{\partial X_j} &= \frac{1}{3} \frac{\partial A^{(e)}}{\partial X_j} \mathsf{F}^{(e)}. \end{aligned}$$

It is obvious that

$$\frac{\partial \mathsf{K}(\mathsf{X})}{\partial X_j} = \sum_{e=N1}^{N2} \frac{\partial \mathsf{K}^{(e)}(\mathsf{X})}{\partial X_j}, \tag{2.5}$$

$$\frac{\partial \mathsf{f}(\mathsf{X})}{\partial X_j} = \sum_{e=N1}^{N2} \frac{\partial \mathsf{f}^{(e)}(\mathsf{X})}{\partial X_j}. \tag{2.6}$$

Here we sum only over the elements at least one node of which lies on the j-th optimization parameter (see Fig. 3).

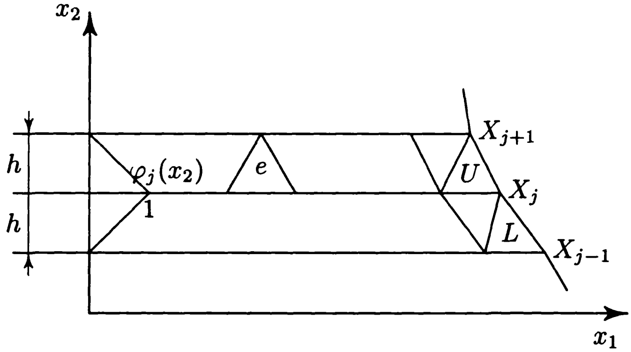

Figure 3. Detail of the computational model

The partial derivative of the cost functional $J(\mathsf{X},\mathsf{q}(\mathsf{X}))$ with respect to the j-th optimization parameter is given by (see Fig. 3)

(2.7)

$$\begin{aligned}\frac{\partial J(\mathsf{X},\mathsf{q})}{\partial X_j} &= h + \frac{1}{\varepsilon}\Bigg\{\frac{h}{2}\Big[(\sigma_{11}^{(U)}-k)^+ + (\sigma_{11}^{(L)}-k)^+ + (\sigma_{22}^{(U)}-k)^+ + (\sigma_{22}^{(L)}-k)^+ \\ &+ \Big((\sigma_{11}^{(U)}-k)(\sigma_{22}^{(U)}-k) - \sigma_{12}^{(U)2}\Big)^- + \Big((\sigma_{11}^{(L)}-k)(\sigma_{22}^{(L)}-k) - \sigma_{12}^{(L)2}\Big)^-\Big] \\ &+ \sum_{e=N1}^{N2}\Big[H(\sigma_{11}^{(e)}-k)\sum_{r=1}^{6}(CB)_{1,r}^{(e)'}q_r^{(e)} + H(\sigma_{22}^{(e)}-k)\sum_{r=1}^{6}(CB)_{2,r}^{(e)'}q_r^{(e)} \\ &- H\Big(-\det(\sigma-\kappa)^{(e)}\Big)\sum_{r=1}^{6}\Big((\sigma_{22}^{(e)}-k)(CB)_{1,r}^{(e)'} + (\sigma_{11}^{(e)}-k)(CB)_{2,r}^{(e)'} \\ &- 2\sigma_{12}^{(e)}(CB)_{3,r}^{(e)'}\Big)q_r^{(e)}\Big]A^{(e)}\Bigg\}, \qquad 0 < j < N.\end{aligned}$$

Here $(CB)'_{i,r}$ denotes the component (i,r) of the partial derivative of the product of matrices C and B with respect to X_j $\big(\frac{\partial}{\partial X_j}(\mathsf{CB})\big)$. It is important to note that for $j=0$ and $j=N$ the formula (2.7) has to be modified.

The vector $\nabla_q J(\mathsf{X},\mathsf{q})$ is obtained by summing over all finite elements

$$\nabla_q J(\mathsf{X},\mathsf{q}) = \sum_{e=1}^{NE}\nabla_q J^{(e)}(\mathsf{X},\mathsf{q}). \tag{2.8}$$

For the e-th finite element we obtain

$$\begin{aligned}\frac{\partial J^{(e)}(\mathsf{X},\mathsf{q})}{\partial q_r} = \frac{1}{\varepsilon}\Big[&H(\sigma_{11}^{(e)} - k)(CB)_{1,r}^{(e)} + H(\sigma_{22}^{(e)} - k)(CB)_{2,r}^{(e)} \\ &- H\big(-\det(\sigma-\kappa)^{(e)}\big)\big((CB)_{1,r}^{(e)}(\sigma_{22}^{(e)} - k) \\ &+ (CB)_{2,r}^{(e)}(\sigma_{11}^{(e)} - k) - 2\sigma_{12}^{(e)}(CB)_{3,r}^{(e)}\big)\Big]A^{(e)}, \\ &r = 1,\dots,6,\end{aligned}$$

where $(CB)_{i,r}$ denotes the component (i,r) of the product of matrices C and B.

3. Formulation of problem 2

On the basis of the numerical results obtained from the solution of problem 1, the second model is suggested.

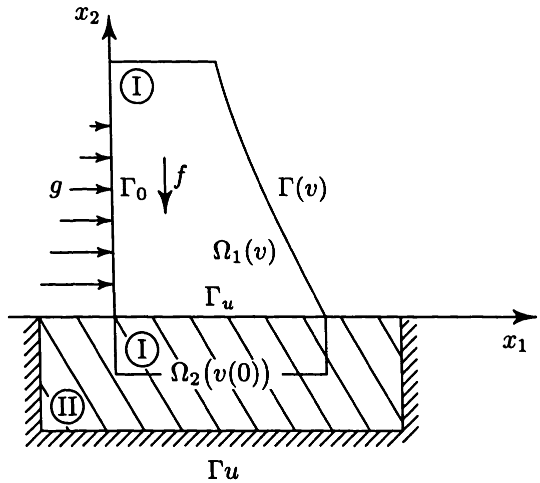

Figure 4. Problem 2—system under study

The cross section of the body considered represents the domain Ω, where

$$\begin{aligned}\Omega(v) &= \Omega_1(v) \cup \Omega_2\big(v(0)\big), \quad \big(\Omega_2\big(v(0)\big)\text{—dashed region}\big), \\ \Omega_1(v) &= \big\{x = (x_1,x_2) \mid 0 < x_1 < v(x_2),\ 0 < x_2 < L\big\}, \\ \Omega_2\big(v(0)\big) &= \big\{x = (x_1,x_2) \mid -\gamma_1 < x_1 < v(0) + \gamma_1,\ -\gamma_2 < x_2 < 0\big\}\end{aligned}$$

and the function $v(x_2)$ belongs to the set U_{ad} which was defined above, γ_1, γ_2 are given positive constants. The boundary value problem (1.1) and the weak solution

(1.2) remain unchanged, only the domain $\Omega(v)$ and the boundary Γ_u are different (see Fig. 4). It is important to note that now two different types of material are under consideration (material I — dam, material II — foundation).

Let us define the penalized cost functional (1.3) on the region $\Omega_1(v)$. The Optimal Design Problem is defined as in (1.4).

4. Numerical realization of problem 2

The numerical realization of problem 2 is similar to that of problem 1. We shall concentrate only on the differences in the sensitivity analysis, which result from the fact that the cost functional is defined on the region Ω_1. The computation of the vector $\nabla_q J(\mathsf{X}, \mathsf{q})$ is as in Section 2.1, only the summation in (2.8) is over elements belonging to the region Ω_1.

The computation of

$$\frac{\partial J(\mathsf{X}, \mathsf{q})}{\partial X_j}, \quad 0 \leqslant j \leqslant N$$

and

$$\frac{\partial \mathsf{K}(\mathsf{X})}{\partial X_j}, \frac{\partial \mathsf{f}(\mathsf{X})}{\partial X_j}, \quad 0 < j \leqslant N$$

is the same as in Section 2.1. For $j = 0$, in the formulas (2.5) and (2.6) we have to sum also over the elements belonging to the region Ω_2 at least one node of which lies on the optimization parameter X_0.

5. Numerical examples

In this section we present numerical results of the problem 1 and 2. The state problem (2.1) and the adjoint problem (2.4) were solved using symmetric Gauss-Doolitle decomposition. In optimization we have used the method of steepest descent with the adaptive step size determination. The geometrical constraints were treated by means of a penalty method as well. All computations were done in double precision using VAX 4000/VLC computer.

The following results (Examples 5.1–5.4) have been obtained for $L = 10$ m, $\alpha = 1$ m, $\beta = 20$ m, $\gamma_1 = 0.2\,X_0^0$ (X_0^0—initial length of X_0), $\gamma_2 = 0.4\,L$, $C_1 = 1.5$, $C_2 = 1.$, $k = 2\,000$ Pa, $E = 2.1 \times 10^{10}$ Pa (Young's modulus), $\mu = 0.2$ (Poisson's ratio), $f_1 = 0$, $f_2 = 21\,582$ Nm^{-3}, $g_1(x_2) = 9\,810 \cdot (L - x_2)$ $[\mathrm{Nm}^{-2}]$, $g_2 = 0$. As to problem 2, the material of the foundation was considered to be the same as the material of the dam.

E x a m p l e 5.1. Problem 1
Number of degrees of freedom — 160, number of iterations — 24, CPU — 4 min.

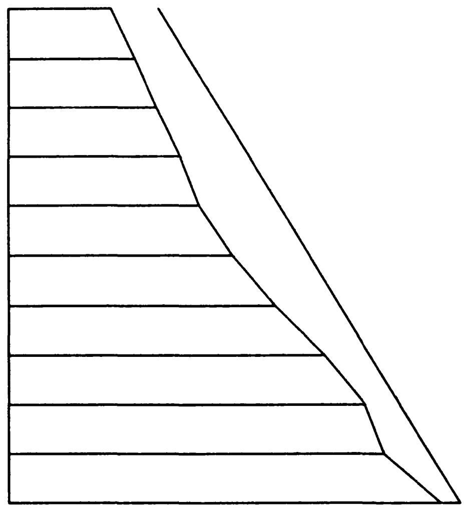

Figure 5. ($h = L/10$). Initial cost = 66.245. Final cost = 53.525.

E x a m p l e 5.2. Problem 2
Number of degrees of freedom — 286, number of iterations — 81, CPU — 14 min.

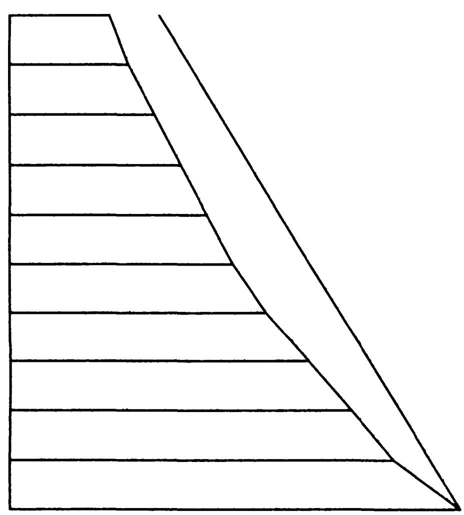

Figure 6. ($h = L/10$). Initial cost = 66.245. Final cost = 53.118.

E x a m p l e 5.3. Problem 1

Number of degrees of freedom — 500, number of iterations — 35, CPU — 25 min.

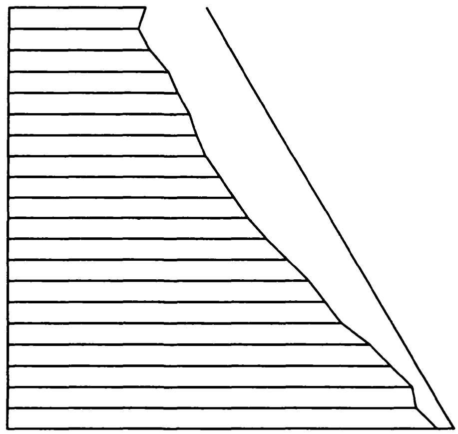

Figure 7. ($h = L/20$). Initial cost = 86.243. Final cost = 68.747.

E x a m p l e 5.4. Problem 2

Number of degrees of freedom — 996, number of iterations — 62, CPU — 63 min.

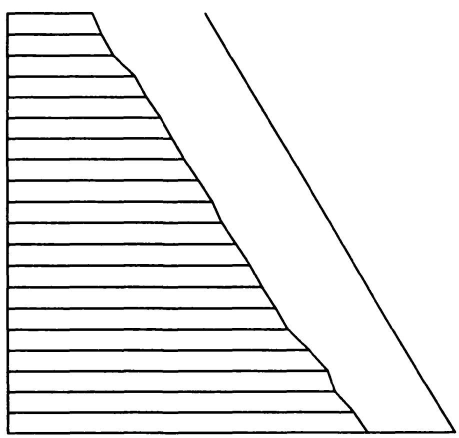

Figure 8. ($h = L/20$). Initial cost = 86.243. Final cost = 57.765.

6. Conclusions

Problem 2 better represents physical reality since the elastic body is considered with sufficiently large surroundings — the foundation. As to problem 1, the prescribed zero displacements along the bottom of the optimized domain cause an unrealistically high stress peak. Namely, at the place loaded with maximum hydrostatic pressure the zero displacement is prescribed as well.

Numerical results confirm this hypothesis. While for $h = L/10$ the final value of the cost functional for problem 1 is greater than that of problem 2 by 0.7 percent, for refined mesh — $h = L/20$ the difference increases up to 19 percent. The numerical results presented show that it is not sufficient for the conclusions to consider mesh characterized by $h = L/10$, because such a mesh is not fine enough to express the stress singularity in problem 1.

The authors intend to investigate elastic bodies with two curved sides (design variables) in a forthcoming paper.

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