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WEIGHT MINIMIZATION OF AN ELASTIC PLATE WITH A UNILATERAL INNER OBSTACLE BY A MIXED FINITE ELEMENT METHOD

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Summary. Unilateral deflection problem of a clamped plate above a rigid inner obstacle is considered. The variable thickness of the plate is to be optimized to reach minimal weight under some constraints for maximal stresses. Since the constraints are expressed in terms of the bending moments only, Herrmann-Hellan finite element scheme is employed. The existence of an optimal thickness is proved and some convergence analysis for approximate penalized optimal design problem is presented.

Keywords: weight minimization, penalty method, unilateral plate bending, mixed finite elements

AMS classification: 65N30, 65K10, 49A29, 73K10

INTRODUCTION

The weight minimization problem for elastic plates is usually constrained by a prescribed upper bound for some stress invariant. This constraint can be equivalently expressed in terms of bending moments. Therefore a mixed variational formulation seems to be suitable, which enables to compute moments and deflection function simultaneously. It is the task of the present paper to employ results of the recent paper [4], where the Herrmann-Hellan-Johnson finite element model has been extended to variational inequalities of the fourth order. Piecewise linear and piecewise constant functions over triangulations are used for approximations of deflections and bending moments, respectively. For the set of admissible thickness functions we choose Lipschitz continuous functions and approximate them by affine triangular finite elements. Section 1 contains basic assumptions and formulation of the optimal design problem. We prove the existence of at least one optimal solution in Section 2, using the concept of a penalized optimal design problem to remove the statical constraints. Here the crucial role is played by Proposition 2.1 on the continuous dependence of the deflection function on the design variable. In Section 3 we define a finite element discretization of the penalized optimal design problem and prove its solvability. Main result of Section 4 is, that any sequence of approximate solutions, with mesh size decreasing to zero, contains a subsequence, converging to a solution of the penalized optimal design problem. Having a sequence of the latter solutions with the penalization parameter tending to zero, any limit point is proved to coincide with a solution of the original weight minimization problem.

1. Assumptions and definitions

Throughout the paper we shall consider clamped elastic nonhomogeneous and anisotropic plates, the middle plane of which occupies a given bounded domain $\Omega \subset \mathbb{R}^2$ with *polygonal* boundary $\partial\Omega$.

Let the bending moments q_{ij} be linked with the second derivatives of the deflection by the following relation

$$q_{ij} = e^3 C^0_{ijkm} w_{,km}, \quad i,j = 1,2, \tag{1}$$

where e denotes the thickness of the plate, $w_{,km} = \partial^2 w/\partial x_k \partial x_m$, the coefficients $C^0_{ijkm} \in L^\infty(\Omega)$ and repeated indices imply summation within the range 1, 2. We assume that

$$C^0_{ijkm} = C^0_{jikm} = C^0_{kmij}$$

and

$$C^0_{ijkm}\tau_{ij}\tau_{km} \geqslant c_0 \tau_{ij}\tau_{ij}$$

holds for all symmetric matrices τ almost everywhere in Ω with some positive constant c_0.

The thickness e will be sought in the following set of admissible functions

$$U_{ad} = \left\{ e \in C^{(0),1}(\bar{\Omega}) \mid e_{\min} \leqslant e(x) \leqslant e_{\max}, \left|\frac{\partial e}{\partial x_i}\right| \leqslant C_i, \ i = 1,2 \right\},$$

where $C^{(0),1}(\bar{\Omega})$ denotes the set of Lipschitz functions, $e_{\min}$, $e_{\max}$ and c_i are given positive parameters.

It is not difficult to verify that U_{ad} is a compact subset of $C(\bar{\Omega})$, making use of the Arzelà theorem (cf. [5]).

Let us introduce the functional space

$$\mathcal{Z} = W_0^{1,p}(\Omega) \quad \text{with some } p \in (2, \infty).$$

Let a loading functional $f \in \mathcal{Z}'$ and a function $\varphi \in C(\bar{\Omega})$ be given, describing a lower unilateral obstacle. We assume that

(A1) $\max\limits_{x \in \partial\Omega} \varphi(x) + \frac{1}{2} e_{\max} < 0.$

For any given $e \in U_{ad}$ we define the solution $w(e)$ of the state problem as follows:

$$w(e) = \operatorname*{argmin}_{v \in K_e} \left\{ \frac{1}{2} \int_\Omega e^3 C^0_{ijkm} v_{'ij} v_{'km} - \langle f, Ev \rangle \right\}, \tag{2}$$

where

$$K_e = \left\{ v \in W_0^{2,2}(\Omega) \mid I_0 v \geqslant \varphi + \tfrac{1}{2} e \right\},$$

I_0 and E is the embedding of $W_0^{2,2}(\Omega)$ into $L^\infty(\Omega)$ and $\mathcal{Z}$, respectively.

It is well-known that a unique $w(e)$ exists for any $e \in U_{ad}$. Since we intend to use a non-standard (mixed) variational approach for numerical approximate solutions, some additional regularity of $w(e)$ will be required. Namely, we assume that

(A2) there exists a triangulation $\mathcal{T}_{h_0}$ of the domain Ω such that each triangle $T \in \mathcal{T}_{h_0}$ has two sides parallel with the coordinate axes, and for all $T \in \mathcal{T}_{h_0}$

(i) $\varphi|_T \in P_1(T)$,

(ii) $C^0_{ijkm}|_T \in P_0(T)$,

(where $P_n(T)$ is the set of polynomials of the degree at most n),

(iii) $w(e)|_T \in W^{3,2}(T)$, $\nu_i \nu_j C^0_{ijkm} w(e)_{'km}$ is continuous at each interelement edge (ν_i are components of the unit normal to the edge) and

$$\sum_{T \in \mathcal{T}_{h_0}} \|w(e)\|^2_{3,2,T} \leqslant C$$

holds for all $e \in U_{ad}$ with some constant C independent of e.

Let a specific weight $\gamma \in L^\infty(\Omega)$ be given, $\gamma > 0$. Thus the weight of the plate is

$$j(e) = \int_\Omega \gamma e \, \mathrm{d}x\,. \tag{3}$$

Moreover, the following constraints will be considered (cf. e.g. [6]):

$$\psi_K(e, q(e)) \leqslant 0, \quad K = 1, 2, \ldots, \overline{K}, \ \overline{K} < +\infty, \tag{4}$$

where

$$\psi_K(e, q(e)) = \frac{36}{\operatorname{meas} \Delta_K} \int_{\Delta_K} e^{-4} \left[q_{11}^2(e) + q_{22}^2(e) + \left(\frac{\sigma_d}{\tau} \right)^2 q_{12}^2(e) \right] \mathrm{d}x - \sigma_d^2,$$

$\Delta_K \subset \bar{\Omega}$ are given subdomains, σ_d, τ given positive constants and $q(e)$ is the bending moment tensor, derived by the relations (1) from the solution $w(e)$ of (2).

Let us introduce the set of statically admissible design variables

$$\mathcal{E}_{ad} = \Big\{ e \in U_{ad} \mid \sum_{K=1}^{\bar{K}} \left[\psi_K\big(e, q(e)\big)\right]^+ = 0 \Big\}$$

and assume

(A3) $\quad \mathcal{E}_{ad} \neq \emptyset$.

Our main task is to solve the *Optimal Design Problem*

$$e_0 = \operatorname*{argmin}_{e \in \mathcal{E}_{ad}} j(e). \tag{5}$$

2. Existence of an optimal solution

We shall remove the constraints (4) by means of a penalty method. To this end we introduce a penalized cost functional

$$\mathcal{J}_\varepsilon\big(e, q(e)\big) = j(e) + \varepsilon^{-1} \sum_{K=1}^{\bar{K}} \left[\psi_K\big(e, q(e)\big)\right]^+, \quad \varepsilon > 0,$$

and a *penalized optimal design problem*

$$e_\varepsilon = \operatorname*{argmin}_{e \in U_{ad}} \mathcal{J}_\varepsilon\big(e, q(e)\big). \tag{6}$$

In the following we shall prove the solvability of the problem (6). The crucial role is played by the continuity of the mapping $e \mapsto w(e)$ of U_{ad} into $W_0^{2,2}(\Omega)$.

Proposition 2.1. *Let $e_n \to e$ in $C(\bar{\Omega})$ as $n \to \infty$, $e_n \in U_{ad}$. Then*

$$w(e_n) \to w(e) \quad \text{in } W_0^{2,2}(\Omega).$$

P r o o f. 1^0 For any $v \in K_e$ there exists a sequence

(7) $\quad \{v_n\}$, such that $v_n \in W_0^{2,2}(\Omega)$, $v_n \in K_{e_n}$ for n sufficiently great and

$$v_n \to v \text{ in } W_0^{2,2}(\Omega), \text{ as } n \to \infty.$$

Indeed, let us define

$$\omega = v - \left(\varphi + \frac{1}{2}e\right)$$

so that $\omega \in C(\bar{\Omega})$, $\omega \geqslant 0$ in $\bar{\Omega}$, and

$$\eta_n = \frac{1}{2}(e_n - e) - \omega = \frac{1}{2}e_n - v + \varphi \in C(\bar{\Omega}),$$
$$G_n = \{x \in \Omega \mid \eta_n(x) > \frac{1}{2}c^*\},$$

where

$$c^* = \max_{x \in \partial\Omega} \varphi(x) + \frac{1}{2}e_{\max} < 0$$

by assumption (A1).

There exists an open set $G \subset \bar{G} \subset \Omega$ such that

$$G_n \subset G \quad \forall n. \tag{8}$$

To see this, we realize that

$$\eta_n = \varphi + \frac{1}{2}e_n \leqslant c^*$$

on the boundary $\partial\Omega$. The continuity of η_n and the constraints $|\partial e_n/\partial x_i| \leqslant C_i$ imply that

$$\bigcup_{n=1}^{\infty} G_n \subset\subset \Omega$$

and (8) follows.

Obviously, there exists a function $\psi \in C^\infty(\bar{\Omega})$ such that $\psi(x) = 1$ for $x \in G$ and $\psi(x) = 0$, $\partial\psi(x)/\partial\nu = 0$ for $x \in \partial\Omega$, $0 < \psi(x) \leqslant 1$ for $x \in \Omega$.

Let us set

$$v_n = v + \frac{1}{2}\|e_n - e\|_{0,\infty}\psi.$$

Then $v_n \in W_0^{2,2}(\Omega)$ and

$$\|v - v_n\|_2 = \frac{1}{2}\|e_n - e\|_{0,\infty}\|\psi\|_2 \to 0 \quad \text{as } n \to \infty.$$

We can show that there exists $n_0 > 0$ such that

$$n > n_0 \implies v_n \geqslant \varphi + \frac{1}{2}e_n \quad \text{in } \bar{\Omega} \implies v_n \in K_{e_n}. \tag{9}$$

Indeed, let

(i) $x \in G$. Then

$$v_n = v + \frac{1}{2}\|e_n - e\|_{0,\infty} \geqslant v + \frac{1}{2}(e_n - e) \geqslant \varphi + \frac{1}{2}e_n.$$

(ii) Let $x \in \bar{\Omega} - G$. Then

$$v_n \geqslant \varphi + \frac{1}{2}e + \omega + \frac{1}{2}|e_n - e|\varphi. \tag{10}$$

Since $x \notin G$, $x \notin G_n$ $\forall n$ and $\eta_n \leqslant \frac{1}{2}c^*$, so that

$$\frac{1}{2}(e_n - e) - \omega \leqslant \frac{1}{2}c^*,$$
$$-\frac{1}{2}c^*\psi + (1-\psi)\omega \leqslant \omega + \frac{1}{2}|e_n - e|\psi.$$

Inserting into (10), we obtain

$$v_n \geqslant \varphi + \frac{1}{2}e + \mathcal{Z},$$

where

$$\mathcal{Z} = -\frac{1}{2}\psi c^* + (1-\psi)\omega.$$

The function $\mathcal{Z}$ is continuous and attains a positive minimum in the compact set $\bar{\Omega} - G$,

$$m = \mathcal{Z}(x_0) = \min_{\bar{\Omega}-G} \mathcal{Z} > 0.$$

Indeed, let $\psi(x_0) = 0$. Then $x_0 \in \partial\Omega$ and

$$\mathcal{Z}(x_0) = \omega(x_0) = -\big(\varphi(x_0) + \frac{1}{2}e(x_0)\big) \geqslant -c^* > 0.$$

If $\psi(x_0) > 0$, then

$$\mathcal{Z}(x_0) \geqslant -\frac{1}{2}c^*\psi(x_0) > 0.$$

There exists $n_0(m)$ such that

$$n > n_0(m) \implies \frac{1}{2}\|e_n - e\|_{0,\infty} \leqslant m.$$

Then

$$\mathcal{Z}(x) \geqslant \mathcal{Z}(x_0) \geqslant \frac{1}{2}\|e_n - e\|_{0,\infty} \geqslant \frac{1}{2}\big(e_n(x) - e(x)\big)$$

so that

$$v_n(x) \geqslant \varphi(x) + \frac{1}{2}e_n(x), \quad n > n_0(m),$$

and (9) is verified. The proof of (7) is completed.

2^0 Denoting

$$A(e;u,v) = \int_\Omega e^3 C^0_{ijkm} w'_{ij} v'_{km}\, \mathrm{d}x,$$

we may introduce the variational inequality for $w(e) \in K_e$

$$A\big(e; w(e), v - w(e)\big) \geqslant \langle f, Ev - Ew(e)\rangle \quad \forall v \in K_e, \tag{11}$$

which is equivalent to the state problem (2).

For brevity, we shall write $w_n = w(e_n)$. Inserting the sequence $\{v_n\}$ from (7) into the variational inequality for e_n, we obtain

$$A(e_n; w_n, v_n - w_n) \geqslant \langle f, Ev_n - Ew_n\rangle\,, \quad n > n_0. \tag{12}$$

Recall that

$$A(e; v, v) \geqslant C_0 \|v\|_2^2 \quad \forall e \in U_{ad},\ \forall v \in W_0^{2,2}(\Omega)$$

with the constant C_0, independent of e, v. Consequently, we may write

$$\begin{aligned} C_0\|w_n\|_2^2 &\leqslant A(e_n; w_n, v_n) + \langle f, Ev_n - Ew_n\rangle \\ &\leqslant C\big(\|w_n\|_2\|v_n\|_2 + \|f\|_{Z'}(\|v_n\|_2 + \|w_n\|_2)\big) \\ &\leqslant \tilde{C}(\|w_n\|_2 + 1) \end{aligned}$$

and

$$\|w_n\|_2 \leqslant C \quad \forall n.$$

There exists $w \in W_0^{2,2}(\Omega)$ and a subsequence $\{w_k\} \subset \{w_n\}$, such that

$$w_k \rightharpoonup w \quad (\text{weakly in } W_0^{2,2}(\Omega)),\ \text{as } k \to \infty.$$

Since

$$w_k \geqslant \varphi + \frac{1}{2} e_k \quad \text{in } \bar{\Omega}$$

and the embedding $W^{2,2}(\Omega) \hookrightarrow C(\bar{\Omega})$ is compact, we may pass to the limit with $k \to \infty$ to obtain that

$$w \geqslant \varphi + \frac{1}{2} e \quad \text{in } \bar{\Omega},$$

i.e., $w \in K_e$.

We shall verify that w satisfies the variational inequality (11). Let us consider an arbitrary $v \in K_e$.

The functional $u \mapsto A(\eta; u, u)$ is weakly lower semicontinuous on $W_0^{2,2}(\Omega)$ for any $\eta \in U_{ad}$. Consequently,

$$\liminf_{k\to\infty} A(e; w_k, w_k) \geqslant A(e; w, w),$$

since $e \in U_{ad}$. Moreover, we have

$$|A(e_k; w_k, w_k) - A(e; w_k, w_k)| \leqslant C\|e_k^3 - e^3\|_{0,\infty}\|w_k\|_2^2 \to 0, \tag{13}$$

so that

$$\begin{aligned}\liminf_{k\to\infty} & A(e_k; w_k, w_k) \\ &= \liminf \Big(A(e; w_k, w_k) + \big[A(e_k; w_k, w_k) - A(e; w_k, w_k)\big]\Big) \\ &\geqslant \liminf A(e; w_k, w_k) \geqslant A(e; w, w).\end{aligned} \tag{14}$$

We derive easily that

$$\lim_{k\to\infty} A(e_k; w_k, g) = A(e; w, g) \quad \forall g \in W_0^{2,2}(\Omega), \tag{15}$$

using the decomposition

$$A(e_k; w_k, g) - A(e; w, g) = \big[A(e_k; w_k, g) - A(e; w_k, g)\big] + A(e; w_k - w, g)$$

and the weak convergence of $\{w_k\}$.

Making use of (7), we obtain

$$|A(e_k; w_k, v_k - v)| \leqslant C\|w_k\|_2\|v_k - v\|_2 \to 0 \quad \text{as } k \to \infty. \tag{16}$$

Combining (15) and (16), we arrive at

$$\begin{aligned}&|A(e_k; w_k, v_k) - A(e; w, v)| \\ &\qquad \leqslant |A(e_k; w_k, v_k - v)| + |A(e_k; w_k, v) - A(e; w, v)| \to 0.\end{aligned} \tag{17}$$

The weak convergence of $\{w_k\}$ and (7) yield that

$$\langle f, E(v_k - w_k)\rangle \to \langle f, E(v - w)\rangle\,. \tag{18}$$

From the inequality (12) we deduce that

$$A(e_k; w_k, w_k) + \langle f, E(v_k - w_k)\rangle \leqslant A(e_k; w_k, v_k). \tag{19}$$

Passing to the liminf on both sides and using (14), (18), (17), we obtain

$$A(e; w, w) + \langle f, E(v - w)\rangle \leqslant A(e; w, v).$$

Consequently, w satisfies the inequality (11). Since the solution $w(e)$ of (11) is unique, $w = w(e)$ follows and the whole sequence $\{w_n\}$ converges to $w(e)$ weakly in $W_0^{2,2}(\Omega)$.

3^0 It remains to verify the strong convergence. First we prove that

$$\lim_{n\to\infty} A(e_n; w_n, w_n) = a(e; w, w). \tag{20}$$

In fact, we obtain from (19), (17) and (18)

$$\limsup_{n\to\infty} A(e_n; w_n, w_n) \leqslant A(e; w, v) + \langle f, E(w - v)\rangle \tag{21}$$

for any $v \in K_e$. Inserting $v := w$, (21) and (14) imply

$$\begin{aligned} A(e; w, w) &\leqslant \liminf A(e_n; w_n, w_n) \leqslant \limsup A(e_n; w_n, w_n) \\ &\leqslant A(e; w, w), \end{aligned}$$

so that (20) follows.

Combining (13) and (20), we arrive at

$$\lim A(e; w_n, w_n) = A(e; w, w). \tag{22}$$

If $W_0^{2,2}(\Omega)$ is equipped with the scalar product $A(e; u, v) = (u, v)_A$, then (22) implies that the associated norms $\|w_n\|_A$ tend to $\|w\|_A$. Since the norms $\|.\|_A$ and $\|.\|_2$ are equivalent, we are led to the strong convergence $\|w_n - w\|_2 \to 0$. □

Corollary 2.1. *Let $e_n \to e$ in $C(\overline{\Omega})$ as $n \to \infty$, $e_n \in U_{ad}$. Then*

$$q(e_n) \to q(e) \quad \textit{in } [L^2(\Omega)]^4.$$

Proof. We may write

$$\begin{aligned} \|q_{ij}(e_n) - q_{ij}(e)\|_{0,\Omega} &\leqslant \|(e_n^3 - e^3)C^0_{ijkm} w(e_n)_{,km}\|_{0,\Omega} \\ &\quad + \|e^3 C^0_{ijkm}\big(w(e_n) - w(e)\big)_{,km}\|_{0,\Omega} \\ &= I_{1n} + I_{2n}. \end{aligned}$$

Proposition 2.1 yields that both I_{1n} and I_{2n} tends to zero, as $n \to \infty$. □

Lemma 2.1. *Let $e_n \to e$ in $C(\bar{\Omega})$ as $n \to \infty$, $e_n \in U_{ad}$. Then for any $K = 1, \ldots, \bar{K}$*
$$[\psi_K(e_n, q(e_n))]^+ \to [\psi_K(e, q(e))]^+.$$

Proof. Since
$$|a^+ - b^+| \leqslant |a - b|,$$
we may write
$$\begin{aligned}
&\left|[\psi_K(e_n, q(e_n))]^+ - [\psi_K(e, q(e))]^+\right| \\
&\leqslant \left|\psi_K(e_n, q(e_n)) - \psi_K(e, q(e))\right| \\
&\leqslant \frac{36}{\operatorname{meas} \Delta_K} \int_{\Delta_K} \Bigl| e_n^{-4}\Bigl(q_{11}^2(e_n) + q_{22}^2(e_n) + \Bigl(\frac{\sigma_d}{T}\Bigr)^2 q_{12}^2(e_n)\Bigr) \\
&\qquad - e^{-4}\Bigl(q_{11}^2(e) + q_{22}^2(e) + \Bigl(\frac{\sigma_d}{T}\Bigr)^2 q_{12}^2(e)\Bigr)\Bigr| \,\mathrm{d}x \\
&\leqslant C \int_{\Delta_K} \left| e_n^{-4}(q_{11}^2(e_n) - q_{11}^2(e) + \ldots) + (e_n^{-4} - e^{-4})(q_{11}^2(e) + \ldots)\right| \mathrm{d}x \\
&\leqslant C e_{\min}^{-4} \Bigl\{ \int_{\Delta_K} |q_{11}(e_n) + q_{11}(e)| . |q_{11}(e_n) - q_{11}(e)| \,\mathrm{d}x + \ldots \Bigr\} \\
&\qquad + C(e) . \|e_n^{-4} - e^{-4}\|_{0,\infty} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$
making use of Corollary 2.1. □

Proposition 2.2. *The penalized optimal design problem (6) has a solution for any $\varepsilon > 0$.*

Proof. Since the functionals $j(e)$ and $[\psi_K(e, q(e))]^+$ are continuous in U_{ad} and U_{ad} is compact in $C(\bar{\Omega})$, there exists a minimizer e_ε of $\mathcal{J}_\varepsilon(e, q(e))$ in U_{ad}. □

Theorem 2.1. *Assume that $\mathcal{E}_{ad} \neq \emptyset$. Let $\{\varepsilon\}$, $\varepsilon \to 0_+$, be a sequence and $\{e_\varepsilon\}$ a sequence of solutions of the penalized optimal design problems (6), $\{q(e_\varepsilon)\}$ the sequence of corresponding moment fields.*

Then there exist a subsequence $\{\tilde{\varepsilon}\} \subset \{\varepsilon\}$ and an element $e_0 \in \mathcal{E}_{ad}$ such that

$$e_{\tilde{\varepsilon}} \to e_0 \quad \text{in } C(\bar{\Omega}), \tag{23}$$
$$q(e_{\tilde{\varepsilon}}) \to q(e_0) \quad \text{in } [L^2(\Omega)]^4, \tag{24}$$

where e_0 is a solution of the optimal design problem (5).

Proof. Since U_{ad} is compact in $C(\bar{\Omega})$, there exists a subsequence $\{e_{\tilde{\varepsilon}}\} \subset \{e_\varepsilon\}$ such that (23) holds with $e_0 \in U_{ad}$. Corollary 2.1 then implies (24). Let us show that e_0 is a solution of (5).

It follows from the definition that

$$(25)\qquad j(e_{\tilde\varepsilon}) + (\tilde\varepsilon)^{-1}\sum_{K=1}^{\overline{K}}\left[\psi_K\big(e_{\tilde\varepsilon}, q(e_{\tilde\varepsilon})\big)\right]^+ \leqslant j(e) + (\tilde\varepsilon)^{-1}\sum_{K=1}^{\overline{K}}\left[\psi_K\big(e, q(e)\big)\right]^+$$

holds for any $e \in U_{ad}$.

Taking now an element $e \in \mathcal{E}_{ad}$, we obtain

$$\tilde\varepsilon j(e_{\tilde\varepsilon}) + \sum_K\left[\psi_K\big(e_{\tilde\varepsilon}, q(e_{\tilde\varepsilon})\big)\right]^+ \leqslant \tilde\varepsilon j(e),$$
$$0 \leqslant \sum_K\left[\psi_K\big(e_{\tilde\varepsilon}, q(e_{\tilde\varepsilon})\big)\right]^+ \leqslant \tilde\varepsilon j(e).$$

Passing to the limit with $\tilde\varepsilon \to 0$ and using Lemma 2.1, we arrive at

$$\sum_K\left[\psi_K\big(e_0, q(e_0)\big)\right]^+ = 0,$$

so that $e_0 \in \mathcal{E}_{ad}$ follows. Then (25) implies

$$j(e_{\tilde\varepsilon}) \leqslant j(e_{\tilde\varepsilon}) + (\tilde\varepsilon)^{-1}\sum_K\left[\psi_K\big(e_{\tilde\varepsilon}, q(e_{\tilde\varepsilon})\big)\right]^+ \leqslant j(e) \quad \forall e \in \mathcal{E}_{ad}.$$

Passing to the limit with $\tilde\varepsilon \to 0$ and using (23), we deduce that

$$j(e_0) \leqslant j(e) \quad \forall e \in \mathcal{E}_{ad}.$$

□

Corollary 2.2. *If* $\mathcal{E}_{ad} \neq \emptyset$ *then there exists at least one solution of the optimal design problem* (5).

P r o o f follows immediately from Proposition 2.2 and Theorem 2.1. □

3. Approximate optimal design problem

Since the constraints are expressed in terms of the bending moments q, it seems to be suitable to employ a mixed finite element model in which both the deflections and moments are computed simultaneously. We shall use the so-called Herrmann-Hellan finite element scheme with piecewise linear approximations for deflections and piecewise constant for bending moments. An extension of this approach to the inner obstacle problem has been developed in [4] on the basis of some results of Brezzi and Raviart [1], Comodi [2] and Glowinski, Lions and Trémolières [3].

We consider a regular family of triangulations $\{\mathcal{T}_h\}$, $h \to 0_+$, refining the original triangulation $\mathcal{T}_{h_0}$ (see (A2) in Sec. 1) and such that each triangle has two sides parallel with the coordinate axes. We shall need the space of tensor-valued functions

$$S = \big\{(q_{ij})_{i,j=1,2} \mid q_{ij} \in L^2(\Omega),\ i,j = 1,2,\ q_{12} = q_{21}\big\}.$$

Let us introduce the following spaces of finite elements

$$\begin{aligned}
\mathcal{Z}_h &= \big\{z_h \in C(\bar{\Omega}) \mid z_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h,\ z_h = 0 \text{ on } \partial\Omega\big\}\\
Q_h &= \big\{q_h \in S \mid q_{hij}|_T \in P_0(T) \quad \forall T \in \mathcal{T}_h,\\
&\qquad q_{hij}\nu_i\nu_j \text{ are continuous at each interelement edge}\big\},\\
W_h(e_h) &= \big\{[z_h, q_h] \in \mathcal{Z}_h \times Q_h \mid a(e_h; q_h, p_h) + b(p_h, z_h) = 0 \quad \forall p_h \in Q_h\big\}
\end{aligned}$$

where

$$\begin{aligned}
a(e; q, p) &= \int_\Omega e^{-3} B_{ijkm} q_{ij} p_{km} \,\mathrm{d}x, \quad B = (C^0)^{-1},\\
b(p, z) &= \sum_{T \in \mathcal{T}_h} \Big(\int_T p_{ij,j} z_{,i} \,\mathrm{d}x - \int_{\partial T} p_{ij}\nu_i t_j \partial z/\partial t \,\mathrm{d}s\Big),
\end{aligned}$$

(t is the unit tangential vector to ∂T).

Let us consider the problem

$$(26) \qquad [\bar{z}_h(e_h), \bar{q}_h(e_h)] = \operatorname*{argmin}_{[z_h, q_h] \in K_h(e_h)} \Big\{\frac{1}{2} a(e_h; q_h, q_h) - \langle f, z_h\rangle\Big\},$$

where

$$K_h(e_h) = \big\{[z_h, q_h] \in W_h(e_h) \mid z_h(P) \geqslant \varphi(P) + \tfrac{1}{2} e_h(P) \quad \forall P \in \Sigma_h^0\big\}$$

and Σ_h^0 is the set of all vertices of $\mathcal{T}_h$ inside Ω.

By [4, Theorem 2.1], (26) has a unique solution for any $e_h \in U_{ad}$. Let us recall the following result ([4, Theorem 2.2). There exists a unique *saddle point* $\{[\bar{z}_h(e_h), \bar{q}_h(e_h)], \bar{\lambda}_h\}$ on $W_h(e_h) \times \mathbb{R}_+^{m_h}$ of

$$\mathcal{L}_h(e_h\,;\,[z_h, q_h], \lambda_h) = \frac{1}{2}a(e_h; q_h, q_h) - \langle f, z_h\rangle + \left\langle \lambda_h, \varphi + \frac{1}{2}e_h - z_h \right\rangle_h,$$

where

$$\langle \lambda_h, \xi_h\rangle_h = \frac{1}{3}\sum_{P\in\Sigma_h^0} A(P)\lambda_h(P)\xi_h(P)$$

and $A(P)$ denotes the sum of the areas of the triangles in $\mathcal{T}_h$, which admit P as common vertex, m_h is the number of vertices in Σ_h^0 and $\mathbb{R}_+^{m_h}$ is the subset of non-negative coordinates in $\mathbb{R}^{m_h}$.

The first component $[\bar{z}_h(e_h), \bar{q}_h(e_h)]$ coincides with the solution of the problem (26).

Remark 3.1. The saddle point can be computed iteratively by means of an algorithm of Uzawa's type. Its convergence has been proved in [4, Theorem 2.3].

Instead of U_{ad} we introduce an approximate set

$$U_{ad}^h = \{e_h \in U_{ad} \mid e_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h\}$$

and the *approximate optimal design problem*

$$e_h^\varepsilon = \operatorname*{argmin}_{e_h \in U_{ad}^h}\Big\{j(e_h) + \varepsilon^{-1}\sum_{K=1}^{\bar{K}}\left[\psi_K(e_h, \bar{q}_h(e_h))\right]^+\Big\}, \tag{27}$$

where $\bar{q}_h(e_h)$ is the component of the saddle point mentioned above.

We shall prove the solvability of the problem (27). To this end we first establish the following lemma.

Lemma 3.1. *Let $\mathcal{T}_h$ be fixed and let $e_h^n \in U_{ad}^h$, $n = 1, 2, \ldots$,*

$$e_h^n \to e_h \quad \text{in } C(\bar{\Omega}), \quad \text{as } n \to \infty.$$

Then

$$\bar{q}_h(e_h^n) \to \bar{q}_h(e_h) \quad \text{in } S.$$

Proof. 1^0 We shall drop out some subscripts h in what follows. Let us recall that $\{[\bar{z}(e), \bar{q}(e)], \bar{\lambda}(e)\}$ is the (unique) saddle point of $\mathcal{L}_h(e)$ on $W_h(e) \times \mathbb{R}_+^{m_h}$, if and only if the following three conditions are fulfilled:

$$b(\bar{q}(e), z) + \langle f, z\rangle + \langle \bar{\lambda}(e), z\rangle_h = 0 \quad \forall z \in \mathcal{Z}_h, \tag{28}$$

$$a(e; \bar{q}(e), p) + b(p, \bar{z}(e)) = 0 \quad \forall p \in Q_h, \tag{29}$$

$$\left\langle \lambda - \bar{\lambda}(e), \varphi + \frac{1}{2}e - \bar{z}(e)\right\rangle_h \leqslant 0 \quad \forall \lambda \in \mathbb{R}_+^{m_h}. \tag{30}$$

Analogous conditions hold for the saddle point $\{[\bar{z}(e^n), \bar{q}(e^n)], \bar{\lambda}(e^n)\}$ of $\mathcal{L}_h(e^n)$ on $W_h(e^n) \times \mathbb{R}_+^{m_h}$. Subtracting, we obtain

$$b(\bar{q}(e^n) - \bar{q}(e), z) + \langle \bar{\lambda}(e^n) - \bar{\lambda}(e), z\rangle_h = 0 \quad \forall z \in \mathcal{Z}_h, \tag{31}$$

$$a(e^n; \bar{q}(e^n), p) - a(e; \bar{q}(e), p) + b(p, \bar{z}(e^n) - \bar{z}(e)) = 0 \quad \forall p \in Q_h. \tag{32}$$

Let us substitute $z := \bar{z}(e^n) - \bar{z}(e)$ in (31) and $p := \bar{q}(e^n) - \bar{q}(e) \equiv \Delta_n$ in (32). Thus we get

$$b(\Delta_n, \bar{z}(e^n) - \bar{z}(e)) + \langle \bar{\lambda}(e^n) - \bar{\lambda}(e), \bar{z}(e^n) - \bar{z}(e)\rangle_h = 0,$$
$$a(e^n; \bar{q}(e^n), \Delta_n) - a(e; \bar{q}(e), \Delta_n) + b(\Delta_n, \bar{z}(e^n) - \bar{z}(e)) = 0,$$

so that

$$a(e; \bar{q}(e), \Delta_n) - a(e^n; \bar{q}(e^n), \Delta_n) + \langle \bar{\lambda}(e^n) - \bar{\lambda}(e), \bar{z}(e^n) - \bar{z}(e)\rangle_h = 0. \tag{33}$$

Let us substitute $\lambda := \bar{\lambda}(e^n)$ in (30) and $\lambda := \bar{\lambda}(e)$ in the counterpart of (30). Thus we arrive at the inequalities, the addition of which yields

$$\left\langle \bar{\lambda}(e) - \bar{\lambda}(e^n), \frac{1}{2}(e^n - e) + \bar{z}(e) - \bar{z}(e^n)\right\rangle_h \leqslant 0$$

and therefore we have

$$\langle \bar{\lambda}(e^n) - \bar{\lambda}(e), \bar{z}(e^n) - \bar{z}(e)\rangle_h \leqslant \left\langle \bar{\lambda}(e^n) - \bar{\lambda}(e), \frac{1}{2}(e^n - e)\right\rangle_h. \tag{34}$$

Substituting (34) into (33), we obtain

$$a(e^n; \bar{q}(e^n), \Delta_n) - a(e; \bar{q}(e), \Delta_n) \leqslant \frac{1}{2}\langle \bar{\lambda}(e^n) - \bar{\lambda}(e), e^n - e\rangle_h. \tag{35}$$

2^0 Next let us show that a constant $C > 0$ exists such that

$$\|\bar{q}(e^n)\|_S + \|\bar{z}(e^n)\|_Z + \|\bar{\lambda}(e^n)\|_h \leqslant C \quad \forall n, \tag{36}$$

where $\|\lambda\|_h = \left(\langle\lambda,\lambda\rangle_h\right)^{1/2}$.

In fact, let us point out that the inequalities

$$a(e^n; q, q) \geqslant a_0\|q\|_S^2 \tag{37}$$

$$a(e^n; p, q) \leqslant C\|p\|_S\|q\|_S \tag{38}$$

hold for all p, $q \in Q_h$ and for all $e^n \in U_{ad}$, with constants a_0 and C independent of p, q, e^n. Then it is easy to verify that Lemma 2.1 of [4] holds with constants C, $\tilde{C}$ in [4, (2.2)] independent of e^n.

Obviously, there exists a function $z^0 \in \mathcal{Z}_h$ such that $z^0(P) \geqslant \varphi(P) + \frac{1}{2}e_{\max}$ at all points $P \in \Sigma_h^0$. Using Lemma 2.1 of [4], we obtain a unique $q^{0n} \in Q_h$ such that $[z^0, q^{0n}] \in W_h(e^n)$. Since $z^0 \geqslant \varphi + \frac{1}{2}e^n$ at $P \in \Sigma_h^0$, $[z^0, q^{0n}] \in K_h(e^n)$ and

$$\|q^{0n}\|_S \leqslant C^{-1}\|z^0\|_Z = \hat{C} \quad \forall n. \tag{39}$$

By definition (26), we have

$$\frac{1}{2}a\left(e^n; \bar{q}(e^n), \bar{q}(e^n)\right) - \langle f, \bar{z}(e^n)\rangle \leqslant \frac{1}{2}a(e^n; q^{0n}, q^{0n}) - \left\langle f, z^0\right\rangle.$$

Making use of (37), (38), (39) and Lemma 2.1 of [4], we get

$$\frac{1}{2}a_0\|\bar{q}(e^n)\|_S^2 \leqslant C_0 + \|f\|_{Z'}\|\bar{z}(e^n)\|_Z \leqslant C_2 + C_3\|\bar{q}(e^n)\|_S.$$

Consequently, the norms $\|\bar{q}(e^n)\|_S$ are bounded. Lemma 2.1 of [4] yields the boundedness of $\bar{z}(e^n)$. Since $\mathbb{R}_+^{m_h}$ is isomorphic with the subset $\Lambda_h = \{\lambda_h \in \mathcal{Z}_h \mid \lambda_h \geqslant 0\}$, we may insert $z := \bar{\lambda}(e^n)$ in (28) to deduce that

$$\begin{aligned}\|\bar{\lambda}(e^n)\|_h^2 &= -b\left(\bar{q}(e^n), \bar{\lambda}(e^n)\right) - \left\langle f, \bar{\lambda}(e^n)\right\rangle\\ &\leqslant C\left(\|\bar{q}(e^n)\|_S + \|f\|_{Z'}\right)\|\bar{\lambda}(e^n)\|_Z \leqslant \tilde{C}\|\bar{\lambda}(e^n)\|_h.\end{aligned}$$

Thus we arrive at (36).

3^0 It is readily seen that

$$\left|a\left(e^n; \bar{q}(e), \Delta_n\right) - a\left(e; \bar{q}(e), \Delta_n\right)\right| \leqslant C\|(e^n)^{-3} - e^{-3}\|_{0,\infty} \to 0 \tag{40}$$

as $n \to \infty$, realizing that (36) implies

$$\|\Delta_n\|_S \leqslant \|\bar{q}(e^n)\|_S + \|\bar{q}(e)\|_S \leqslant C' \quad \forall n.$$

On the basis of (37) and (35), we may write

$$\begin{aligned}a_0\|\Delta_n\|_S^2 &\leqslant a(e^n;\bar q(e^n),\Delta_n) - a(e^n;\bar q(e),\Delta_n)\\ &= \big[a(e^n;\bar q(e^n),\Delta_n) - a(e;\bar q(e),\Delta_n)\big]\\ &\quad + \big[a(e;\bar q(e),\Delta_n) - a(e^n;\bar q(e),\Delta_n)\big]\\ &\leqslant \frac{1}{2}\left\langle\bar\lambda(e^n) - \bar\lambda(e), e^n - e\right\rangle_h + |I_n|\\ &\leqslant \frac{1}{2}\big(\|\bar\lambda(e^n)\|_h + \|\bar\lambda(e)\|_h\big)\|e^n - e\|_h + |I_n|.\end{aligned}$$

Since the right-hand side tends to zero with $n\to\infty$ by virtue of (36) and (40), we get

$$\|\bar q(e^n) - \bar q(e)\|_S = \|\Delta_n\|_S \to 0, \quad \text{as } n\to\infty.$$

□

Theorem 3.1. *The approximate problem (27) has at least one solution for any fixed triangulation $\mathcal{T}_h$ and any $\varepsilon > 0$.*

Proof. Making use of Lemma 3.1, we prove that the functions $\big[\psi_K(e_h,\bar q_h(e_h))\big]^+$ are continuous in U_{ad}^h (cf. the proof of an analogous Lemma 2.1). Consequently, the cost functional in (27) is continuous, as well.

Obviously,

$$e_h\in U_{ad}^h \iff \big\{e_h(P_i)\big\}_{i=1}^{r_h}\in\mathcal{A}_h\subset\mathbb{R}^{r_h},$$

where P_i are the vertices of $\mathcal{T}_h$. The set $\mathcal{A}_h$ is compact in $\mathbb{R}^{r_h}$, being bounded and closed. Hence the cost functional attains its minimum in U_{ad}^h. □

4. Finite element analysis

In the present section we study the convergence of the solutions e_h^ε of the approximate optimal design problem (27), when the mesh size h tends to zero. To this end, we shall need the following result.

Lemma 4.1. *Let $e_h\in U_{ad}^h$. Then*

$$\|\bar q_h(e_h) - q(e_h)\|_S \leqslant Ch^{1/p}. \tag{41}$$

holds with some constant C independent of e_h and h.

Proof. Let us denote $q(e_h) = q$, $\bar z(e_h)\equiv w(e_h) = z$, $\bar q_h(e_h) = q_h$, $\bar z_h(e_h) = z_h$, for brevity.

Let us introduce the space of tensor fields

$$Q(\mathcal{T}_h) = \{p \in S \mid p_{ij}|_T \in H^1(T) \quad \forall T \in \mathcal{T}_h,\ i,j = 1,2,$$
$$p_{ij}\nu_i\nu_j \text{ is continuous at each interelement edge}\}.$$

By assumption (A2) (ii) and (iii), the relation (1) implies

$$q(e_h) \in Q(\mathcal{T}_h), \tag{42}$$

as the triangulations $\mathcal{T}_h$ under consideration refine $\mathcal{T}_{h_0}$. Moreover, (A2) implies

$$\begin{aligned}\|q(e_h)\|^2_{Q(\mathcal{T}_h)} &= \sum_{i,j=1,2} \sum_{T\in\mathcal{T}_h} \|q_{ij}(e_h)\|^2_{1,2,T} \\ &\leqslant \hat{C} \sum_{T\in\mathcal{T}_{h_0}} \|w(e_h)\|^2_{3,2,T} \leqslant \hat{C}C\end{aligned} \tag{43}$$

Obviously, we may write

$$\begin{aligned}a_0\|q - q_h\|^2_S &\leqslant a(e_h; q - q_h, q - \Pi_h q) + a(e_h; q - q_h, \Pi_h q - q_h) \\ &= I_1 + I_2,\end{aligned} \tag{44}$$

where $\Pi_h : Q(\mathcal{T}_h) \to Q_h$ is the (linear) mapping from [1-Lemma 4].

Recall that

$$b(p - \Pi_h p, y_h) = 0 \quad \forall p \in Q(\mathcal{T}_h),\ y_h \in \mathcal{Z}_h. \tag{45}$$

Using the relation (see [4, Theorem 1.2])

$$a(e_h; q, p) + b(p, z) = 0 \quad \forall p \in Q(\mathcal{T}_h),$$

the definition of $W_h(e_h)$, the Lagrange linear interpolation $I_h \colon \mathcal{Z} \to \mathcal{Z}_h$ and (45) we get

$$\begin{aligned}I_2 &= a(e_h; q, \Pi_h q - q_h) - a(e_h; q_h, \Pi_h q - q_h) \\ &= -b(\Pi_h q - q_h, z) + b(\Pi_h q - q_h, z_h) = b(\Pi_h q - q_h, z_h - z) \\ &= b(\Pi_h q - q, I_h z - z) + b(\Pi_h q - q, z_h - I_h z) + b(q - q_h, z_h - z) \\ &= -b(q, I_h z - z) + b(q - q_h, z_h - z),\end{aligned} \tag{46}$$

since

$$b(p_h, I_h z - z) = 0 \quad \forall p_h \in Q_h. \tag{47}$$

We have $z_h \in \mathcal{Z}_h \subset \mathcal{Z}$ and $z_h \geqslant \varphi + \frac{1}{2}e_h$ in $\bar{\Omega}$ by virtue of the assumption (A2)(i). Consequently, Lemma 3.2 of [4] yields that

$$-b(q, z_h - z) \geqslant \langle f, z_h - z \rangle . \tag{48}$$

From the definition of the problem (26), we get

$$-b(q_h, y_h - z_h) \geqslant \langle f, y_h - z_h \rangle \tag{49}$$

for all $y_h \in \mathcal{Z}_h$ such that $y_h(P) \geqslant \varphi(P) + \frac{1}{2}e_h(P)\ \ \forall P \in \Sigma_h^0$. Using (48), (49) (with $y_h := I_h z$) and (47), we obtain

$$\begin{aligned} b(q - q_h, z_h - z) &= b(q, z_h - z) - b(q_h, z_h - I_h z) - b(q_h, I_h z - z) \\ &\leqslant \langle f, z - z_h \rangle + \langle f, z_h - I_h z \rangle = \langle f, z - I_h z \rangle . \end{aligned} \tag{50}$$

Combining (44), (46) and (50), we arrive at

$$a_0 \|q - q_h\|_S^2 \leqslant a(e_h; q - q_h, q - \Pi_h q) + b(q, z - I_h z) + \langle f, z - I_h z \rangle . \tag{51}$$

Making use of (43) and the continuity of the form b, we may write

$$\begin{aligned} b(q, z - I_h z) + \langle f, z - I_h z \rangle &\leqslant \left(C\|q(e_h)\|_{Q(\mathcal{T}_h)} + \|f\|_{Z'} \right) \|z - I_h z\|_Z \\ &\leqslant C \|z - I_h z\|_{1,p,\Omega}. \end{aligned} \tag{52}$$

The interpolation theory yields

$$\|z - I_h z\|_{1,p,\Omega} \leqslant \hat{C} h^{2/p} |z|_{2,2,\Omega} \leqslant \tilde{C} h^{2/p}, \tag{53}$$

since $z \equiv w(e_h)$ is bounded in $W^{2,2}(\Omega)$ for all $e_h \in U_{ad}$ by virtue of assumption (A2)(iii).

A slight modification of Lemma 4 in [1] and (43) implies

$$\|q - \Pi_h q\|_S \leqslant \tilde{C} h \|q(e_h)\|_{Q(\mathcal{T}_h)} \leqslant C h.$$

Consequently, from (38) we derive that

$$\begin{aligned} a(e_h; q - q_h, q - \Pi_h q) &\leqslant C \|q - q_h\|_S \|q - \Pi_h q\|_S \\ &\leqslant \frac{1}{2} a_0 \|q - q_h\|_S^2 + C_3 h^2. \end{aligned} \tag{54}$$

Combining (51)–(54), we get

$$\frac{1}{2} a_0 \|q - q_h\|_S^2 \leqslant C_3 h^2 + \tilde{C} h^{2/p} \leqslant C h^{2/p},$$

so that (41) follows. □

Proposition 4.1. *Let $\{e_h\}$, $h \to 0_+$, be a sequence of $e_h \in U^h_{ad}$, such that*

$$e_h \to e \quad \text{in } C(\bar{\Omega}), \text{ as } h \to 0_+.$$

Then

$$\bar{q}_h(e_h) \to q(e) \quad \text{in } S, \text{ as } h \to 0_+. \tag{55}$$

Proof. By triangle inequality and Lemma 4.1

$$\begin{aligned}\|\bar{q}_h(e_h) - q(e)\|_S &\leqslant \|\bar{q}_h(e_h) - q(e_h)\|_S + \|q(e_h) - q(e)\|_S \\ &\leqslant Ch^{1/p} + \|q(e_h) - q(e)\|_S.\end{aligned}$$

Then (55) follows from Corollary 2.1, since $U^h_{ad} \subset U_{ad}$. □

Proposition 4.2. *Let the assumptions of Proposition 4.1 be fulfilled. Then*

$$\mathcal{J}_\varepsilon\big(e_h, \bar{q}_h(e_h)\big) \to \mathcal{J}_\varepsilon\big(e, q(e)\big), \quad \text{as } h \to 0_+.$$

Proof is analogous to that of Lemma 2.1, being based on Proposition 4.1. □

Lemma 4.2. *For any $e \in U_{ad}$ there exists a sequence $\{\eta_h\}$, $h \to 0_+$, such that $\eta_h \in U^h_{ad}$ and*

$$\eta_h \to e \quad \text{in } C(\bar{\Omega}), \text{ as } h \to 0_+.$$

Proof. Let us set $\eta_h = I_h e$, i.e., the Lagrange linear interpolate of e over $\mathcal{T}_h$. Since $e \in W^{1,\infty}(\Omega)$, the interpolation theory yields

$$\|e - I_h e\|_{0,\infty} \leqslant Ch\|e\|_{1,\infty}.$$

Obviously, $e_{\min} \leqslant I_h e \leqslant e_{\max}$ everywhere in $\bar{\Omega}$. Finally, we have, for $\overline{P_i P_{i+1}}$ parallel with the x_j-axis, $j = 1, 2$,

$$\left|\frac{\partial I_h e}{\partial x_j}\right| = \frac{1}{h_j}|e(P_{i+1}) - e(P_i)| \leqslant \frac{1}{h_j}\int_{P_i}^{P_{i+1}} \left|\frac{\partial e}{\partial x_j}\right| \mathrm{d}x_j \leqslant C_j.$$

□

Theorem 4.1. *Let $\{e_h^\varepsilon\}$, $h \to 0_+$, be a sequence of solutions of the approximate optimal design problems* (27).

Then there exists a subsequence $\{e_{\hat{h}}^\varepsilon\} \subset \{e_h^\varepsilon\}$ and an element $e_\varepsilon \in U_{ad}$ such that

$$e_{\hat{h}} \to e_\varepsilon \quad \text{in } C(\bar{\Omega}), \tag{56}$$

$$\bar{q}_{\hat{h}}(e_{\hat{h}}) \to q(e_\varepsilon) \quad \text{in } S \tag{57}$$

and e_ε is a solution of the penalized optimal design problem (6). *Each uniformly convergent subsequence of $\{e_h^\varepsilon\}$ tends to a solution of* (6) *and* (57) *holds.*

Proof. Since $U_{ad}^h \subset U_{ad}$ and U_{ad} is compact in $C(\bar{\Omega})$, there exists a subsequence of $\{e_h^\varepsilon\}$, such that (56) holds with $e_\varepsilon \in U_{ad}$. Then (57) follows from Proposition 4.1.

Let us prove that e_ε is a solution of the problem (6). Consider any $e \in U_{ad}$ and use Lemma 4.2 to obtain $\{\eta_{\hat{h}}\}$, $\eta_{\hat{h}} \in U_{ad}^{\hat{h}}$, such that $\eta_{\hat{h}} \to e$ in $C(\bar{\Omega})$. By definition (6),

$$\mathcal{J}_\varepsilon\big(e_{\hat{h}}^\varepsilon, \bar{q}_{\hat{h}}(e_{\hat{h}}^\varepsilon)\big) \leqslant \mathcal{J}_\varepsilon\big(\eta_{\hat{h}}, \bar{q}_{\hat{h}}(\eta_{\hat{h}})\big).$$

Passing to the limit with $\hat{h} \to 0$ and employing Proposition 4.2 to both sides, we arrive at

$$\mathcal{J}_\varepsilon\big(e_\varepsilon, q(e_\varepsilon)\big) \leqslant \mathcal{J}_\varepsilon\big(e, q(e)\big).$$

Consequently, e_ε solves the problem (6). The rest of the assertion is obvious. □

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