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ERROR ESTIMATES OF AN ITERATIVE METHOD FOR A QUASISTATIC ELASTIC-VISCO-PLASTIC PROBLEM

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Summary. This paper deals with an initial and boundary value problem describing the quasistatic evolution of rate-type viscoplastic materials. Using a fixed point property, an iterative method in the study of this problem is proposed. A concrete algorithm as well as some numerical results in the one-dimensional case are also presented.

Keywords: rate-type models, viscoelasticity, viscoplasticity, fixed point, iterative method, error estimates, finite element method

1. INTRODUCTION

In this paper we present a numerical method for a nonlinear evolution problem in the study of viscoplastic rate-type models. Only the case of small deformations and small rotations is considered hence in this case the Cauchy stress tensor and the two Piola-Kirchhoff stress tensors coincide. With these assumptions the constitutive equation considered here is of the form

$$\dot{\sigma} = \mathcal{E}\dot{\varepsilon} + F(\sigma, \varepsilon) \tag{1.1}$$

in which σ is the stress tensor, ε is the small strain tensor and $\mathcal{E}$, F are given constitutive functions (in (1.1) and everywhere in this paper the dot above a quantity represents the derivative with respect to the time variable of that quantity).

Such type of equations generalizes some classical models used in viscoelasticity and viscoplasticity and is used for describing the behaviour of real materials like rubbers, metals, rocks and so on. Various results, mechanical interpretations as well as concrete examples concerning constitutive laws of the form (1.1) may be found for instance in the papers of Geiringer and Freudenthal [1], Cristescu and Suliciu [2], Suliciu [3].

In the paper of Ionescu and Sofonea [4], a quasistatic initial and boundary value problem for this type of materials is considered. Results concerning existence, stability, asymptotic and large time behaviour of the solution are obtained. The main idea used in this paper in order to obtain existence and uniqueness of the solution is the equivalence between the mechanical problem and an ordinary differential equation in a product Hilbert space followed by classical Cauchy-Lipschitz arguments. This idea was used also in Ionescu [5] where a numerical approach to the problem based on a Euler method is presented.

A new demonstration of the existence result of [4] was given in the paper of Djabi and Sofonea [6]. This demonstration is based only on classical existence results of linear elasticity followed by a fixed point technique.

The purpose of this paper is to continue the ideas of [6] and to present an iterative method in the study of the quasistatic problem of [4]. So, in Section 2 the necessary notation is introduced and some preliminary results are recalled; in Section 3 the mechanical problem is stated and, for the convenience of the reader, some results and techniques from Djabi and Sofonea [6] that will be useful in this work, are briefly presented; in Section 4 we present a semi-discretisation method and give an estimate of the error (Theorem 4.1); a final algorithm for the numerical approach of the solution is presented in Section 5 and finally some numerical results are discussed in Section 6.

2. Notation and preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N = 1, 2, 3)$ with a Lipschitz boundary $\partial\Omega = \Gamma$ (see for instance Nečas and Hlaváček [7] p. 17). Let Γ_1 be an open subset of Γ such that $\operatorname{meas}\Gamma_1 > 0$. Let $\Gamma_2 = \Gamma - \overline{\Gamma}_1$; we denote by ν the outward unit normal vector on Γ and by S_N the set of second order symmetric tensors on $\mathbb{R}^N$. Let " $\cdot$ " denote the inner product on the spaces $\mathbb{R}^N$ and S_N and let $|\cdot|$ stand for the Euclidean norms on these spaces. The following notation is also used:

$$H = [L^2(\Omega)]^N,\ H_1 = [H^1(\Omega)]^N,\ H_\Gamma = [H^{\frac{1}{2}}(\Gamma)]^N,$$
$$\mathcal{H} = [L^2(\Omega)]_s^{N\times N},\ \mathcal{H}_1 = \{\sigma \in \mathcal{H} \mid \operatorname{Div}\sigma \in H\}$$

where $\operatorname{Div}\sigma$ is the divergence of the vector-valued function σ. The spaces H, H_1, H_Γ, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products denoted by $\langle\cdot,\cdot\rangle_H$, $\langle\cdot,\cdot\rangle_{H_1}$, $\langle\cdot,\cdot\rangle_{H_\Gamma}$, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$, respectively. The norms on these spaces will be denoted by $|\cdot|_H$, $|\cdot|_{H_1}$, $|\cdot|_{H_\Gamma}$, $|\cdot|_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$.

We also consider the closed subspace of H_1 defined by

$$V = \{u \in H_1 \mid \gamma u = 0 \text{ on } \Gamma_1\}$$

where $\gamma\colon H_1 \to H_\Gamma$ is the trace map. Let E be the subspace of H_Γ defined by

$$E = \gamma(V) = \{\xi \in H_\Gamma \mid \xi = 0 \text{ on } \Gamma_1\}.$$

We denote by V' the strong dual of V, by $|\cdot|_V$ the restriction of $|\cdot|_{H_1}$ to V and by $\langle\cdot,\cdot\rangle_{V'\times V}$ the duality between V' and V.

The deformation operator $\varepsilon\colon H_1 \to \mathcal{H}$ defined by

$$\varepsilon(u) = \tfrac{1}{2}(\nabla u + \nabla^t u)$$

is a linear and continuous operator. Moreover, since $\operatorname{meas}\Gamma_1 > 0$, Korn's inequality holds:

$$|\varepsilon(v)|_{\mathcal{H}} \geqslant C|v|_{H_1} \quad \text{for all } v \in V \tag{2.1}$$

where C is a strictly positive constant which depends only on Ω and Γ_1 (everywhere in this paper C will represent strictly positive generic constants that depend on Ω, Γ_1, $\mathcal{E}$, F and do not depend on time or on input data).

Let $H'_\Gamma = \left[H^{-\frac{1}{2}}(\Gamma)\right]^N$ be the strong dual of the space H_Γ and let $\langle\cdot,\cdot\rangle_{H'_\Gamma\times H_\Gamma}$ denote the duality between H'_Γ and H_Γ. If $\tau \in \mathcal{H}_1$ there exists an element $\gamma_\nu\tau \in H'_\Gamma$ such that

$$\langle\gamma_\nu\tau, \gamma v\rangle_{H'_\Gamma\times H_\Gamma} = \langle\tau, \varepsilon(v)\rangle_{\mathcal{H}} + \langle\operatorname{Div}\tau, v\rangle_H \quad \text{for all } v \in H_1. \tag{2.2}$$

By $\tau\nu|_{\Gamma_2}$ we shall understand the element of E' (the strong dual of E), that is the restriction of $\gamma_\nu\tau$ to E, and $\langle\cdot,\cdot\rangle_{E'\times E}$ will denote the duality between E' and E.

Let us now denote by $\mathcal{V}$ the following subspace of $\mathcal{H}_1$:

$$\mathcal{V} = \{\tau \in \mathcal{H}_1 \mid \operatorname{Div}\tau = 0 \text{ in } \Omega,\ \tau\nu = 0 \text{ on } \Gamma_2\}.$$

As it follows from Nečas and Hlaváček [7] p. 105, $\varepsilon(V)$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$, hence

$$\langle\tau, \varepsilon(v)\rangle_{\mathcal{H}} = 0 \quad \text{for all } v \in V,\ \tau \in \mathcal{V}. \tag{2.3}$$

In the sequel, for every real Hilbert space X we denote by $|\cdot|_X$ the norm on X and, for $T > 0$ and $j \in \{0,1\}$, $C^j(0,T,X)$ will denote the spaces defined as follows:

$$C^0(0,T,X) = \{z\colon [0,T] \to X \mid z \text{ is continuous}\},$$
$$C^1(0,T,X) = \{z\colon [0,T] \to X \mid \text{the derivative } \dot z \text{ of } z \text{ exists and } \dot z \in C^0(0,T,X)\}.$$

In a similar way the spaces $C^0(\mathbb{R}_+, X)$ and $C^1(\mathbb{R}_+, X)$, where $\mathbb{R}_+ = [0, +\infty)$, can be defined.

Finally, let us recall that $C^j(0, T, X)$ are real Banach spaces endowed with the norms

$$|z|_{0,T,X} = \max_{t\in[0,T]} |z(t)|_X \tag{2.4}$$

$$|z|_{1,T,X} = |z|_{0,T,X} + |\dot{z}|_{0,T,X}. \tag{2.5}$$

3. Problem statement. An existence and uniqueness result

Let us consider the mixed problem

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + F\big(\sigma, \varepsilon(u)\big) \quad \text{in } \Omega \times (0, T) \tag{3.1}$$

$$\operatorname{Div} \sigma + b = 0 \quad \text{in } \Omega \times (0, T) \tag{3.2}$$

$$u = f \quad \text{on } \Gamma_1 \times (0, T) \tag{3.3}$$

$$\sigma\nu = g \quad \text{on } \Gamma_2 \times (0, T) \tag{3.4}$$

$$u(0) = u_0, \ \sigma(0) = \sigma_0 \quad \text{in } \Omega \tag{3.5}$$

in which $T > 0$ is a time interval and the unknowns are the displacement function u: $\Omega \times [0, T] \to \mathbb{R}^N$ and the stress function σ: $\Omega \times [0, T] \to S_N$. This problem models the quasistatic evolution of a continuous body that occupies the domain Ω in its present configuration, subjected to given body forces, to an imposed displacement on Γ_1 and to given surface tractions applied to the part Γ_2 of its boundary. (3.1) represents the constitutive equation in which $\mathcal{E}$ is a fourth order tensor and F is a given function, (3.2) is the Cauchy equilibrium equation, (3.3)–(3.4) represent the boundary conditions and finally (3.5) represents the initial conditions.

In the study of the problem (3.1)–(3.5), we consider the following assumptions:

$$\begin{cases} \mathcal{E}\colon \Omega \times S_N \to S_N \text{ is a symmetric and positively definited tensor, i.e.} \\ \text{(a) } \mathcal{E}_{ijkh} \in L^\infty(\Omega) \quad \text{for all } i, j, k, h = \overline{1, N} \\ \text{(b) } \mathcal{E}\sigma \cdot \tau = \sigma \cdot \mathcal{E}\tau \quad \forall \sigma, \tau \in S_N, \text{ a.e. in} \Omega \\ \text{(c) there exists } \alpha > 0 \text{ such that } \mathcal{E}\sigma \cdot \sigma \geqslant \alpha|\sigma|^2 \\ \quad \text{for all } \sigma \in S_N, \text{ a.e. in } \Omega \end{cases} \tag{3.6}$$

$$
(3.7)\qquad \begin{cases} F\colon \Omega \times S_N \times S_N \to S_N \text{ and} \\ \text{(a) there exists } L > 0 \text{ such that} \\ \quad |F(x,\sigma_1,\varepsilon_1) - F(x,\sigma_2,\varepsilon_2)| \leqslant L(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|) \\ \quad \text{for all } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2 \in S_N, \text{ a.e. in } \Omega \\ \text{(b) } x \to F(x,\sigma,\varepsilon) \text{ is a measurable function with respect to} \\ \quad \text{the Lebesgue measure on } \Omega, \quad \text{for all } \sigma,\varepsilon \in S_N \\ \text{(c) } x \to F(x,0,0) \in \mathcal{H} \end{cases}
$$

$$
(3.8)\qquad b \in C^1(0,T,H),\ f \in C^1(0,T,H_\Gamma),\ g \in C^1(0,T,E')
$$

$$
(3.9)\qquad u_0 \in H_1,\ \sigma_0 \in \mathcal{H}_1
$$

$$
(3.10)\qquad \operatorname{Div}\sigma_0 + b(0) = 0 \text{ in } \Omega,\ u_0 = f(0) \text{ on } \Gamma_1,\ \sigma_0\nu = g(0) \text{ on } \Gamma_2.
$$

In the paper of Ionescu and Sofonea [4] it is proved that, under the assumptions (3.6)–(3.10), problem (3.1)–(3.5) has a unique solution $u \in C^1(0,T,H_1)$, $\sigma \in C^1(0,T,\mathcal{H}_1)$. Moreover, as results from the paper of Djabi and Sofonea [6], the existence of this solution can be obtained in the following way: let $\eta \in C^0(0,T,\mathcal{H})$ be an arbitrary function and let $z_\eta \in C^1(0,T,\mathcal{H})$ be the function defined by

$$
(3.11)\qquad z_\eta(t) = \int_0^t \eta(s)\,\mathrm{d}s + z_0 \quad \text{for all } t \in [0,T]
$$

where

$$
(3.12)\qquad z_0 = \sigma_0 - \mathcal{E}\varepsilon(u_0).
$$

Let also $u_\eta \in C^1(0,T,H_1)$, $\sigma_\eta \in C^1(0,T,\mathcal{H}_1)$ be the solution of the elastic problem

$$
(3.13)\qquad \sigma_\eta = \mathcal{E}\varepsilon(u_\eta) + z_\eta \quad \text{in } \Omega \times (0,T)
$$

$$
(3.14)\qquad \operatorname{Div}\sigma_\eta + b = 0 \quad \text{in } \Omega \times (0,T)
$$

$$
(3.15)\qquad u_\eta = f \quad \text{on } \Gamma_1 \times (0,T)
$$

$$
(3.16)\qquad \sigma_\eta\nu = g \quad \text{on } \Gamma_2 \times (0,T).
$$

and let $\Lambda\colon C^0(0,T,\mathcal{H}) \to C^0(0,T,\mathcal{H})$ be the operator defined by

$$
(3.17)\qquad \Lambda\eta(t) = F\big(\sigma_\eta(t), \varepsilon\big(u_\eta(t)\big)\big) \quad \text{for all } t \in [0,T].
$$

Denoting by Λ^p the powers of the operator Λ ($p \in \mathbb{N}$), for p large enough Λ^p is a contraction in $C^0(0,T,\mathcal{H})$, hence Λ has a unique fixed point $\eta^* \in C^0(0,T,\mathcal{H})$. It results that $u_{\eta^*} \in C^1(0,T,H_1)$, $\sigma_{\eta^*} \in C^1(0,T,\mathcal{H}_1)$ is a solution of (3.1)–(3.5).

Remark 3.1. Similar fixed point techniques in the study of elastic-inelastic materials with internal state variable were also used by Kratochvíl and Nečas [8], John [9] and Laborde [10], [11].

Remark 3.2. Problem (3.1)–(3.5) may also be considered in the case of the infinite time interval $(0, +\infty)$ instead of $(0, T)$. In this case, if (3.6), (3.7), (3.9), (3.10) are fulfilled and

$$b \in C^1(\mathbb{R}_+, H), \ f \in C^1(\mathbb{R}_+, H_\Gamma), \ g \in C^1(\mathbb{R}_+, E')$$

then problem (3.1)–(3.5) has a unique solution (u, σ) having the regularity $u \in C^1(\mathbb{R}_+, H_1)$, $\sigma \in C^1(\mathbb{R}_+, \mathcal{H}_1)$ (for the proof of this result see Ionescu and Sofonea [4]).

4. A NUMERICAL APPROACH

As follows from the previous section, the existence and uniqueness of the solution for the problem (3.1)–(3.5) may obtained in two steps: the study of the elastic problem (3.13)–(3.16) defined for every $\eta \in C^0(0, T, \mathcal{H})$ and the fixed point property of the operator Λ defined by (3.17). So, in order to obtain a numerical approximation of the solution for the problem (3.1)–(3.5), we start by presenting an approximation in the space of the elastic problem (3.13)–(3.16).

Let us suppose in the sequel that (3.6)–(3.10) hold. Let $\eta \in C^0(0, T, \mathcal{H})$ and let $z_\eta \in C^1(0, T, \mathcal{H})$ be defined by (3.11), (3.12). Using (3.8) we obtain that there exists $\tilde{u} \in C^1(0, T, H_1)$ such that

$$\tilde{u} = f \text{ on } \Gamma_1 \times (0, T). \tag{4.1}$$

Let $a \colon V \times V \to \mathbb{R}$ and $l_\eta \colon [0, T] \to V'$ be defined by

$$a(u, v) = \langle \mathcal{E}\varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}} \tag{4.2}$$

$$\begin{cases} \langle l_\eta(t), v \rangle_{V' \times V} = \langle b(t), v \rangle_H + \langle g(t), \gamma v \rangle_{E' \times E} \\ - \langle \mathcal{E}\varepsilon(\tilde{u}(t)), \varepsilon(v) \rangle_{\mathcal{H}} - \langle z_\eta(t), \varepsilon(v) \rangle_{\mathcal{H}} \end{cases} \tag{4.3}$$

for all $u, v \in V$ and $t \in [0, T]$.

Using (3.6)–(3.8) we get that a is a bilinear symmetric and coercive form on V, $l_\eta \in C^1(0, T, V')$ and, by a standard argument, it results that $u_\eta \in C^1(0, T, H_1)$, $\sigma_\eta \in C^1(0, T, \mathcal{H})$ is a solution of the elastic problem (3.13)–(3.16) if and only if

$$\begin{cases} u_\eta = \overline{u}_\eta + \tilde{u} \\ \overline{u}_\eta \in V, \ a(\overline{u}_\eta, v) = \langle l_\eta, v \rangle_{V' \times V} \quad \text{for all } v \in V \\ \sigma_\eta = \mathcal{E}\varepsilon(u_\eta) + z_\eta \end{cases} \tag{4.4}$$

for all $t \in [0,T]$.

Let now V_h be a closed subspace included in V. We consider the following problem: find $u_\eta^h\colon [0,T] \to H_1$, $\sigma_\eta^h\colon [0,T] \to \mathcal{H}$ such that

$$(4.5)\qquad \begin{cases} u_\eta^h = \overline{u}_\eta^h + \tilde{u} \\ \overline{u}_\eta^h \in V_h,\ a(\overline{u}_\eta^h, v) = \langle l_\eta, v_h\rangle_{V'\times V} \quad \text{for all } v_h \in V_h \\ \sigma_\eta^h = \mathcal{E}\varepsilon(u_\eta^h) + z_\eta \end{cases}$$

for all $t \in [0,T]$.

Using a standard argument we obtain that (4.5) has a unique solution $u_\eta^h \in C^1(0,T,H_1)$, $\sigma_\eta^h \in C^1(0,T,\mathcal{H})$. Moreover, if $(u_{\eta_1}^h, \sigma_{\eta_1}^h)$ and $(u_{\eta_2}^h, \sigma_{\eta_2}^h)$ are the solutions of (4.5) for $\eta = \eta_1$ and $\eta = \eta_2$, there exists $C > 0$ which depends only on Ω, Γ_1 and $\mathcal{E}$ such that

$$(4.6)\qquad |u_{\eta_1}^h(t) - u_{\eta_2}^h(t)|_{H_1} + |\sigma_{\eta_1}^h(t) - \sigma_{\eta_1}^h(t)|_{\mathcal{H}} \leqslant C|z_{\eta_1}(t) - z_{\eta_2}(t)|_{\mathcal{H}} \quad \text{for all } t \in [0,T]$$

$$(4.7)\qquad |u_{\eta_1}^h - u_{\eta_2}^h|_{j,T,H_1} + |\sigma_{\eta_1}^h - \sigma_{\eta_2}^h|_{j,T,\mathcal{H}} \leqslant C|z_{\eta_1} - z_{\eta_2}|_{j,T,\mathcal{H}} \quad \text{for all } j = 0,1.$$

Let us denote by (u_η, σ_η) the solution of (4.4) and let $S_\eta^h(j,T)$ be the quantities defined by

$$(4.8)\quad S_\eta^h(0,T) = \sup_{t\in[0,T]} \big(\inf_{v_h\in V_h} |\overline{u}_\eta(t) - v_h|_V\big)$$

$$(4.9)\quad S_\eta^h(1,T) = \sup_{t\in[0,T]} \big(\inf_{v_h\in V_h} |\overline{u}_\eta(t) - v_h|_V\big) + \sup_{t\in[0,T]} \big(\inf_{v_h\in V_h} |\dot{\overline{u}}_\eta(t) - v_h|_V\big).$$

The distance between the couples $(u_\eta^h, \sigma_\eta^h)$ and (u_η, σ_η) is given by the following result:

Lemma 4.1. *There exists $\tilde{C}$ which depends only on Ω, Γ_1 and $\mathcal{E}$ such that*

$$(4.10)\qquad |u_\eta^h - u_\eta|_{j,T,H_1} + |\sigma_\eta^h - \sigma_\eta|_{j,T,\mathcal{H}} \leqslant \tilde{C} S_\eta^h(j,T) \quad \textit{for all } j = 0,1.$$

Proof. Using classical results for elliptic variational inequalities (see for instance Ciarlet [12] p. 186), from (4.4) and (4.5) we get

$$(4.11)\qquad |u_\eta^h(t) - u_\eta(t)|_{H_1} \leqslant C_1 \inf_{v_h\in V_h} |\overline{u}_\eta(t) - v_h|_V \quad \text{for all } t \in [0,T]$$

where $C_1 > 0$ depends only on Ω, Γ_1 and $\mathcal{E}$. In a similar way, taking the derivative with respect to the time variable in (4.4) and (4.5) we obtain

$$|\dot{u}_\eta^h(t) - \dot{u}_\eta(t)|_{H_1} \leqslant C_1 \inf_{v_h \in V_h} |\dot{\bar{u}}_\eta(t) - v_h|_V \quad \text{for all } t \in [0,T]. \tag{4.12}$$

Using now the notation (4.8), (4.9), the inequalities (4.11), (4.12) imply

$$|u_\eta^h - u_\eta|_{j,T,H_1} \leqslant C_1 S_\eta^h(j,T) \quad \text{for } j = 0,1. \tag{4.13}$$

Moreover, from (4.4), (4.5) we obtain

$$|\sigma_\eta^h - \sigma_\eta|_{j,T,\mathcal{H}} \leqslant C_2 |u_\eta^h - u_\eta|_{j,T,H_1} \quad \text{for all } j = 0,1 \tag{4.14}$$

where C_2 depends only on $\mathcal{E}$ and, using (4.13), we conclude that

$$|\sigma_\eta^h - \sigma_\eta|_{j,T,\mathcal{H}} \leqslant C_1 C_2 S_\eta^h(j,T) \quad \text{for all } j = 0,1. \tag{4.15}$$

The inequality (4.10) is now a consequence of (4.13), (4.15). □

We now study the discrete version of the fixed point property of the operator Λ defined by (3.17). As in the continuous case, let us now define an operator $\Lambda_h\colon C^0(0,T,\mathcal{H}) \to C^0(0,T,\mathcal{H})$ by the equality

$$\Lambda_h \eta(t) = F\big(\sigma_\eta^h(t), \varepsilon(u_\eta^h(t))\big) \quad \text{for all } \eta \in C^0(0,T,\mathcal{H}) \text{ and } t \in [0,T]. \tag{4.16}$$

Let $\eta_1, \eta_2 \in C^0(0,T,\mathcal{H})$; using (3.7), (4.6) and (3.11) we obtain

$$|\Lambda_h \eta_1(t) - \Lambda_h \eta_2(t)|_{\mathcal{H}} \leqslant CL \int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}} \, \mathrm{d}s \quad \text{for all } t \in [0,T] \tag{4.17}$$

and, by recurrence, denoting by Λ_h^p the powers of the operator Λ_h, we get

$$|\Lambda_h^p \eta_1 - \Lambda_h^p \eta_2|_{0,T,\mathcal{H}} \leqslant \frac{(CLT)^p}{p!} |\eta_1 - \eta_2|_{0,T,\mathcal{H}} \quad \text{for all } p \in \mathbb{N}. \tag{4.18}$$

The inequality (4.18) shows that for p large enough the operator Λ_h^p is a contraction in $C^0(0,T,\mathcal{H})$, hence the operator Λ_h has a unique fixed point $\eta_h^* \in C^0(0,T,\mathcal{H})$.

Now let η^* be the fixed point of the operator Λ defined by (3.17); as results from Section 3, the solution $(u_{\eta^*}, \sigma_{\eta^*})$ of the elastic problem (3.13)–(3.16) for $\eta = \eta^*$ is the solution of the viscoplastic problem (3.1)–(3.5), i.e.

$$u_{\eta^*} = u, \quad \sigma_{\eta^*} = \sigma. \tag{4.19}$$

For this reason we are interested in examining the distance between the couples $(u^h_{\eta^*_h}, \sigma^h_{\eta^*_h})$ and $(u_{\eta^*}, \sigma_{\eta^*})$.

Lemma 4.2. *Let C and $\tilde{C}$ be the constants of* (4.6) *and* (4.10) *and $K = CLT$. Then*

$$\begin{cases} |u^h_{\eta^*_h} - u_{\eta^*}|_{j,T,H_1} + |\sigma^h_{\eta^*_h} - \sigma_{\eta^*}|_{j,T,\mathcal{H}} \\ \leqslant C(T+j)L\tilde{C}e^K S^h_{\eta^*}(0,T) + \tilde{C}S^h_{\eta^*}(j,T) \quad \text{for } j = 0,1. \end{cases} \tag{4.20}$$

P r o o f. Since $\eta^*_h = \Lambda_h \eta^*_h$ and $\eta^* = \Lambda\eta^*$, using (3.17), (4.16), (4.17) and (3.7) we obtain

$$\begin{aligned} |\eta^*_h(t) - \eta^*(t)|_{\mathcal{H}} &\leqslant |\Lambda_h\eta^*_h(t) - \Lambda_h\eta^*(t)|_{\mathcal{H}} + |\Lambda_h\eta^*(t) - \Lambda\eta^*(t)|_{\mathcal{H}} \\ &\leqslant CL\int_0^t |\eta^*_h(s) - \eta^*(s)|_{\mathcal{H}}\,\mathrm{d}s + L\big(|u^h_{\eta^*}(t) \\ &\quad - u_{\eta^*}(t)|_{H_1} + |\sigma^h_{\eta^*}(t) - \sigma_{\eta^*}(t)|_{\mathcal{H}}\big) \end{aligned}$$

for all $t \in [0,T]$.

If we apply (4.10) for $\eta = \eta^*$ and $j = 0$, this inequality becomes

$$|\eta^*_h(t) - \eta^*(t)|_{\mathcal{H}} \leqslant L\tilde{C}S^h_{\eta^*}(0,T) + CL\int_0^t |\eta^*_h(s) - \eta^*(s)|_{\mathcal{H}}\,\mathrm{d}s \quad \text{for all } t \in [0,T]$$

and, using a Gronwall-type inequality, we get

$$|\eta^*_h(t) - \eta^*(t)|_{\mathcal{H}} \leqslant L\tilde{C}e^K S^h_{\eta^*}(0,T) \quad \text{for all } t \in [0,T]. \tag{4.21}$$

Let us also remark that from (3.11) we obtain

$$|z_{\eta^*_h} - z_{\eta^*}|_{j,T,\mathcal{H}} \leqslant (T+j)|\eta^*_h - \eta^*|_{0,T,\mathcal{H}} \quad \text{for all } j = 0,1,$$

hence by (4.21) it results that

$$|z_{\eta^*_h} - z_{\eta^*}|_{j,T,\mathcal{H}} \leqslant (T+j)L\tilde{C}e^K S^h_{\eta^*}(0,T) \quad \text{for all } j = 0,1. \tag{4.22}$$

Using now (4.7) for $\eta_1 = \eta^*_h$ and $\eta_2 = \eta^*$ we obtain

$$|u^h_{\eta^*_h} - u^h_{\eta^*}|_{j,T,H_1} + |\sigma^h_{\eta^*_h} - \sigma^h_{\eta^*}|_{j,T,\mathcal{H}} \leqslant C|z_{\eta^*_h} - z_{\eta^*}|_{j,T,\mathcal{H}} \quad \text{for all } j = 0,1 \tag{4.23}$$

and using again (4.10) for $\eta = \eta^*$ we conclude

$$|u^h_{\eta^*} - u_{\eta^*}|_{j,T,H_1} + |\sigma^h_{\eta^*} - \sigma_{\eta^*}|_{j,T,\mathcal{H}} \leqslant \tilde{C}S^h_{\eta^*}(j,T) \quad \text{for all } j = 0,1. \tag{4.24}$$

The inequality (4.20) is now a consequence of (4.22)–(4.24). □

We now consider the iterative part of the method. Let η_0 be an arbitrary element of $C^0(0,T,\mathcal{H})$ and let $(\eta_h^n) \subset C^0(0,T,\mathcal{H})$ be the sequence defined by

$$\eta_h^0 = \eta_0, \; \eta_h^{n+1} = \Lambda_h \eta_h^n \quad \text{for all } n \in \mathbb{N}. \tag{4.25}$$

Let $(u^h_{\eta_h^n}, \sigma^h_{\eta_h^n})$ be the solution of (4.5) for $\eta = \eta_h^n$ and recall that $(u^h_{\eta_h^*}, \sigma^h_{\eta_h^*})$ is the solution of (4.5) for $\eta = \eta_h^*$. The distance between the couples $(u^h_{\eta_h^n}, \sigma^h_{\eta_h^n})$ and $(u^h_{\eta_h^*}, \sigma^h_{\eta_h^*})$ is given by

Lemma 4.3. *Let C be the strictly positive constant defined in* (4.6) *and let $K = CLT$. Then*

$$\begin{cases} |u^h_{\eta_h^n} - u^h_{\eta_h^*}|_{j,T,H_1} + |\sigma^h_{\eta_h^n} - \sigma^h_{\eta_h^*}|_{j,T,\mathcal{H}} \\ \leqslant C(T+j)e^K \frac{K^n}{n!}|\Lambda_h\eta_0 - \eta_0|_{0,T,\mathcal{H}} \quad \textit{for all } j = 0,1 \textit{ and } n \in \mathbb{N}. \end{cases} \tag{4.26}$$

Proof. We start by estimating the distance between η_h^n and η_h^*; we remark that for every $m, n \in \mathbb{N}$, $m \geqslant n$, (4.25) and (4.18) yield

$$\begin{aligned} |\eta_h^n - \eta_h^m|_{0,T,\mathcal{H}} &\leqslant |\eta_h^n - \eta_h^{n+1}|_{0,T,\mathcal{H}} + \cdots + |\Lambda_h^{m-n-1}\eta_h^n - \Lambda_h^{m-n-1}\eta_h^{n+1}|_{0,T,\mathcal{H}} \\ &\leqslant \Big(1 + \frac{K}{1!} + \cdots + \frac{K^{m-n-1}}{(m-n-1)!}\Big)|\eta_h^n - \eta_h^{n+1}|_{0,T,\mathcal{H}}. \end{aligned}$$

This inequality implies

$$|\eta_h^n - \eta_h^m|_{0,T,\mathcal{H}} \leqslant e^K |\eta_h^n - \eta_h^{n+1}|_{0,T,\mathcal{H}}$$

and, passing to the limit when $m \to +\infty$, since $\eta_h^m \to \eta_h^*$ in $C^0(0,T,\mathcal{H})$ (consequence of (4.25) and (4.18)), we get

$$|\eta_h^n - \eta_h^*|_{0,T,\mathcal{H}} \leqslant e^K |\eta_h^n - \eta_h^{n+1}|_{0,T,\mathcal{H}}.$$

By (4.25) we get $\eta_h^n = \Lambda_h^n \eta_0$, $\eta_h^{n+1} = \Lambda_h^{n+1}\eta_0$, hence using again (4.18) the last inequality leeds to

$$|\eta_h^n - \eta_h^*|_{0,T,\mathcal{H}} \leqslant e^K \frac{K^n}{n!}|\Lambda_h\eta_0 - \eta_0|_{0,T,\mathcal{H}}. \tag{4.27}$$

Let us denote by $z_{\eta_h^n}$ and $z_{\eta_h^*}$ the elements defined by (3.11) for $\eta = \eta_h^n$ and $\eta = \eta_h^*$. We have

$$|z_{\eta_h^n} - z_{\eta_h^*}|_{j,T,\mathcal{H}} \leqslant (T+j)|\eta_h^n - \eta_h^*|_{0,T,\mathcal{H}} \quad \text{for all } j = 0,1 \text{ and } n \in \mathbb{N} \tag{4.28}$$

and using (4.7) we get

$$|u^h_{\eta_h^n} - u^h_{\eta_h^*}|_{j,T,H_1} + |\sigma^h_{\eta_h^n} - \sigma^h_{\eta_h^*}|_{j,T,\mathcal{H}} \leqslant C|z_{\eta_h^n} - z_{\eta_h^*}|_{j,T,\mathcal{H}} \quad \text{for all } j = 0,1 \text{ and } n \in \mathbb{N}. \tag{4.29}$$

The estimate (4.26) now follows from (4.27)–(4.29). □

In order to come to a conclusion we use (4.19), (4.20), (4.26) and obtain the following estimate of the difference between the solution (u, σ) of the viscoplastic problem (3.1)–(3.5) and the solution $(u^h_{\eta^n_h}, \sigma^h_{\eta^n_h})$ of the approximate problem (4.5) for $\eta = \eta^n_h$:

Theorem 4.1. *There exist C, $\tilde{C}$ which depend only on Ω, Γ_1 and $\mathcal{E}$ such that*

$$\left\{ \begin{array}{l} |u^h_{\eta^n_h} - u|_{j,T,H_1} + |\sigma^h_{\eta^n_h} - \sigma|_{j,T,\mathcal{H}} \leqslant C(T+j)L\tilde{C}e^K S^h_{\eta^*}(0,T) \\ + \tilde{C}S^h_{\eta^*}(j,T) + C(T+j)e^K \frac{K^n}{n!} |\Lambda_h \eta_0 - \eta_0|_{0,T,\mathcal{H}} \end{array} \right. \tag{4.30}$$

for all $j = 0, 1$, $n \in \mathbb{N}$ where $K = CLT$.

5. The final algorithm

In this section we propose a numerical algorithm which can be directly run on a computer, in order to approximate the solution (u, σ) of the viscoplastic problem (3.1)–(3.5). This algorithm is based on the approximation of the unknowns in space and time. As results from Section 4, the approximation in space is realized by considering a closed subspace V_h of V and replacing problem (3.1)–(3.5) by the following sequence of linear problems:

Find $u^n_h \colon [0,T] \to H_1$, $\sigma^n_h \colon [0,T] \to \mathcal{H}$ such that

$$\left\{ \begin{array}{l} u^n_h = \overline{u}^n_h + \tilde{u} \\ \overline{u}^n_h \in V_h,\ a(\overline{u}^n_h, v_h) = \langle l^n_h, v_h \rangle_{V' \times V} \quad \text{for all } v_h \in V_h \\ \sigma^n_h = \mathcal{E}\varepsilon(u^n_h) + z^n_h \end{array} \right. \tag{5.1}$$

for all $t \in [0,T]$, where $l^n_h \colon [0,T] \to V'$ is the functional defined by (4.3) for $\eta = \eta^n_h$, i.e.

$$\left\{ \begin{array}{l} \langle l^n_h(t), v \rangle_{V' \times V} = \langle b(t), v \rangle_H + \langle g(t), \gamma v \rangle_{E' \times E} \\ -\langle \mathcal{E}\varepsilon(\tilde{u}(t)), \varepsilon(v) \rangle_{\mathcal{H}} - \langle z^n_h(t), \varepsilon(v) \rangle_{\mathcal{H}} \end{array} \right. \tag{5.2}$$

for all $u, v \in V$ and $t \in [0,T]$. In (5.2) we have

$$z^n_h(t) = \int_0^t \eta^n_h(s)\, \mathrm{d}s + z_0 \quad \text{for all } t \in [0,T], \tag{5.3}$$

η^n_h is recursively defined by the equalities

$$\eta^n_h(t) = \Lambda_h \eta^{n-1}_h(t) = F\big(\sigma^{n-1}_h(t), \varepsilon\big(u^{n-1}_h(t)\big)\big) \quad \text{for all } t \in [0,T] \text{ and } n \in \mathbb{N} \tag{5.4}$$

and $\eta_h^0 = \eta_0$ is an arbitrary element of the space $C^0(0,T,\mathcal{H})$.

In practice V_h is a finite dimensional subspace of V (constructed for instance by the finite element method), hence (5.1) is in fact a linear algebraic system.

Let us now consider $M \in \mathbb{N}$ and let $k = T/M$ be the time step. The approximation in space and time must enable us to compute the elements $u_h^n(mk)$, $\sigma_h^n(mk)$ for every $n \in \mathbb{N}$ and $m = \overline{0,M}$. For this reason let us denote by $P_h^k(n,m)$ the set defined by

$$P_h^k(n,m) = \{\eta_h^n(mk), z_h^n(mk), u_h^n(mk), \sigma_h^n(mk)\} \tag{5.5}$$

for all $n \in \mathbb{N}$ and $m = \overline{0,M}$ and let us split the computing of $P_h^k(n,m)$ into the following steps:

(a) Computing the set $P_h^k(n,0)$.

For every $\eta_h^0 = \eta_0 \in C^0(0,T,\mathcal{H})$ we get $z_h^n(0) = z_0$ by (5.3) for every $n \in \mathbb{N}$, hence by (5.1), (5.2) and (5.4) we obtain $u_h^n(0)$, $\sigma_h^n(0)$ and $\eta_h^n(0)$ for all $n \in \mathbb{N}$.

(b) Computing the set $P_h^k(0,m)$.

Since η_h^0 is given, the values $\eta_h^0(mk)$ are known for all $m = \overline{0,M}$. The elements $z_h^0(mk)$ can be obtained using the trapezoidal rule for approximating (5.3):

$$z_h^0(0) = z_0,\ z_h^0(mk) = z_h^0\big((m-1)k\big) + \frac{k}{2}\Big[\eta_h^0(mk) + \eta_h^0\big((m-1)k\big)\Big] \tag{5.6}$$

for all $m = \overline{1,M}$, and finally $u_h^0(mk)$, $\sigma_h^0(mk)$ are determined from (5.1), (5.2) and (5.6) for all $m = \overline{0,M}$.

(c) Computing the set $P_h^k(n+1,m)$.

Let us suppose that the sets $P_h^k(n+1,m-1)$, $P_h^k(n,m)$ are known for a given $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $1 \leqslant m \leqslant M$. Using (5.4) we get

$$\eta_h^{n+1}(mk) = F\big(\sigma_h^n(mk), \varepsilon(u_h^n(mk))\big)$$

and using again the trapezoidal rule, from (5.3) we obtain

$$z_h^{n+1}(mk) = z_h^{n+1}\big((m-1)k\big) + \frac{k}{2}\Big[\eta_h^{n+1}(mk) + \eta_h^{n+1}\big((m-1)k\big)\Big].$$

Finally, $u_h^{n+1}(mk)$, $\sigma_h^{n+1}(mk)$ can be obtained by (5.1), (5.2).

Using now the steps (a), (b), (c) we compute the set $P_h^k(n,m)$ for all $n \in \mathbb{N}$ and $m = \overline{0,M}$; in this way the approximate solution $u_h^n(t)$, $\sigma_h^n(t)$ is computed for all $t = mk$, $m = \overline{0,M}$.

6. Numerical results

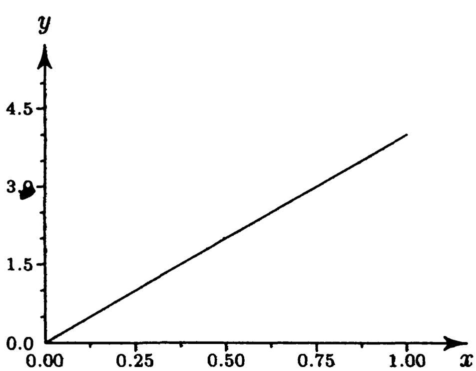
Let us consider a viscoelastic problem of the form (3.1)–(3.5) defined on the infinite time interval $(0, +\infty)$ in the following context:
$\Omega = (0,1)$, $\Gamma_1 = \{0\}$, $\Gamma_2 = \{1\}$, $b(x,t) = 0$, $f(t) = 0$, $g(t) = 15$, $u_0(x) = 2x^2$, $\sigma_0(x) = 15 \ \forall x \in (0,1)$ and $t > 0$, $\mathcal{E} = 20$, $F(\sigma, \varepsilon) = -10(\sigma - G(\varepsilon))$,

$$G(\varepsilon) = \begin{cases} 10\varepsilon & \text{for } \varepsilon \leqslant 2 \\ -5\varepsilon + 30 & \text{for } 2 < \varepsilon < 4 \\ 10\varepsilon - 30 & \text{for } \varepsilon \geqslant 4 \end{cases}$$

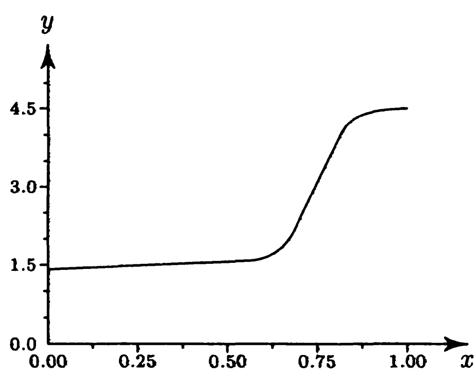
$\forall \sigma, \varepsilon \in \mathbb{R}$. Let (u, σ) be the solution of this problem (see Remark 3.2) and let $\varepsilon = \varepsilon(u)$. We have $\sigma(x,t) = 15 \ \forall x \in [0,1]$, $t > 0$ and, after some computation, we get

$$\lim_{t \to +\infty} \varepsilon(x,t) = \begin{cases} 1.5 & \text{if } 0 \leqslant x < 0.75 \\ 3 & \text{if } x = 0.75 \\ 4.5 & \text{if } 0.75 < x \leqslant 1. \end{cases} \tag{6.1}$$

In order to illustrate the algorithm (5.1) let $V_h \subset H_0^1(\Omega)$ be the finite element space constructed with a polynomial function of degree 1, Ω being divided into 100 finite elements. The initial value considered for η_0 is $\eta_0 = 0$ and the number of iterations made was $n = 10$ (the numerical experiments show that for $n \geqslant 10$ the numerical solution stabilizes). The time step chosen was $k = 0.05$. The computed solution $\varepsilon(u_h^n)$ obtained by using the algorithm (5.1) for different moments t are plotted in Fig. 6.1. The results obtained agree with the behaviour of the exact solution given by (6.1).



a)



b)

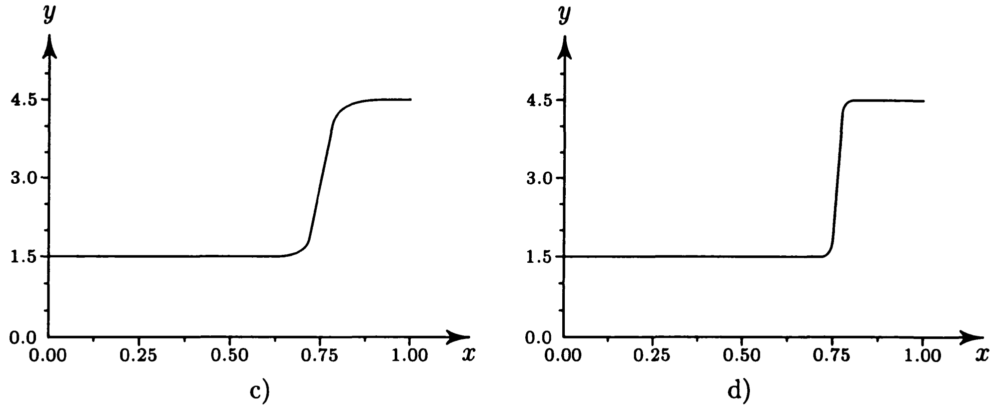


Fig. 6.1. The computed strain field $\varepsilon(u_h^n(x,t))$ for different values of t: a) $t = 0$; b) $t = 0.5$; c) $t = 1$; d) $t = 1.5$.

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