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MODELLING OF SINGULARITIES IN ELASTOPLASTIC MATERIALS WITH FATIGUE

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Summary. The hypothesis that, on the macroscopic level, the accumulated fatigue of an elastoplastic material with kinematic hardening can be identified from the mathematical point of view with the dissipated energy, is used for the construction of a new constitutive elastoplastic fatigue model. Its analytical investigation characterizes conditions for the formation of singularities in a finite time. The corresponding constitutive law is then coupled with the dynamical equation of motion of a one-dimensional continuum and the resulting hyperbolic problem is solved via space-discretization method.

Keywords: hysteresis, elastoplasticity, fatigue, hyperbolic system

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INTRODUCTION

The aim of this paper is to propose a model for the accumulation of fatigue in elastoplastic materials that enables us to predict the apparition of singularities (in space and time) as a result of oscillatory loading. The idea is based on the mathematical identification of the accumulated microscopical damage with the dissipation of energy. This hypothesis is experimentally justified by the so-called *rainflow method of damage evaluation* which is one of the most efficient and most successful engineering methods of estimation of material fatigue ([12]). It has been shown in [4] that the rainflow method is based on a law of accumulation of relative damage (Palmgren-Miner linear damage accumulation law) which is identical to the standard dissipation law resulting from the second principle of thermodynamics.

The rainflow method is a scalar method. Its extension to the vector (tensor) case seems to be rather difficult (cf. [1]). Even in the scalar case, the rainflow method does not explain the following experimental facts (cf. [11]).

- the elasticity modulus decreases during the accumulation of fatigue,
- singularities (cracks) occur after some critical time.

Our model consists in introducing a constitutive operator $\varepsilon = F(\sigma)$ between the stress and strain tensors σ, ε, respectively, such that F depends implicitly on the dissipated energy (identified with the accumulated fatigue). This constitutive law satisfies the requirements above, in particular it can develop singularities in a finite time. Moreover, the operator F is a hysteresis operator (it is causal and rate independent) which is continuous with respect to the uniform convergence.

We pay special attention to the uniaxial (scalar) case. We introduce a dimensionless *fatigue characteristic number* Q depending only on material constants. It turns out that for small values of Q, closed hysteresis loops produced by a uniaxial loading and unloading are convex in the corresponding $\sigma - \varepsilon$ plane. This is exactly what we need (cf. [10]) for solving for instance the equations of forced longitudinal or torsional oscillations of a beam

$$(1) \qquad \begin{cases} v_t = \sigma_x + g(x,t) \\ \varepsilon_t = v_x \\ \varepsilon = F(\sigma) \end{cases}$$

with a given forcing term g and with suitable initial and boundary conditions. Let us note that the problem of convexity of the hysteresis operator F is related to the so-called *strong form of the second principle of thermodynamics* used in the theory of plasticity (see [2]).

System (1) is hyperbolic in the sense of bounded speed of propagation. The only singularities which occur are those due to the fatigue (no shocks!). Here again (as in [10]), the convexity of loops (ε follows a convex path when σ increases and a concave path when σ decreases) prevents the system from the formation of shocks.

I. A model of elastoplasticity with kinematic hardening

We start with a standard elastoplastic model with kinematic hardening defined by the relations ([11], [13])

$$(2) \qquad \begin{cases} \varepsilon = \varepsilon^p + \varepsilon^e, \quad \sigma = \sigma^p + \sigma^e, \\ \varepsilon^p = A\sigma^e, \quad \varepsilon^e = B\sigma, \\ \sigma^p \in Z, \quad \langle \dot{\varepsilon}^p, \sigma^p - \tilde{\sigma} \rangle \geqslant 0 \quad \forall \tilde{\sigma} \in Z, \end{cases}$$

where ε^e, σ^e, ε^p, σ^p are the elastic and plastic components of the strain and stress tensor, respectively, A, B are given positive definite symmetric matrices over the space of symmetric tensors $\mathbf{T}$, $Z \subset \mathbf{T}$ is a given convex closed constraint, $0 \in \operatorname{Int} Z$ (its boundary ∂Z represents the yield surface) and dot denotes the time derivative. The *hardening rule* $\sigma \mapsto \sigma^e$ is then defined by the variational inequality

$$(3) \qquad \begin{cases} \sigma = \sigma^e + \sigma^p, \\ \sigma^p \in Z, \\ \langle A\dot{\sigma}^e, \sigma^p - \tilde{\sigma}\rangle \geqslant 0 \ \ \forall \tilde{\sigma} \in Z, \\ \sigma^p(0) = \sigma_0^p \in Z \ \text{ given.} \end{cases}$$

Let us note that the relation $\langle \xi, \eta\rangle_A := \langle A\xi, \eta\rangle$ defines another scalar product $\langle\cdot,\cdot\rangle_A$ in $\mathbf{T}$. The following result is an easy application of the standard technique of evolution variational inequalities.

Proposition 1.1. ([8]) *Let* $\sigma \in W^{1,1}(0,T;\mathbf{T})$ *be given. Then the problem* (3) *has a unique solution* $\sigma^e \in W^{1,1}(0,T;\mathbf{T})$.

The solution operator $\sigma^e = \ell(\sigma)$ is called *multidimensional play* ([6]). The constitutive relation $\sigma \mapsto \varepsilon$ defined by (2) can then be written in the form

$$(4) \qquad \varepsilon = B\sigma + A\ell(\sigma).$$

Theorem 1.2. ([6], [8], [9]). *The operator* ℓ

(i) *is continuous in* $W^{1,1}(0,T;\mathbf{T})$,

(ii) *admits a continuous extension to* $C([0,T];\mathbf{T})$,

(iii) *maps* $C([0,T];\mathbf{T})$ *into* $C([0,T],\mathbf{T}) \cap BV(0,T;\mathbf{T})$.

(iv) *If* $\sigma_n \to \sigma$ *uniformly,* $\sigma_n, \sigma \in C([0,T];\mathbf{T})$, *then* $\operatorname{Var}\ell(\sigma_n) \leqslant \text{const}$.

(v) *If moreover* Z *is bounded and* ∂Z *is smooth, then* $\operatorname{Var}\ell(\sigma_n) \to \operatorname{Var}\ell(\sigma)$.

II. A MODEL OF FATIGUE

In the constitutive relation (4) we modify the elastic law by putting

$$(5) \qquad \varepsilon(t) = (1 + \alpha q^2(t))B\sigma(t) + A\ell(\sigma)(t),$$

(the spatial variable x plays the role of a parameter), where $q(t)$ is the dissipated energy during the interval $[0,t]$ and $\alpha > 0$ is given constant.

The motivation for (5) is very transparent. As mentioned in the introduction, the elasticity modulus decreases during the accumulation of fatigue. Having identified the accumulated fatigue with the dissipated energy q, we are led quite naturally to the assumption that the elasticity modulus is a decreasing function of q. The quadratic expression αq^2 has been choosen as a first guess because of its simplicity.

We define the *internal energy* U in a standard form ([14])

$$U := \frac{1}{2}(\langle \varepsilon^e, \sigma \rangle + \langle \varepsilon^p, \sigma^e \rangle) = \frac{1}{2}[(1 + \alpha q^2)\langle B\sigma, \sigma \rangle + \langle A\ell(\sigma), \ell(\sigma)\rangle]. \tag{6}$$

The rate of dissipation $\dot{q}$ is defined by the formula

$$\dot{q} = \langle \dot{\varepsilon}, \sigma \rangle - \dot{U} = \alpha q \dot{q} \langle B\sigma, \sigma \rangle + \left\langle A\dot{\ell(\sigma)}, \sigma - \ell(\sigma) \right\rangle. \tag{7}$$

This yields the following ordinary differential equation for q

$$\begin{cases} \dot{q}(1 - \alpha q \langle B\sigma, \sigma \rangle) = \left\langle A\dot{\ell(\sigma)}, \sigma - \ell(\sigma) \right\rangle, \\ q(0) = 0. \end{cases} \tag{8}$$

The second principle of thermodynamics requires $\dot{q}$ to be nonnegative.

Obvious observation.

1. The constitutive relation (5) is well defined provided q is a solution of (8).
2. The expression $\left\langle A\dot{\ell(\sigma)}, \sigma - \ell(\sigma) \right\rangle$ is nonnegative a.e. for $\sigma \in W^{1,1}(0, T; \mathbf{T})$ by definition (3) of the multidimensional play ℓ; therefore, the second principle of thermodynamics is satisfied provided $\alpha q(t) \langle B\sigma(t), \sigma(t) \rangle < 1$.
3. A singularity occurs as soon as $\alpha q(t) \langle B\sigma(t), \sigma(t) \rangle = 1$.
4. The relationship $\sigma \mapsto q$ (and consequently also $\sigma \mapsto \varepsilon$) is rate independent.

More precisely, the following easy statement holds.

Proposition 2.1. *For every $\sigma \in W^{1,1}(0, T; \mathbf{T})$ there exist $T^* > 0$ and a unique nondecreasing absolutely continuous function $q\colon [0, T^*) \to [0, \infty)$ satisfying (8); the maximal value of T^* is*

$$T^* = \sup\{t \in [0, T]\,;\, \alpha q(t) \langle B\sigma(t), \sigma(t) \rangle < 1\}.$$

In order to pass to arbitrary continuous inputs σ we introduce for each $\sigma \in W^{1,1}(0, T; \mathbf{T})$ an auxiliary function

$$\begin{aligned} V(\sigma)(t) &:= \int_0^t \left\langle A\dot{\ell(\sigma)}(\tau), \sigma(\tau) - \ell(\sigma)(\tau) \right\rangle \mathrm{d}\tau \\ &= \int_0^t \langle \sigma(\tau) - \ell(\sigma)(\tau), \mathrm{d}\ell(\sigma)(\tau) \rangle_A \,. \end{aligned}$$

Proposition 2.2. *The operator* V
(i) *maps continuously* $W^{1,1}(0,T;\mathbf{T})$ *into* $W^{1,1}(0,T)$,
(ii) *admits a continuous extension* $C([0,T];\mathbf{T}) \to C([0,T])$.

Proof. Part (i) is obvious. Part (ii) follows from Theorem 1.2. Indeed, the Stieltjes integral (9) is well defined for each $\sigma \in C([0,T];\mathbf{T})$ and $V(\sigma)$ is continuous, since $\ell(\sigma) \in BV(0,T;\mathbf{T}) \cap C([0,T];\mathbf{T})$. Let $\{\sigma_n\} \subset C([0,T];\mathbf{T})$ be a given sequence, $\sigma_n \to \sigma$ uniformly. By Theorem 1.2 we have $\operatorname{Var} \ell(\sigma_n) \leqslant$ const., $\ell(\sigma_n) \to \ell(\sigma)$ uniformly, hence by Theorem II.15.3 of [5] we obtain $V(\sigma_n)(t) \to V(\sigma)(t)$ for all $t \in [0,T]$. The sequence $\{V(\sigma_n)\}$ is a sequence of nondecreasing continuous functions which converges pointwise to a nondecreasing continuous function, hence $V(\sigma_n) \to V(\sigma)$ uniformly. □

Equation (8) can be rewritten in the form

$$q(t) = \int_0^t \frac{1}{1 - \alpha q(\tau) \langle B\sigma(\tau), \sigma(\tau) \rangle} \, \mathrm{d}V(\sigma)(\tau). \tag{10}$$

Proposition 2.3. *Let* $\sigma \in C([0,T];\mathbf{T})$ *be given. Put* $D := \{(t,q) \in [0,T) \times [0,\infty);\ \alpha q \langle B\sigma(t), \sigma(t)\rangle < 1\}$. *Let* $(t_0, q_0) \in D$ *be given. Then there exists* $t_1 > t_0$ *and a unique solution* $q\colon [t_0,t_1] \to [0,\infty)$ *to the equation*

$$q(t) = q_0 + \int_{t_0}^t \frac{1}{1 - \alpha q(\tau) \langle B\sigma(\tau), \sigma(\tau) \rangle} \, \mathrm{d}V(\sigma)(\tau). \tag{10*}$$

The function q *is continuous and nondecreasing in* $[t_0,t_1]$ *and* $(t, q(t)) \in D$ *for all* $t \in [t_0, t_1]$.

Proof. Put $\delta := \frac{1}{2}(1 - \alpha q_0 \langle B\sigma(t_0), \sigma(t_0)\rangle) > 0$. We find $t_1 > t_0$ such that
(11)
$$\delta \alpha q_0 \mid \langle B\sigma(t), \sigma(t)\rangle - \langle B\sigma(t_0), \sigma(t_0)\rangle \mid + \alpha \langle B\sigma(t), \sigma(t)\rangle (V(\sigma)(t) - V(\sigma)(t_0)) < \delta^2$$

for all $t \in [t_0, t_1]$.

We next define a (convex) closed set $U_\delta \subset C([t_0,t_1])$ and an operator $A\colon U_\delta \to C([t_0,t_1])$ by the formulae

$$U_\delta := \{u \in C([t_0,t_1]);\ u(t_0) = q_0, 1 - \alpha u(t) \langle B\sigma(t), \sigma(t)\rangle \geqslant \delta \ \forall t \in [t_0,t_1]\},$$

$$A(u)(t) := q_0 + \int_{t_0}^t \frac{1}{1 - \alpha u(\tau) \langle B\sigma(\tau), \sigma(\tau) \rangle} \, \mathrm{d}V(\sigma)(\tau).$$

Using (11) we check easily that U_δ is nonempty (the constant function $u(t) \equiv q_0$ belongs to U_δ), and that A is a contraction which maps U_δ into U_δ. The assertion now follows from a standard fixed point argument. □

Corollary 2.4. *For every $\sigma \in C([0,T];\mathbf{T})$ there exists $T^* > 0$ and a unique maximal continuous nondecreasing solution $q\colon [0,T^*) \to [0,\infty)$ to the equation* (10), *$T^* = \sup\{t \in [0,T];\ \alpha q(t)\langle B\sigma(t),\sigma(t)\rangle < 1\}$.*

The following Theorem is the most important result of this section. It states that q depends on σ continuously with respect to the uniform convergence.

Theorem 2.5. *Let $\sigma \in C([0,T];\mathbf{T}]$ be given and let $q\colon [0,T^*) \to [0,\infty)$ be the maximal solution to* (10). *For an arbitrary $\eta > 0$ put*

$$\delta := \frac{1}{2}\max_{[0,T^*-\eta]}(1 - \alpha q(t)\langle B\sigma(t),\sigma(t)\rangle) > 0.$$

Let $\{\sigma_n\} \subset C([0,T];\mathbf{T})$ be a sequence, $\sigma_n \to \sigma$ uniformly in $[0,T]$, and let $q_n\colon [0,T_n^*) \to [0,\infty)$ be the corresponding maximal solutions to (10). Then there exists $n_0 > 0$ such that for all $n \geqslant n_0$ we have

(i) $T_n^* > T^* - \eta$,

(ii) $1 - \alpha q_n(t)\langle B\sigma_n(t),\sigma_n(t)\rangle \geqslant \delta \qquad \forall t \in [0,T^*-\eta]$,

(iii) *$q_n \to q$ uniformly in $[0,T^*-\eta]$.*

Remark. We have in particular $\liminf_{n\to\infty} T_n^* \geqslant T^*$. Time $t = T^*$ will be called *critical time* for q. The proof of Theorem 2.5 relies on Gronwall's inequality in the following form.

Lemma 2.6. *Let w, U be nonnegative continuous functions in $[0,T]$, $U(0) = 0$, U nondecreasing, and let M, N be nonnegative constants. Assume that*

$$w(t) \leqslant M + N\int_0^t w(\tau)\,\mathrm{d}U(\tau) \quad \forall t \in [0,T].$$

Then

$$w(t) \leqslant M\mathrm{e}^{NU(t)} \quad \forall t \in [0,T].$$

We just recall that Lemma 2.6 follows immediately from the integration-by-parts formula

$$\int_0^s \mathrm{e}^{-NU(t)}w(t)\,\mathrm{d}U(t) = N\int_0^s \mathrm{e}^{-NU(t)}\Big(\int_0^t w(\tau)\,\mathrm{d}U(\tau)\Big)\,\mathrm{d}U(t) + \mathrm{e}^{-NU(s)}\int_0^s w(t)\,\mathrm{d}U(t).$$

Proof of Theorem 2.5. Let us assume that for some n there exists $t_n \in [0,T_n^*) \cap [0,T^*-\eta]$ such that

$$1 - \alpha q_n(t_n)\langle B\sigma_n(t_n),\sigma_n(t_n)\rangle < \delta. \tag{12}$$

Put $\hat{T}_n := \min\{t \in [0, T_n^*) ; 1 - \alpha q_n(t) \langle B\sigma_n(t), \sigma_n(t)\rangle \leqslant \delta\}$. For $t \in [0, T^* - \eta]$ put

$$\begin{aligned} M_n(t) := \Bigg| \int_0^t & \frac{1}{1 - \alpha q(\tau) \langle B\sigma(\tau), \sigma(\tau)\rangle} \, \mathrm{d}V(\sigma_n)(\tau) \\ & - \int_0^t \frac{1}{1 - \alpha q(\tau) \langle B\sigma(\tau), \sigma(\tau)\rangle} \, \mathrm{d}V(\sigma)(\tau) \Bigg| \\ & + \frac{\alpha}{2\delta^2} \int_0^t \left| \langle B\sigma_n(\tau), \sigma_n(\tau)\rangle - \langle B\sigma(\tau), \sigma(\tau)\rangle \right| q(\tau) \, \mathrm{d}V(\sigma_n)(\tau) \end{aligned}$$

and

$$N := \frac{\alpha}{2\delta^2} \sup\{\langle B\sigma_n(t), \sigma_n(t)\rangle \; ; \; n \geqslant 1, t \in [0, T]\}.$$

A straightforward computation shows that the inequality

$$|q_n(t) - q(t)| \leqslant M_n(t) + N \int_0^t |q_n(\tau) - q(\tau)| \, \mathrm{d}V(\sigma_n)(t)$$

holds for all $t \in [0, \hat{T}_n]$.

Lemma 2.6 then yields

$$|q_n(t) - q(t)| \leqslant \|M_n\|_{[0, T^* - \eta]} \mathrm{e}^{NV(\sigma_n)(t)} \tag{13}$$

for all $t \in [0, \hat{T}_n]$. We have indeed $\|M_n\|_{[0,T^*-\eta]} \to 0$ as $n \to \infty$, hence (13) implies in particular

$$\alpha \|q_n \langle B\sigma_n, \sigma_n\rangle - q \langle B\sigma, \sigma\rangle \|_{[0, \hat{T}_n]} < \delta$$

for n sufficiently large. This shows that condition (12) can hold only for finitely many n. Consequently, for n sufficiently large we have

$$1 - \alpha q_n(t) \langle B\sigma_n(t), \sigma_n(t)\rangle \geqslant \delta \quad \forall t \in [0, T^* - \eta]$$

and the uniform convergence of q_n to q follows from (13). □

III. The scalar case

In this section we study in detail particular properties of the constitutive relation (4) when ε, σ are scalar-valued functions. It is natural to assume that the material was not subject to any plastic deformation in the past. If ℓ_h is the scalar play operator defined by (3) for $Z = [-h, h]$, where $h > 0$ is a given constant, the above requirement means

$$|\sigma(0)| < h, \ \ell_h(\sigma)(0) = 0. \tag{14}$$

The constitutive relation (4) has the form $\varepsilon = F(\sigma)$ with

$$F(\sigma) := \frac{1}{E}(1 + \alpha q^2)\sigma + A\ell_h(\sigma), \tag{15}$$

where q is the solution of the equation

$$\dot{q}(t) = \frac{AE}{E - \alpha q(t)\sigma^2(t)}\,(\sigma(t) - \ell_h(\sigma)(t))\,\dot{\ell_h(\sigma)}(t) = \frac{EAh}{E - \alpha q(t)\sigma^2(t)}|\dot{\ell_h(\sigma)}(t)|. \tag{16}$$

We will see below that in the one-dimensional case and under appropriate assumptions, the operator F has a particular convexity property which plays a crucial role in the theory of hyperbolic equations with hysteresis (cf. [10]).

Proposition 3.1. *Let $\sigma \in C([0,T])$ be given such that $\|\sigma\|_{[0,T]} \leqslant 2h$ and (14) holds, and let $q\colon [0,T^*) \to [0,\infty)$ be the corresponding maximal solution of (16) in the sense of Corollary 2.4. Let us assume that σ is monotone in an interval $[t_1,t_2] \subset [0,T^*)$. Then one of the following cases occurs:*

(i) σ is nondecreasing and there exists a convex increasing function $\Phi_+\colon [\sigma(t_1),\sigma(t_2)] \to \mathbf{R}^1$ such that $\Phi'_+(\xi) \geqslant \frac{1}{E}$ for almost all $\xi \in (\sigma(t_1),\sigma(t_2))$ and $F(\sigma)(t) = \Phi_+(\sigma(t))$ for all $t \in [t_1,t_2]$;

(ii) σ is nonincreasing and there exists a concave increasing function $\Phi_-\colon [\sigma(t_2),\sigma(t_1)] \to \mathbf{R}^1$ such that $\Phi'_-(\xi) \geqslant \frac{1}{E}$ for almost all $\xi \in (\sigma(t_2),\sigma(t_1))$ and $F(\sigma)(t) = \Phi_-(\sigma(t)$ for all $t \in [t_1,t_2]$.

Remark 3.2. Condition $\|\sigma\|_{[0,T]} \leqslant 2h$ in the case $\sigma \in W^{1,1}(0,T)$ is necessary and sufficient for the validity of the strong version of the second law of thermodynamics (see [2])

$$\dot{\varepsilon}^p \cdot \sigma \geqslant 0 \text{ a.e.} \tag{17}$$

Indeed, it suffices to notice that we have here $\varepsilon^p = A\ell_h(\sigma)$ and (cf. [6])

$$\|\ell_h(\sigma)\|_{[0,T]} = \max\{0, \|\sigma\|_{[0,T]} - h\}, \text{ and} \tag{18}$$

$$\dot{\ell_h}(\sigma)(t) > 0 \Rightarrow \ell_h(\sigma)(t) = \sigma(t) - h, \quad \dot{\ell_h}(\sigma)(t) < 0 \Rightarrow \ell_h(\sigma)(t) = \sigma(t) + h. \tag{19}$$

Further discussion about the condition (17) in the context of Mróz' model can be found in [3].

Proof of Proposition 3.1. We consider just the case of σ nondecreasing using an alternative definition of $\ell_h(\sigma)$ (see [6]), namely

$$\ell_h(\sigma)(t) = \max\{\ell_h(\sigma)(t_1), \sigma(t) - h\} \text{ for } t \in [t_1, t_2].$$

In particular, there exists a point $\tau \in [t_1, t_2]$ such that

$$\ell_h(\sigma)(t) = \begin{cases} \ell_h(\sigma)(t_1), & t \in [t_1, \tau], \\ \sigma(t) - h, & t \in (\tau, t_2]. \end{cases} \tag{20}$$

We next define an auxiliary function R: $[\sigma(t_1), \sigma(t_2)] \to [0, \infty)$ as the solution of the problem

$$\begin{cases} R(s) = q(t_1) & \text{for } s \in [\sigma(t_1), \sigma(\tau)] \\ \dfrac{\mathrm{d}R}{\mathrm{d}S} = \dfrac{EAh}{E - \alpha s^2 R} & \text{for } s \in (\sigma(\tau), \sigma(t_2)]. \end{cases} \tag{21}$$

The case $\tau = t_2$ is trivial. In the nontrivial case $\tau < t_2$ we have $\sigma(\tau) = \ell_h(\sigma)(\tau) + h$, hence $\sigma(\tau) \geqslant 0$ by (18).

Comparing (21) to (16) and using (20) we see that we have

$$q(t) = R(\sigma(t)) \qquad \text{for all} \qquad t \in [t_1, t_2].$$

Condition $[t_1, t_2] \subset [0, T^*)$ guarantees that the solution R of (21) is defined in $[\sigma(t_1), \sigma(t_2)]$.

This enables us to give an explicit formula for Φ_+, namely

$$\Phi_+(s) = \frac{1}{E}(1 + \alpha R^2(s))s + A\max\{\ell_h(\sigma)(t_1), s - h\} \tag{22}$$

for $s \in [\sigma(t_1), \sigma(t_2)]$.

The rest of the proof is an easy exercise of differentiation. $\square$

We now formulate a sufficient condition for the validity of the strong version (17) of the second law of thermodynamics in terms of material constants.

We first introduce the set

$$(23) \qquad \Omega := \{(s,r) \in \mathbf{R}^1 \times [0,\infty);\, \alpha r s^2 < E,\ -S_0(r) \leqslant s \leqslant S_0(r)\},$$

where $S_0\colon [0,\infty) \to [0,\infty)$ is the solution of the problem

$$(24) \qquad \frac{\mathrm{d}S_0}{\mathrm{d}r} = \frac{1}{EAh}(E - \alpha r S_0^2), \qquad S_0(0) = h.$$

Let us note that the equation $\alpha r S_0^2(r) = E$ has a unique solution $r_0 > 0$, hence $\Omega = \Omega_- \cup \Omega_+$, where

$$\Omega_- = \{(s,r) \in \mathbf{R}^1 \times [0,r_0];\, |s| \leqslant S_0(r)\}\Omega_+ = \{(s,r) \in \mathbf{R}^1 \times [r_0,\infty);\, \alpha r s^2 < E\}.$$

The function S_0 is increasing in $[0,r_0]$. This implies in particular

$$(25) \qquad \Omega \subset [-S_0(r_0), S_0(r_0)] \times [0,\infty).$$

Lemma 3.3. *Let $\sigma \in C([0,T])$ be given such that* (14) *holds and let $q\colon [0,T^*) \to [0,\infty)$ be the maximal solution of* (16) *in the sense of Corollary* 2.4. *Then the following statements hold for all $t \in [0,T^*)$.*

(i) $q(t) < r_0 \Rightarrow h - S_0(q(t)) \leqslant l_h(\sigma)(t) \leqslant S_0(q(t)) - h$,

(ii) $(\sigma(t), q(t)) \in \Omega$.

Proof. It suffices to assume that σ is absolutely continuous and piecewise monotone; the general case then follows from Theorem 2.5.

We prove (i) by induction. Let $0 = \hat{t}_0 < \hat{t}_1 < \ldots < \hat{t}_N = T^*$ be a partition of $[0,T^*)$ such that σ is monotone in $[\hat{t}_{k-1}, \hat{t}_k]$, $k = 1,\ldots,N$. The assertion holds for $t = 0$. We prove the following implication:

If (i) holds for $t = t_1$ and σ is monotone in $[t_1,t_2] \subset [0,T^*)$, then (i) holds for $t = t_2$.

It suffices again to consider the case of σ nondecreasing. Using (2), (21) we see that $\ell_h(\sigma)$, q are constant in $[t_1,\tau]$, hence (i) holds. In the nontrivial case $\tau < t_2$ we have $\ell_h(\sigma)(t) = \sigma(t) - h$ and $q(t) = R(\sigma(t))$ for $t \in [\tau, t_2]$.

The function R is increasing in $[\sigma(\tau), \sigma(t_2)]$ and the inverse function

$$S := R^{-1}\colon [q(\tau), q(t_2)] \to [\sigma(\tau), \sigma(t_2)]$$

satisfies

$$(26) \qquad \frac{\mathrm{d}S}{\mathrm{d}r} = \frac{1}{EAh}(E - \alpha r S^2),$$

$$\sigma(t) = S(q(t)), \quad t \in [\tau, t_2]. \tag{27}$$

We therefore have $\ell_h(\sigma)(t) = S(q(t)) - h$ for $t \in [\tau, t_2]$, and $S(q(\tau)) = \sigma(\tau) = \ell_h(\sigma)(\tau) + h \leqslant S_0(q(\tau))$.

The uniqueness property of the equation (26) guarantees

$$\ell_h(\sigma)(t) + h = \sigma(t) = S(q(t)) \leqslant S_0(q(t))$$

for all $t \in [\tau, t_2]$, and (i) is proved.

Part (ii) is an easy consequence of (i). Indeed, for $q(t) < r_0$ we have

$$|\sigma(t)| \leqslant |\ell_h(\sigma)(t)| + h \leqslant S_0(q(t)).$$

For $t < T^*$ and $q(t) \geqslant r_0$ we have by definition $E - \alpha\sigma^2(t)q(t) > 0$.

Lemma 3.3 is proved. □

Let us introduce a dimensionless *fatigue characteristic number* Q defined by the formula

$$Q := \frac{E}{A\alpha h^4}. \tag{28}$$

An "almost necessary and sufficient" condition for the validity of the strong version of the second law of thermodynamics reads as follows.

Proposition 3.4. *Let the assumptions of Lemma 3.3 hold. If $Q \leqslant \frac{192}{31}$, then $\|\sigma\|_{[0,T^*]} < 2h$.*

Conversely, if $Q \geqslant \frac{64}{e^2+1}$ then for $\sigma(t) = t$ we have $\|\sigma\|_{[0,T^*]} = T^* > 2h$.

Proof. By Lemma 3.3 and (25), it suffices to prove the implications

$$Q \leqslant \frac{192}{31} \Rightarrow S_0(r_0) < 2h, \tag{29}$$

$$Q \geqslant \frac{64}{e^2+1} \Rightarrow S_0(r_0) > 2h. \tag{30}$$

We first transform the equation (24) into a dimensionless form by introducing new variables

$$y(x) := aS_0(bx), \quad a := \left(\frac{A\alpha h}{E}\right)^{1/3}, \quad b := \frac{E}{\alpha}a^2.$$

The function y is the solution of the problem

$$(31) \qquad \begin{cases} y'(x) = 1 - xy^2(x), \\ y(0) = Q^{-1/3}. \end{cases}$$

Let us first assume $Q \leqslant \frac{192}{31}$. Putting $x_0 = \frac{1}{6}r_0$ and $y_0 = y(x_0)$ we check easily that $x_0 y_0^2 = 1$.

For $x \in (0, x_0)$ we have

$$y''(x) = -\left(y^2(x) + xy(x)y'(x)\right) < 0,$$

hence

$$y(x) > Q^{-1/3} + \frac{x}{x_0}(y_0 - Q^{-1/3}),$$

and (31) yields

$$y'(x) < 1 - x\left[Q^{-1/3} + \frac{x}{x_0}(y_0 - Q^{-1/3}\right]^2.$$

Integrating the last inequality from 0 to x_0 we obtain

$$y_0 - Q^{-1/3} < x_0 - x_0^2\left[\frac{1}{2}Q^{-1/3} + \frac{2}{3}Q^{-1/3}(y_0 - Q^{-1/3}) + \frac{1}{4}(y_0 - Q^{-1/3})^2\right].$$

We rewrite this inequality using the identity $x_0 y_0^2 = 1$ and an auxiliary quantity $c = y_0 Q^{1/3}$ in the form

$$c^4(c-1) < Q\left[\frac{3}{4}c^2 - \frac{1}{6}c - \frac{1}{12}\right] \leqslant \frac{16}{31}[9c^2 - 2c - 1],$$

therefore

$$(c-2)(31c^4 + 31c^3 + 62c^2 - 20c - 2) < 0.$$

This implies $c < 2$, hence (29) holds.

Let us assume now $Q \geqslant \frac{64}{e^2+1}$. The function $z(x) := \frac{1}{y(x)}$ satisfies the equation

$$(32) \qquad \begin{cases} z'(x) + z^2(x) = x, \\ z(0) = Q^{1/3}, \end{cases}$$

hence $z''(x) = 1 - 2z(x)z'(x) \geqslant 1 - 2z_0 z'(x)$ for $x \in (0, x_0)$, where $z_0 = z(x_0) = \sqrt{x_0}$.

We have $z'(x_0) = 0$, therefore

$$-z'(x) \geqslant \frac{1}{2z_0}\left(e^{2z_0(x_0 - x)} - 1\right) \qquad \text{for} \quad x \in (0, x_0).$$

After integration from 0 to x_0 we obtain

$$\frac{2Q^{1/3}}{z_0} - 1 \geqslant \frac{e^{2z_0^3} - 1}{2z_0^3}.$$

Assuming $z_0 \geqslant \frac{1}{2}Q^{1/3}$ we infer from this inequality $3 \geqslant \frac{e^{\frac{Q}{4}}-1}{\frac{Q}{4}}$. The function $f(p) = \frac{e^p-1}{p}$ is strictly convex and increasing in $[0,\infty)$; let p_0 be the solution of the equation $f(p) = 3$. We have $f(2) > 3$, hence

$$\frac{Q}{4} \leqslant p_0 < 2 - \frac{f(2) - 3}{f'(2)} = \frac{16}{e^2 + 1},$$

which is a contradiction. We must therefore have $z_0 < \frac{1}{2}Q^{1/3}$, hence (30) holds. $\square$

Remark. Propositions 3.1 and 3.4 guarantee that the condition $Q \leqslant \frac{192}{31}$ is sufficient for the convexity of the operator F. In fact, the convexity of F is preserved even beyond the domain of validity of the strong version of the second law of thermodynamics. We conjecture (and this is to be verified by a detailed analysis of the equation (21)) that there exists a precise upper bound for Q ensuring the convexity of F of the order of $Q \approx 200$.

Condition $Q \leqslant \frac{192}{31}$ can be interpreted as a lower bound for admissible values of α. In the case of $A316$ stainless steel (see [11]) we have for instance $E = 196,000[MPa], h = 260[MPa], \frac{1}{A} = 2,100[MPa]$, hence α is not allowed to be smaller than approximately $\frac{1}{70}\left[(MPa)^{-2}\right]$.

The following energy-type inequality is a variant of Theorem 3.8 and Lemma 3.2 of [10].

Proposition 3.5. *Let us assume $Q \leqslant \frac{192}{31}$ and let $\sigma \in W^{1,\infty}(0,T)$ be given. Let T^* be as in Proposition 2.1. Assume that $F(\sigma) \in W^{2,1}(0,\hat{T})$ for some $\hat{T} < T^*$ and put $P_2(\sigma)(t) := \frac{1}{2}\dot{F}(\sigma)(t)\dot{\sigma}(t)$ for $t \in (0,\hat{T})$. Then*

(i) $P_2(\sigma) \in BV(0,\hat{T})$,

(ii) *for all $t_1, t_2 \in [0,\hat{T}]$, $t_1 < t_2$, we have $\int_{t_1}^{t_2} \ddot{F}(\sigma)(t)\dot{\sigma}(t)\,dt \geqslant P_2(\sigma)(t_2-) - P_2(\sigma)(t_1+)$.*

In this section we construct a solution to the system (1) with F given by (15). We prescribe initial and boundary conditions

$$\sigma(x,0)=\sigma^0(x),\ v(x,0)=v^0(x),\ \sigma(0,t)=v(1,t)=0. \tag{33}$$

We extend easily Corollary 2.4 to the case of functions $\sigma\colon[0,1]\times[0,T]\to\mathbf{R}^1$, where the spatial variable $x\in[0,1]$ is considered as a parameter.

Proposition 4.1. *Let $\sigma\in C([0,1]\times[0,T])$ be given. Then there exists $T^*\in(0,T]$ and a unique continuous function $q\colon[0,1]\times[0,T^*)\to\mathbf{R}^1$ such that*

$$q(x,t)=\int_0^t\frac{EAh}{E-\alpha q(x,\tau)\sigma^2(x,\tau)}\,\mathrm{d}_\tau V(\sigma)(x,\tau), \tag{34}$$

where $V(\sigma)(x,t)=\operatorname*{Var}_{[0,t]}(\ell_h(\sigma(x,.)))$.

Proof. Equation (34) is the integral form of (16). By Corollary 2.4, for each $x\in[0,1]$ there exists $T^*(x)>0$ such that $q(x,t)$ is defined for $t\in[0,T^*(x))$. By Theorem 2.5, function $x\to T^*(x)$ is lower semicontinuous in [0,1], hence $T^*(x)$ attains its minimum $T^*=\min_{[0,1]}T^*(x)>0$.

The continuity of q follows also from Theorem 2.5. If $x_n\to x, t_n\to t$ are arbitrary sequences, $x_n,x\in[0,1], t_n,t\in[0,T^*)$, then in the triangle inequality

$$|q(x_n,t_n)-q(x,t)|\leqslant|q(x_n,t_n)-q(x,t_n)|+|q(x,t_n)-q(x,t)|$$

we have $q(x_n,.)\to q(x,.)$ uniformly in $[0,\sup_k t_k]$ and the assertion follows easily. □

Our main result for the system (1), (15), (33) reads as follows.

Theorem 4.2. *Let us assume $Q\leqslant\frac{192}{31}$, $g,g_t\in L^1(0,T;L^2(0,1))$ and let $\sigma^0,v^0\in W^{1,2}(0,1)$ be given, $\sigma^0(0)=v^0(1)=0$, $\|\sigma^0\|_{[0,1]}<h$. Then there exist $T^*>0$ and functions σ,v continuous in $[0,1]\times[0,T^*)$ such that*

(i) $\sigma_t,\sigma_x,v_t,v_x,\varepsilon_t\in L^\infty(0,T^*-\delta;L^2(0,1))$ *for all* $\delta\in(0,T^*)$,

(ii) *equations (1) are satisfied almost everywhere in* $(0,1)\times(0,T^*)$,

(iii) *conditions (33) hold for all* $x\in[0,1]$ *and* $t\in[0,T^*)$,

(iv) *we have either* $T^*=T$ *or there exists* $x\in[0,1]$ *such that* $\lim_{t\to T^*-}\alpha q(x,t)\sigma^2(x,t)=E$.

Remarks 4.3.

(i) Hyperbolicity in the sense of bounded speed of propagation for the system (1), (15) can be proved analogously as in [7].

(ii) Assertion (iv) of Theorem 4.2 says that the solution exists globally until a singularity due to the fatigue occurs at some point $x \in [0, 1]$.

(iii) We choose mixed boundary conditions in (33) in order to simplify the construction.

The rest of the paper is devoted to the proof of Theorem 4.2. The construction is somewhat awkward. We first discretize the system (1), (15), (33) in space, and for constructing a solution of the system of ordinary differential equations thus obtained, we discretize it in time. The time-space discrete system admits only weak estimates which are nevertheless sufficient for passing to the limit with respect to the time step. The space-discrete system enables us afterwards to obtain estimates of higher order (as in [10]) and to pass to the limit with respect to the space step. The main difficulty here is that we have to maintain the time of breakdown under control.

Space discretization

Let $n > 0$ be a given integer. For $j = 1, \ldots, n-1$ and $t \in [0,T]$ put $g_j(t) := n\int_{\frac{j-1}{n}}^{\frac{j}{n}} g(x,t)\,dx$. We replace (1), (15), (33) by the system of equations for $j = 1, \ldots, n-1$

$$\text{(35)}\quad \begin{aligned} &\text{(i)} && \dot{F(\sigma_j)}(t) = n\,(v_{j+1}(t) - v_j(t)),\\ &\text{(ii)} && \dot{v}_j(t) = n\,(\sigma_j(t) - \sigma_{j-1}(t)) + g_j(t) \end{aligned}$$

where dot denotes the derivative with respect to t, with unknown functions $\sigma_1, \ldots, \sigma_{n-1}$, $v_1, \ldots, v_{n-1}$, $\sigma_0 = v_n \equiv 0$. We prescribe for (35) natural initial conditions

$$\text{(36)}\qquad \sigma_j(0) = \sigma^0\left(\frac{j}{n}\right), \quad v_j(0) = v^0\left(\frac{j}{n}\right).$$

One cannot simply refer to the results of [10]. We have here no a priori information about the domain of definition of the solution and about the inverse of F. The method we choose here is based on a time discretization of the system (35), (36).

Time discretization. For a fixed integer $m > 0$ and for all $k = 0, 1, \ldots, m$ we put $\eta := \frac{T}{m}$, $g_j^k := g_j(k\eta)$, $j = 0, \ldots, n$. Our system has the form

$$\text{(37)}\quad \begin{aligned} &\text{(i)} && \tfrac{1}{\eta}\left(\varepsilon_j^{k+1} - \varepsilon_j^k\right) = A_{k+1} n\left(v_{j+1}^k - v_j^k\right),\\ &\text{(ii)} && \tfrac{1}{\eta}\left(v_j^{k+1} - v_j^k\right) = A_{k+1}\left(n(\sigma_j^k - \sigma_{j-1}^k) + g_j^k\right),\\ &\text{(iii)} && v_j^0 = v_j(0),\ \sigma_j^0 = \sigma_j(0),\ \sigma_j^0 = \tfrac{1}{E}\sigma_j^0,\ A_0 = 1,\ v_n^k = \sigma_0^k = 0,\\ & && k = 0, \ldots, m-1,\ j = 1, \ldots, n-1. \end{aligned}$$

We have to couple the system (37) with an algorithm for determining the values of σ_j^{k+1} and A_{k+1}.

Let us suppose that v_j^i, σ_j^i, ε_j^i, A_i are known for all $j = 1,\ldots,n-1$, $i = 0,\ldots,k$ and that $A_i = 1$ for all such i.

For $t \in [i\eta,(i+1)\eta)$ and $j = 1,\ldots,n-1$ put

$$\sigma_j^{(m)}(t) = \sigma_j^i + \frac{1}{\eta}(t - i\eta)(\sigma_j^{i+1} - \sigma_j^i). \tag{38}$$

Let $q_j^{(m)}$ be the solution of (16) for $\sigma = \sigma_j^{(m)}$. We stop the algorithm as soon as the critical quantity $C_j^{(m)}(t) := \alpha q_j^{(m)}(t)\left(\sigma_j^{(m)}(t)\right)^2$ attains the value E for some j.

We therefore assume $C_j^{(m)}(k\eta) < 1$ for all $j = 1,\ldots,n-1$.

The induction hypothesis is complete if we assume

$$\varepsilon_j^i = F(\sigma_j^{(m)})(i\eta), \quad j = 1,\ldots,n-1, \quad i = 0,\ldots,k. \tag{39}$$

Let us note that the choice of initial data guarantees (cf. (14)) that $q_j^{(m)}(0) = C_j^{(m)}(0) = 0$ and (39) holds for $k = 0$.

Algorithm.

1. Try $A_{k+1} = 1$ and compute ε_j^{k+1} from (37) (i).
2. Try to find the solution s to the equation $\varepsilon_j^{k+1} = \Phi_+^j(s)$ if $\varepsilon_j^{k+1} \geqslant \varepsilon_j^k$ (and $\varepsilon_j^{k+1} = \Phi_-^j(s)$ if $\varepsilon_j^{k+1} < \varepsilon_j^k$) where $\Phi_\pm^j$ are the functions from Proposition 3.1 for $t_1 = k\eta$ and $\sigma = \sigma_j^{(m)}$, and put $\sigma_j^{k+1} := s$.

The domain of definition of Φ_+^j, Φ_-^j is the interval $\mathcal{D}_j = [\sigma_j^k, s_j^+)$ $(\mathcal{D}_j = (s_j^-, \sigma_j^k]$, respectively) which is the maximal interval of existence of the solution $R_j^\pm$ to the equation

$$\text{(40)}\quad \text{(i)}\ \frac{\mathrm{d}R_j^+}{\mathrm{d}s} = \frac{EAh}{E - \alpha s^2 R_j^+}\,\frac{\mathrm{d}}{\mathrm{d}s}\max\left\{\ell_h(\sigma_j^{(m)})(k\eta), s - h\right\},$$

$$\text{(ii)}\ \frac{\mathrm{d}R_j^-}{\mathrm{d}s} = \frac{-EAh}{E - \alpha s^2 R_j^-}\,\frac{\mathrm{d}}{\mathrm{d}s}\min\left\{\ell_h(\sigma_j^{(m)})(k\eta), s + h\right\},$$

$$\text{(iii)}\ R_j^\pm(\sigma_j^k) = q_j^{(m)}(k\eta).$$

By definition, the expression $\alpha s^2 R_j^\pm(s)$ tends to E as $s \to s_j^\pm$.

Functions $\Phi_\pm^j$ are increasing and continuous in $\mathcal{D}_j$, hence there are two possibilities:

a) $\quad \varepsilon_j^{k+1} \in \Phi_\pm^j(\mathcal{D}_j)$ **for all** $j = 1,\ldots,n-1$.

Then σ_j^{k+1} are determined uniquely by the relation $\varepsilon_j^{k+1} = \Phi_\pm^j(\sigma_j^{k+1})$. By construction, (39) holds for $i = k+1$. The values of v_j^{k+1} are given by (37) (ii). We have as in the proof of Proposition 3.1 $q_j^{(m)}(t) = R_j^\pm\left(\sigma_j^{(m)}(t)\right)$ for $t \in [k\eta, (k+1)\eta]$, hence $C_j^{(m)}((k+1)\eta) < E$ and the procedure can continue for $i = k+2$.

b) $\quad \varepsilon_j^{k+1} \notin \Phi_\pm^j(\mathcal{D}_j)$ for some j.

For $j = 1, \ldots, n-1$ put

$$\hat{\varepsilon}_j^{k+1} := \begin{cases} \Phi_+^j(s_j^+ -) & \text{if } \varepsilon_j^{k+1} \geqslant \varepsilon_j^k, \\ \Phi_-^j(s_j^- +) & \text{if } \varepsilon_j^{k+1} < \varepsilon_j^k, \end{cases}$$

$$B_{k+1} := \max\left\{\frac{\varepsilon_j^{k+1} - \varepsilon_j^k}{\hat{\varepsilon}_j^{k+1} - \varepsilon_j^k}\,;\, j = 1, \ldots, n-1\right\}.$$

We now update A_{k+1}, ε_j^{k+1} by putting $A_{k+1} := \frac{1}{B_{k+1}}$, ε_j^{k+1} being the solution of (37) (i) for this new value of A_{k+1} for all $j = 1, \ldots, n-1$. We have by hypothesis $A_{k+1} \in (0,1]$. The values σ_j^{k+1} are then defined by the equations $\varepsilon_j^{k+1} = \Phi_+^j(\sigma_j^{k+1}-)$ if $\varepsilon_j^{k+1} \geqslant \varepsilon_j^k$ and $\varepsilon_j^{k+1} = \Phi_-^j(\sigma_j^{k+1}+)$ if $\varepsilon_j^{k+1} < \varepsilon_j^k$. For at least one $j \in \{1, \ldots, n-1\}$ we have by construction $\sigma_j^{k+1} = s_j^\pm$ and $\lim\limits_{t \to (k+1)\eta -} \alpha q_j^{(m)}(t)\left(\sigma_j^{(m)}(t)\right)^2 = E$.

We now stop the algorithm by putting $A_i := 0$, $\varepsilon_j^i = \varepsilon_j^{k+1}$, $\sigma_j^i = \sigma_j^{k+1}$, $v_j^i = v_j^{k+1}$ for all $j = 1, \ldots, n-1$, $i = k+2, \ldots, m$.

We have in fact proved the following result.

Proposition 4.4. *For each choice of integers $m, n > 0$ there exist piecewise linear functions $\sigma_j^{(m)} \in C([0,T])$, $j = 1, \ldots, n-1$ such that*

(i) (38) *holds for all* $i = 0, \ldots, m-1$, $j = 1, \ldots, n-1$,

(ii) *there exists* ${}^nT_{(m)}^* \in (0,T], {}^nT_{(m)}^* = (k^*+1)\eta$, *such that the solution* $q_j^{(m)}$ *of* (16) *corresponding to* $\sigma_j^{(m)}$ *exists in* $[0, {}^nT_{(m)}^*)$ *for all* $j = 1, \ldots, n-1$ *and we have either* ${}^nT_{(m)}^* = T$ *or there exists* $j_0 \in \{1, \ldots, n-1\}$ *such that* $\lim\limits_{t \to {}^nT_{(m)}^* -} \alpha q_{j_0}^{(m)}(t)\left(\sigma_{j_0}^{(m)}(t)\right)^2 = E$,

(iii) *for* $k \leqslant k^*$ *there exist* $\left\{\varepsilon_j^k, v_j^k, A_k\,;\, j = 1, \ldots, n-1\right\}$ *such that* $A_k = 1$, $\varepsilon_j^k = F(\sigma_j^{(m)})(k\eta)$ *for all* $j = 1, \ldots, n-1$ *and* (37) *holds,*

(iv) *for* $k > k^*$ *there exist* $\left\{\varepsilon_j^k, v_j^k, A_k\,;\, j = 1, \ldots, n-1\right\}$ *such that* $A_{k^*+1} \in (0,1]$, $A_k = 0$ *for* $k \geqslant k^*+2$ *and* (37) *holds.*

We now pass to the limit as $m \to \infty$ keeping n fixed.

Estimates I. We multiply (37) (i) by $(\sigma_j^{k+1}-\sigma_j^k)$ and (37) (ii) by $(v_j^{k+1}-v_j^k)$. This yields

$$\frac{1}{\eta}\sum_{j=1}^{n-1}\sum_{k=0}^{m-1}\left[(\varepsilon_j^{k+1}-\varepsilon_j^k)(\sigma_j^{k+1}-\sigma_j^k)+(v_j^{k+1}-v_j^k)^2\right]\leqslant$$

$$\leqslant n\sum_{j=1}^{n-1}\sum_{k=0}^{m-1}\left[(|v_{j+1}^k|+|v_j^k|)|\sigma_j^{k+1}-\sigma_j^k|+(|\sigma_j^k|+|\sigma_{j-1}^k|+\frac{1}{n}|g_j^k|)|v_j^{k+1}-v_j^k|\right].$$

We have by Proposition 3.1 $(\varepsilon_j^{k+1}-\varepsilon_j^k)(\sigma_j^{k+1}-\sigma_j^k)\geqslant\frac{1}{E}(\sigma_j^{k+1}-\sigma_j^k)^2$ and from the elementary Young inequality $ab\leqslant\frac{1}{2c}a^2+\frac{c}{2}b^2$ we infer

$$(41)\qquad\begin{cases}\frac{1}{\eta}\sum\limits_{j=1}^{n-1}\sum\limits_{k=0}^{m-1}\left[\frac{1}{E}(\sigma_j^{k+1}-\sigma_j^k)^2+(v_j^{k+1}-v_j^k)^2\right]\\ \qquad\leqslant\frac{1}{\eta}\sum\limits_{j=1}^{n-1}\sum\limits_{k=0}^{m-1}\left[(\varepsilon_j^{k+1}-\varepsilon_j^k)(\sigma_j^{k+1}-\sigma_j^k)+(v_j^{k+1}-v_j^k)^2\right]\\ \qquad\leqslant 2\eta\sum\limits_{j=1}^{n-1}\sum\limits_{k=0}^{m-1}\left[2n^2E|v_j^k|^2+4n^2|\sigma_j^k|^2+|g_j^k|^2\right]\end{cases}$$

Estimates II.

We multiply (37)(i) by σ_j^k and (37)(ii) by $v_j^k, k=0,1,\dots,k^*$. Using the identities

$$(v_j^{k+1}-v_j^k)v_j^k=\frac{1}{2}(|v_j^{k+1}|^2-|v_j^k|^2)-\frac{1}{2}(v_j^{k+1}-v_j^k)^2$$

and

$$\begin{aligned}(\varepsilon_j^{k+1}-\varepsilon_j^k)\sigma_j^k &= \int_{k\eta}^{(k+1)\eta}\dot{F}(\sigma_j^{(m)})(t)\sigma_j^{(m)}(t)\,\mathrm{d}t\\ &\quad-\frac{1}{\eta}\int_{k\eta}^{(k+1)\eta}\dot{F}(\sigma_j^{(m)})(t)(\sigma_j^{k+1}-\sigma_j^k)(t-k\eta)\,\mathrm{d}t\\ &\geqslant\int_{k\eta}^{(k+1)\eta}\dot{F}(\sigma_j^{(m)})(t)\sigma_j^{(m)}(t)\,\mathrm{d}t-(\varepsilon_j^{k+1}-\varepsilon_j^k)(\sigma_j^{k+1}-\sigma_j^k)\end{aligned}$$

we obtain for each $\ell\in\{0,\dots,k^*\}$.

$$(42)\qquad\begin{cases}\sum\limits_{j=1}^{n-1}\left[\int_0^{(\ell+1)\eta}\dot{F}(\sigma_j^{(m)})(t)\sigma_j^{(m)}(t)\,\mathrm{d}t+\frac{1}{2}|v_j^{\ell+1}|^2\right]\\ \qquad\leqslant\frac{1}{2}\sum\limits_{j=1}^{n-1}|v_j^0|^2+\eta\sum\limits_{j=1}^{n-1}\sum\limits_{k=0}^{\ell}|g_j^k||v_j^k|+\\ \qquad+\sum\limits_{j=1}^{n-1}\sum\limits_{k=0}^{\ell}\left[(\varepsilon_j^{k+1}-\varepsilon_j^k)(\sigma_j^{k+1}-\sigma_j^k)+\frac{1}{2}(v_j^{k+1}-v_j^k)^2\right].\end{cases}$$

Energy inequality (6) entails for all $\tau \in \big(0, (\ell+1)\eta\big)$

$$\int_0^\tau F(\dot{\sigma}_j^{(m)})(t)\sigma_j^{(m)}(t)\,\mathrm{d}t \geqslant U_j^{(m)}(\tau) - U_j^{(m)}(0),$$

where

$$U_j^{(m)}(\tau) := \frac{1}{2}\left[\frac{1}{E}\big(1+\alpha(q_j^{(m)}(\tau))^2\big)\big(\sigma_j^{(m)}(\tau)\big)^2 + A\big(\ell_k(\sigma_j^{(m)})(\tau)\big)^2\right]$$

is the internal energy. We therefore have

$$U_j^{(m)}(\tau) \geqslant \frac{1}{2E}\big(\sigma_j^{(m)}(\tau)\big)^2, \quad U_j^{(m)}(0) = \frac{1}{2E}|\sigma_j^0|^2.$$

Combining (41) and (42) we obtain

$$(43)\qquad \begin{cases} \frac{1}{2}\sum\limits_{j=1}^{n-1}\left[\frac{1}{E}|\sigma_j^{\ell+1}|^2 + |v_j^{\ell+1}|^2\right] \leqslant \sum\limits_{j=1}^{n-1}\left[\frac{1}{E}|\sigma_j^0|^2 + |v_j^0|^2\right] + \eta\sum\limits_{j=1}^{n-1}\sum\limits_{k=0}^{\ell}|g_j^k||v_j^k| \\ \qquad\qquad + c_n\eta^2\sum\limits_{j=1}^{n-1}\sum\limits_{k=0}^{m-1}\left[|v_j^k|^2 + |\sigma_j^k|^2 + |g_j^k|^2\right]. \end{cases}$$

By c_n we denote here and in the sequel any constant dependent possibly on n and independent of m.

Put

$$\begin{aligned} V_j &:= \max\big\{|v_j^k|;\, k=0,\dots,k^*+1\big\} = \max\big\{|v_j^k|;\, k=0,\dots,m\big\}, \\ S_j &:= \max\big\{|\sigma_j^k|;\, k=0,\dots,k^*+1\big\} = \max\big\{|\sigma_j^k|;\, k=0,\dots,m\big\}. \end{aligned}$$

We have obviously

$$\eta\sum_{j=1}^{n-1}\sum_{k=0}^{m-1}|g_j^k| \leqslant \left(Tn\eta\sum_{j=1}^{n-1}\sum_{k=0}^{m-1}|g_j^k|^2\right)^{1/2} \leqslant c_n,$$

hence (43) yields

$$\frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{E}S_j^2 + V_j^2 \leqslant c_n\left[1 + \max_j V_j + \eta\sum_{j=1}^{n-1}(V_j^2 + S_j^2)\right].$$

Taking m sufficiently large (i.e. η sufficiently small) we obtain the final estimates

$$(44)\qquad \begin{cases} \max\limits_k|\sigma_j^k| + \max\limits_k|v_j^k| \leqslant c_n \\ \frac{1}{\eta}\sum\limits_{k=0}^{m-1}\left[(\sigma_j^{k+1}-\sigma_j^k)^2 + (v_j^{k+1}-v_j^k)^2\right] \leqslant c_n \end{cases}$$

for all $j = 1, \ldots, n-1$.

Convergence as $m \to \infty$. We now define the function $v_j^{(m)}$ by a formula analogous to (38), namely

$$v_j^{(m)}(t) = v_j^i + \frac{1}{\eta}(t - i\eta)(v_j^{i+1} - v_j^i)$$

for $i = 0, \ldots, m-1$ and $t \in [i\eta, (i+1)\eta)$.

Estimates (44) show that the sequences $\{v_j^{(m)}\}_{m=1}^{\infty}, \{\sigma_j^{(m)}\}_{m=1}^{\infty}$ are bounded in $W^{1,2}(0,T)$ for each j. Taking a subsequence, if necessary, we find $\sigma_j, v_j \in W^{1,2}(0,T)$ such that $\sigma_j^{(m)} \to \sigma_j, v_j^{(m)} \to v_j$ weakly in $W^{1,2}(0,T)$ and uniformly in $C([0,T])$.

Functions σ_j, v_j obviously satisfy initial conditions (36). Let q_j be the solution of (16) for $\sigma = \sigma_j$ and let T_j^* be the critical time for q_j, ${}^nT^* := \min_j T_j^*$. By Theorem 2.5, we have $\liminf_{m\to\infty} {}^nT^*_{(m)} \geqslant {}^nT^*$ and $F(\sigma_j^{(m)})$ converge to $F(\sigma_j)$ in $[0, {}^nT^*)$ locally uniformly as $m \to \infty$.

We want to show that v_j, σ_j satisfy system (35) in $[0, {}^nT^*)$. For $t \in [k\eta, (k+1)\eta)$ we define functions

$$\tilde{v}_j^{(m)}(t) := v_j^k,\ \tilde{\sigma}_j^{(m)}(t) := \sigma_k^k,\ \tilde{g}_j^{(m)}(t) := g_j^k,$$

$$\mathrm{e}_j^{(m)}(t) := \mathrm{e}_j^k + \frac{1}{\eta}(t - k\eta)(\mathrm{e}_j^{k+1} - \mathrm{e}_j^k), \quad k = 0, \ldots, k^*.$$

Let $\kappa > 0$ be arbitrarily chosen. System (37) can be rewritten for almost all $t \in (0, {}^nT^* - \kappa)$ and for m sufficiently large in the form

$$(45) \qquad \begin{cases} \dot{\varepsilon}_j^{(m)}(t) = n\left(\tilde{v}_{j+1}^{(m)}(t) - \tilde{v}_j^{(m)}(t)\right), \\ \dot{v}_j^{(m)}(t) = n\left(\tilde{\sigma}_j^{(m)}(t) - \tilde{\sigma}_{j-1}^{(m)}(t)\right) + \tilde{g}_j^{(m)}(t), \\ v_j^{(m)}(0) = v_j(0),\ \sigma_j^{(m)}(0) = \sigma_j(0). \end{cases}$$

We just have to prove $\varepsilon_j^{(m)} \to F(\sigma_j), \tilde{v}_j^{(m)} \to v_j, \tilde{\sigma}_j^{(m)} \to \sigma_j, \tilde{g}_j^{(m)} \to g_j$ uniformly in $[0, {}^nT^* - \kappa]$ as $m \to \infty$ for all $j = 1, \ldots, n-1$.

We have for instance for $t \in [k\eta, (k+1)\eta)$

$$|\tilde{\sigma}_j^{(m)}(t) - \sigma_j(t)| \leqslant |\sigma_j^{k+1} - \sigma_j^k| \leqslant \left(\sum_{k-0}^{m-1} |\sigma_j^{k+1} - \sigma_j^k|^2\right)^{1/2} \leqslant c_n \eta^{1/2},$$

$$|\tilde{g}_j^{(m)}(t) - g_j(t)| \leqslant n \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{k\eta}^{(k+1)\eta} |g_t(x,t)|\, \mathrm{d}t\, \mathrm{d}x,$$

$$\begin{aligned}|\varepsilon_j^{(m)}(t) - F(\sigma_j)(t)| \leqslant{}& |\varepsilon_j^{(m)}(t) - \varepsilon_j^{(m)}(k\eta)| + |F(\sigma_j^{(m)})(k\eta) - F(\sigma_j)(k\eta)| \\ &+ |F(\sigma_j)(k\eta) - F(\sigma_j)(t)|,\end{aligned}$$

where

$$\begin{aligned}|\varepsilon_j^{(m)}(t)-\varepsilon_j^{(m)}(k\eta)| &\leqslant |\varepsilon_j^{(m)}((k+1)\eta)-\varepsilon_j^{(m)}(k\eta)| \\ &\leqslant |F(\sigma_j^{(m)})((k+1))\eta)-F(\sigma_j)((k+1)\eta)| \\ &+|F(\sigma_j^{(m)})(k\eta)-F(\sigma_j)(k\eta)|+|F(\sigma_j)((k+1)\eta)-F(\sigma_j)(k\eta)|\end{aligned}$$

and the convergence follows easily. We can summarize the results obtained so far in the following

Proposition 4.5. *Let n be a given integer and let σ^0, v^0 be given functions satisfying the hypotheses of Theorem 4.2. Then there exists ${}^nT^* \in (0,T]$ and absolutely continuous functions $v_j, \sigma_j : [0, {}^nT^*) \to \mathbf{R}^1$, $j = 1,\dots,n-1$ satisfying (35), (36) for $t \in [0, {}^nT^*)$ and we have either ${}^nT^* = T$ or there exists $j \in \{1,\dots,n-1\}$ such that $\lim\limits_{t\to {}^nT^*-} \alpha q_j(t)\sigma_j^2(t) = E$, where q_j is the solution of equation (16) for $\sigma = \sigma_j$.*

To finish the proof of Theorem 4.2, we apply now the standard technique of hyperbolic equations with hysteresis (see e.g. [10]), namely estimates based on Proposition 3.5.

Estimates III.

Functions v_j, σ_j, g_j are absolutely continuous in $[0, {}^nT^*)$, hence we can differentiate equations (35) with respect to t. Then, multiplying the derivative of (35)(i) by $\dot\sigma_j$ and the derivative of (35)(ii) by $\dot v_j$ we obtain from Propositions 3.5 and 3.1 for almost all $t \in (0, {}^nT^*)$

$$\frac{1}{2n}\sum_{j=1}^{n-1}\left(\frac{1}{E}\dot\sigma_j^2+\dot v_j^2\right) \leqslant \frac{1}{2n}\sum_{j=1}^{n-1}(\dot{F(\sigma_j)}(0)\dot\sigma_j(0)+\dot v_j^2(0))+\frac{1}{n}\sum_{j=1}^{n-1}\int_0^t \dot g_j(\tau)\dot v_j(\tau)\,\mathrm{d}\tau,$$

where

$$\dot{F(\sigma_j)}(0)\dot\sigma_j(0) = E\left(\dot{F(\sigma_j)}(0)\right)^2 = E\left|n\int_{\frac{j}{n}}^{\frac{j+1}{n}} v^0(x)\,\mathrm{d}x\right|^2,$$

$$\dot v_j^2(0) = \left|n\int_{\frac{j-1}{n}}^{\frac{j}{k}}\left(\sigma^{0\prime}(x)+g(x,0)\right)\mathrm{d}x\right|^2$$

and

$$\sum_{j=1}^{n-1}\int_0^t \dot g_j(\tau)\dot v_j(\tau)\,\mathrm{d}\tau \leqslant \left(\sum_{j=1}^{n-1}\max_{\tau\in[0,t]}|\dot v_j(\tau)|^2\right)^{1/2}\int_0^t\left(\sum_{j=1}^{n-1}\dot g_j^2(\tau)\right)^{1/2}\mathrm{d}\tau,$$

hence for all $\tau \in (0, {}^nT^*)$ we have

(46)

$$\frac{1}{2n}\sum_{j=1}^{n-1}\left(\dot{\sigma}_j^2(t)+\dot{v}_j^2(t)\right) \leqslant c\left(\|v^0\|^2_{W^{1,2}}+\|\sigma^0\|^2_{W^{1,2}}+\|g(\cdot,0)\|_{L^2}+\|g_t\|_{L^1(0,T;L^2)}\right),$$

where c is a constant independent of n.

Estimate IV. Analogously as in Estimate II, we multiply (35) (i) by σ_j and (35) (ii) by v_j. After integration we obtain from (6) for every $t \in (0, {}^nT^*)$.

$$\frac{1}{2n}\sum_{j=1}^{n-1}\left(\frac{1}{E}\sigma_j^2(t)+v_j^2(t)\right) \leqslant \frac{1}{2n}\sum_{j=1}^{n-1}\left(\frac{1}{E}\sigma_j^2(0)+v_j^2(0)\right)+\frac{1}{n}\sum_{j=1}^{n-1}\int_0^t g_j(\tau)v_j(\tau)\,\mathrm{d}\tau,$$

therefore

$$\frac{1}{2n}\sum_{j=1}^{n-1}\left(\frac{1}{E}\sigma_j^2(t)+v_j^2(t)\right) \leqslant c\left(\|\sigma^0\|_{L^2}+\|v^0\|_{L^2}+\|g\|_{L^2(0,T;L^2)}\right). \tag{47}$$

Convergence as $n \to \infty$.

Estimate (46) shows that σ_j, v_j are Lipschitz continuous in $[0, {}^nT^*)$, hence they can be extended to $[0,T]$ by putting $\sigma_j(t) = \sigma_j({}^nT^*-)$, $v_j(t) = v_j({}^nT^*-)$ for $t \in [{}^nT^*, T]$.

We construct for $x \in [\frac{i}{n}, \frac{i+1}{n})$, $t \in [0,T]$, $j = 0,\dots,n-1$ linear interpolates

$$\sigma^{(n)}(x,t) := \sigma_j(t) + n\Big(x-\frac{j}{n}\Big)(\sigma_{j+1})t) - \sigma_j(t)),$$
$$v^{(n)}(x,t) := v_j(t) + n\Big(x-\frac{j}{n}\Big)(v_{j+1}(t) - v_j(t)).$$

Sequences $\{\sigma^{(n)}\}, \{v^{(n)}\}, \{\sigma_t^{(n)}\}, \{v_t^{(n)}\}, \{\sigma_x^{(n)}\}$ are bounded in $L^\infty(0,T;L^2(0,1))$. Passing to subsequences, if necessary, we find functions $v \in L^\infty(0,T;L^2(0,1))$, $\sigma \in C([0,1]\times[0,T])$ such that $v_t, \sigma_x, \sigma_t \in L^\infty(0,T;L^2(0,1))$ and $v_t^{(n)} \to v_t$, $v^{(n)} \to v$, $\sigma_t^{(n)} \to \sigma_t$, $\sigma_x^{(n)} \to \sigma_x$ in $L^\infty(0,T;L^2(0,1))$ weakly-star and $\sigma^{(n)} \to \sigma$ uniformly in $C([0,1]\times[0,T])$.

Let $q(x,t)$ be the solution of equation (34) and let T^* be the critical time corresponding to q. Using once more Theorem 2.5 we infer that $\liminf_{n\to\infty} {}^nT^* \geqslant T^*$, hence equations (35) are satisfied in $[0, T^*-\delta]$ for arbitrary $\delta > 0$ for n sufficiently large. Moreover $F(\sigma^{(n)}) \to F(\sigma)$ uniformly in $[0,1]\times[0,T^*-\delta]$.

For $t \in [0,T^*-\delta]$ and $x \in [\frac{i}{n}, \frac{i+1}{n})$, $j = 0,\dots,n-1$ we introduce auxiliary functions $\tilde{\varepsilon}^{(n)}(x,t) := F(\sigma_j)(t)$, $\tilde{v}^{(n)}(x,t) := v_{j+1}(t)$, $\tilde{g}^{(n)}(x,t) := g_{j+1}(t)$.

System (35) can be rewritten in the form

$$
(48) \qquad \begin{cases} \tilde{\varepsilon}_t^{(n)} = v_x^{(n)} \\ \tilde{v}_t^{(n)} = \sigma_x^{(n)} + \tilde{g}^{(n)} \end{cases}
$$

a.e. in $(0,1) \times (0, T^* - \delta)$.

Let $q^{(n)}$ be the solution of equation (34) for $\sigma = \sigma^{(n)}$. Since $q^{(n)} \to q$ uniformly in $[0,1] \times [0, T^* - \delta]$ and $q^{(n)}\left(\frac{j}{n}, t\right) = q_j(t)$, $j = 1, \ldots, n-1$, we infer that there exists a constant $c > 0$ dependent possibly on δ and independent of n such that

$$
\max_{\substack{t \in [0, T^* - \delta] \\ j = 1, \ldots, n-1}} q_j(t) \leqslant c, \quad \max_{\substack{t \in [0, T^* - \delta] \\ j = 1, \ldots, n-1}} \frac{1}{E - \alpha a_j(t) \sigma_j^2(t)} \leqslant c.
$$

Relations (47), (15) then imply

$$
\max_{t \in [0, T^* - \delta]} \int_0^1 |\tilde{\varepsilon}_t^{(n)}(x,t)|^2 \, \mathrm{d}x \leqslant c, \quad \text{hence} \quad \max_{t \in [0, T^* - \delta]} \int_0^1 |v_x^{(n)}(x,t)|^2 \, \mathrm{d}x \leqslant c.
$$

Consequently, $v^{(n)} \to v$ uniformly in $[0, T^* - \delta]$, $v_x^{(n)} \to v_x$, $\tilde{\varepsilon}_t^{(n)} \to w$ in $L^\infty\big(0, T^* - \delta; L^2(0,1)\big)$ weakly-star (we pass to a subsequence, if necessary).

To prove that $w = F(\sigma)_t$ a.e., it suffices to verify that $\tilde{\varepsilon}^{(n)} \to F(\sigma)$ uniformly in $[0,1] \times [0, T^* - \delta]$. We have for $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right)$

$$
\left| \tilde{\varepsilon}^{(n)}(x,t) - F(\sigma^{(n)})(x,t) \right| = \left| F(\sigma^{(n)})\Big(\frac{j}{n}, t\Big) - F(\sigma^{(n)})(x,t) \right|
$$

and the assertion follows from the uniform convergence $F(\sigma^{(n)}) \to F(\sigma)$.

Mean Continuity Theorem yields $\int_0^T \int_0^1 |\tilde{g}^{(n)}(x,t) - g(x,t)| \, \mathrm{d}x \, \mathrm{d}t \to 0$ as $n \to \infty$. It is easy to check that σ, v satisfy initial and boundary conditions (33), hence v, σ are solutions to (1), (15), (33).

Let us note that $\delta > 0$ has been chosen arbitrarily, hence the proof of Theorem 4.2 is complete.

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