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# ON KALMBACH MEASURABILITY

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*Summary.* In this note we show that, for an arbitrary orthomodular lattice $L$, when $\mu$ is a faithful, finite-valued outer measure on $L$, then the Kalmbach measurable elements of $L$ form a Boolean subalgebra of the centre of $L$.

*Keywords*: Kalmbach measurability, Boolean algebra, orthomodular lattice

*AMS classification*: 28B, D6E, D6C

## 1. INTRODUCTION

In non-commutative measure theory, which is being developed because of the desire to investigate the mathematical foundations of quantum mechanics, see [1], [3], [6] and [7], one replaces the notion of a Boolean algebra by the notion of an orthomodular lattice.

In [5] Kalmbach considers outer measures defined on an orthomodular lattice $L$ and extends the Caratheodory notion of measurability to this general setting. She proves that when $L$ is a dimension lattice then the Kalmbach measurable elements form a Boolean algebra.

In this note we show that, for an arbitrary orthomodular lattice $L$, when $\mu$ is a faithful, finite-valued outer measure on $L$, then the Kalmbach measurable elements of $L$ form a Boolean subalgebra of the centre of $L$. Throughout this note $L$ will be an orthomodular lattice. Our standard references for orthomodular lattices are [2], [4].

We shall define a function $\mu\colon L \to [0, +\infty)$ to be a *finitely additive outer measure* if the following conditions are satisfied:

(i) $\mu(0) = 0$,
(ii) $\mu(p \vee q) \leqslant \mu(p) + \mu(q)$ whenever $p \perp q$,
(iii) if $p \leqslant q$ then $\mu(p) \leqslant \mu(q)$.

In [5] a stronger condition than (ii) is imposed. The outer measures defined in [5] are clearly finitely additive outer measures. The converse is false, in general. If $\mu(p) = 0$ implies $p = 0$ then $\mu$ is said to be *faithful.* If $\mu(p) < +\infty$ for all $p$, then $\mu$ is said to be *finite valued.*

## 2. Kalmbach measurable sets

Let $\mu$ be an outer measure on $L$. Then, see [5], $f \in L$ is said to be *Kalmbach measurable* (with respect to $\mu$) if

$$(*) \qquad \mu(x) = \mu\big(x \wedge (x' \vee f)\big) + \mu\big(x \wedge (x' \vee f')\big)$$

for each $x$ in $L$.

More generally, whenever $\mu$ is a function from $L$ to an abelian group $G$, we may define $f \in L$ to be *Kalmbach measurable* if $(*)$ holds for each $x$ in $L$. The reader whose primary interest is in real valued measures may interpret $G$ as the additive group of reals.

**Theorem.** *Let $L$ be an orthomodular lattice and let $G$ be an abelian group. Let $\mu$ be a $G$-valued function on $L$ such that $\mu(a) = 0$ precisely when $a = 0$. Then the Kalmbach measurable elements of $L$ form a Boolean subalgebra, $B$, of the centre of $L$. Furthermore the restriction of $\mu$ to $B$ is additive.*

Proof. Let $f$ be a Kalmbach measurable element and let $e$ be an element of $L$. We put

$$e_0 = (e' \vee f') \wedge (e' \vee f) \wedge e$$

and we observe that

$$e_0' \vee f' = (e \wedge f) \vee (e \wedge f') \vee e' \vee f' = 1,$$
$$e_0' \vee f = (e \wedge f) \vee (e \wedge f') \vee e' \vee f = 1.$$

From the Kalmbach measurability of $f$ it follows that

$$\mu(e_0) = \mu\big(e_0 \wedge (e_0' \vee f')\big) + \mu\big(e_0 \wedge (e_0' \vee f)\big) = \mu(e_0) + \mu(e_0).$$

Then $\mu(e_0) = 0$ and hence $e_0 = 0$.

We now write the upper commutator of $e$ and $f$:

$$\begin{aligned}
&(e' \vee f') \wedge (e' \vee f) \wedge (e \vee f) \wedge (e \vee f') = \\
&\Big(\big((e' \vee f') \wedge (e' \vee f) \wedge e\big) \vee \big((e' \vee f') \wedge (e' \vee f) \wedge f\big)\Big) \wedge \\
&\Big(\big((e' \vee f') \wedge (e' \vee f) \wedge e\big) \vee \big((e' \vee f') \wedge (e' \vee f) \wedge f'\big)\Big) = \\
&(e' \vee f') \wedge (e' \vee f) \wedge f \wedge f' = 0.
\end{aligned}$$

Thus $e$ and $f$ commute and we have proved that every Kalmbach measurable element is in the centre of $L$. It follows that an element $p$ of $L$ is Kalmbach measurable if

(i) $p$ is central,
(ii) $\mu(x) = \mu(x \wedge p) + \mu(x \wedge p')$ for every $x$ in $L$.

Suppose $f$ and $g$ are both Kalmbach measurable. Then $f \vee g$ is central and $x$, $f$ and $g$ are mutually commutative. The equality

$$\mu\big(x \wedge (f \vee g)\big) + \mu\big(x \wedge (f \vee g)'\big) = \mu(x)$$

can be proved as in classical measure theory. □

**Corollary.** *Let $L$ be an orthomodular lattice. Let $\mu\colon L \to [0, +\infty)$ be a faithful finite additive outer measure. Then the elements of $L$ which are Kalmbach measurable with respect to $\mu$ form a Boolean sublattice of the centre of $L$.*

*References*


[1] *E. Beltrametti, G. Cassinelli*: The logic of quantum mechanics. Addison Wesley, Reading MA, 1981.
[2] *L. Beran*: Orthomodular Lattices. Algebraic approach. Academia, Prague and Reidel, Dordrecht, 1984.
[3] *S. Gudder*: Stochastic methods in quantum mechanics. North-Holland, Amsterdam, 1979.
[4] *G. Kalmbach*: Orthomodular lattices. Academic Press, London, 1983.
[5] *G. Kalmbach*: Quantum measure spaces. Foundations of Phys. *20* (1990), 801–821.
[6] *P. Pták, S. Pulmannova*: Orthomodular structures as quantum logics. Kluwer, Dordrecht, 1991.
[7] *V. S. Varadarajan*: Geometry of quantum theory. Springer-Verlag, Berlin, 1985.



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