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ON SYSTEMS GOVERNED BY TWO ALTERNATING VECTOR FIELDS

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Summary. We investigate the nonautonomous periodic system of ODE's of the form $\dot{x} = \vec{v}(x) + r_p(t)(\vec{w}(x) - \vec{v}(x))$, where $r_p(t)$ is a $2p$-periodic function defined by $r_p(t) = 0$ for $t \in \langle 0, p \rangle$, $r_p(t) = 1$ for $t \in (p, 2p)$ and the vector fields $\vec{v}$ and $\vec{w}$ are related by an involutive diffeomorphism.

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1. Introduction

There is a rich literature on qualitative behaviour of systems governed by ODE's or PDE's, but so far only a little was written about systems which are governed alternately by two different vector fields. A natural example of such a system was presented in [2] where dynamical behavior of a chemical reactor with periodic flow reversal was studied. However, this paper considers the problem almost exclusively from the engineering point of view. Here we would like to provide a mathematical setting which suits the observed phenomena.

The original problem is described by four nonlinear PDE's of reaction-diffusion type, which were spatially discretized and then solved numerically. Because PDE's are difficult to handle,we shall concentrate on the related problem of a system of ODE's. Basically we have two vector fields which alternately operate on some phase space. Generally speaking these vector fields could be totally unrelated as well as the periods of their "acting", but rather than study the problem in its most general setting we consider only the case when the two vector fields act for the same time periods (usually referred to as the switching period and denoted by p) and are related by some diffeomorphism G. Since simulations of various systems revealed that even for linear G the resulting behavior may be "wild", we decided to restrict the choice of G by the condition which the conjugating mapping of the original problem satisfied, namely G has to be an involution: $G \circ G = \mathrm{Id}$, equivalently $G = G^{-1}$.

Once we fix a switching period p, we obtain a discrete dynamical system, described by a composition of the flows corresponding to the vector fields. We can thus consider p to be a parametr of the system and study the dependence of the asymptotic behavior on this parametr. Of course, the system may have some physical parameters as well and so the other possibility how to study the observed phenomena is to fix p and look at what happens to the invariant sets as the physical parametr varies.

Both approaches show very interesting behavior and it is worthwhile to spend some time on building a proper mathematical background, which would explain and predict the behavior of the system without numerical simulation. Even though this task in its full generality is rather complicated, we tried to give it a start by addressing the most eminent questions.

2. Description

Let us have a sufficiently smooth vector field $\vec{v}$ on $\mathbf{R}^n$, generating a global phase flow $\varphi\colon \mathbf{R}\times\mathbf{R}^n \longrightarrow \mathbf{R}^n$ (that is a 1-parameter subgroup of diffeomorphisms φ^t, $t \in \mathbf{R}$).

Let us further consider a diffeomorphism $G \in \mathrm{Diff}(\mathbf{R}^n)$ satisfying

$$G \circ G = \mathrm{Id}, \tag{1}$$

and the G-related vector field

$$\vec{w} = G_*(\vec{v})$$

(see for instance [1] for basic notation and properties).

We shall denote the flow of the vector field $\vec{w}$ by ψ^t. Then we have (see [1], p. 137, Exercise 4)

$$\psi^t = G \circ \varphi^t \circ G \quad \text{for all } t \in \mathbf{R}. \tag{2}$$

Let $p > 0$ be a period of switching. Consider a $2p$-periodic function $r_p(t)$ defined by

$$r_p(t) = \begin{cases} 0 & \text{for } t \in \langle 0, p\rangle \\ 1 & \text{for } t \in (p, 2p) \end{cases} \tag{3}$$

The purpose of this paper in this setting is to investigate the properties of the $2p$-periodic nonautonomous system of ODE's

$$\dot{x} = \vec{f}(x,t) = \vec{v}(x) + r_p(t)[\vec{w}(x) - \vec{v}(x)]. \tag{4}$$

(whose right hand side $\vec{f}(x,t)$ satisfies the Caratheodory conditions).

The useful tool for the investigation of periodic solutions of (4) is the *period map* (Poincaré map).

Let us denote $\Phi(t;0,x_0)$ the solution of (4), satisfying the initial condition $\Phi(0) = x_0$.

This solution can be expressed for $t \in \langle 0, 2p\rangle$ in the form

$$\Phi(t;0,x_0) = \begin{cases} \varphi^t(x_0) & \text{for } t \in \langle 0, p\rangle \\ \psi^{t-p}(\varphi^p(x_0)) & \text{for } t \in (p, 2p\rangle \end{cases} \tag{5}$$

Let us put

$$P(x) = \Phi(2p;0,x). \tag{6}$$

Then the mapping $P\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is the period map for the system (4) and P is a differentiable mapping.

The relation (5) yields

$$P = \psi^p \circ \varphi^p \tag{7}$$

and with respect to (2) we obtain

$$P = G \circ \varphi^p \circ G \circ \varphi^p. \tag{8}$$

If we put

$$H = G \circ \varphi^p, \tag{9}$$

then

$$P = H \circ H = H^2. \tag{10}$$

Remark 1. It is well-known that the system (4) has a $2p$-periodic solution iff the period map P has a fixed point. With respect to the relation (10) the period map P has two kinds of fixed points: The fixed point of P is a fixed point of H, or the fixed point of P corresponds to a 2-periodic orbit of H.

Theorem 1. *Let $\hat{x}$ be a fixed point of the mapping H and K the closed phase curve corresponding to the fixed point $\hat{x}$. Then the curve K is G-symmetric, i.e. $G(K) = K$.*

Proof. We have $G(\varphi^p(\hat{x})) = \hat{x}$, hence $\varphi^p(\hat{x}) = G(\hat{x})$. Let us put $\hat{y} = G(\hat{x})$. The relation (2) yields

$$G(\varphi^t(\hat{x})) = \psi^t(G(\hat{x})) = \psi^t(\hat{y}). \tag{11}$$

This relation proves Theorem 1. □

Theorem 2. *Let $\{\hat{x}_1, \hat{x}_2\}$ be a 2-periodic orbit of the mapping H (i.e. $H(\hat{x}_1) = \hat{x}_2$ and $H(\hat{x}_2) = \hat{x}_1$. Then $P(\hat{x}_1) = \hat{x}_1$ and $P(\hat{x}_2) = \hat{x}_2$. Let K and L be the closed phase curves corresponding to the fixed points $\hat{x}_1$ and $\hat{x}_2$, respectively.*

Then

$$G(K) = L. \tag{12}$$

Proof. We have

$$G(\varphi^p(\hat{x}_1)) = \hat{x}_2, \quad \varphi^p(\hat{x}_1) = G(\hat{x}_2)$$

and

$$G(\varphi^p(\hat{x}_2)) = \hat{x}_1, \quad \varphi^p(\hat{x}_2) = G(\hat{x}_1).$$

The curves K and L consist each of two arcs:

$$\begin{aligned}
K_1 &= \{\varphi^t(\hat{x}_1),\ t \in \langle 0, p \rangle\},\\
K_2 &= \{\psi^t(G(\hat{x}_2)),\ t \in \langle 0, p \rangle\},\\
L_1 &= \{\varphi^t(\hat{x}_2),\ t \in \langle 0, p \rangle\},\\
L_2 &= \{\psi^t(G(\hat{x}_1)),\ t \in \langle 0, p \rangle\}.
\end{aligned}$$

From the relation $G(\varphi^t(\hat{x}_1)) = \psi^t(G(\hat{x}_1))$ we get

$$G(K_1) = L_2 \quad \text{and} \quad G(K_2) = L_1.$$

□

a) If M is a closed positive invariant set of the flow φ^t, homeomorphic with the closed unit ball in $\mathbf{R}^n$ so that $G(M) = M$, then due to Brouwer's Fixed Point Theorem the mapping $H = G \circ \varphi^p : M \longrightarrow M$ has a fixed point for each $p > 0$.

Example 1. In the well-known Lorenz system (see [3], p. 196) we can take for the set M the ellipsoid E

$$E = \{rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leqslant c\}.$$

If we take for the diffeomorphism G some element of the group of symmetries of the ellipsoid E, then the period map P in question has a fixed point in E.

b) In case we cannot apply the procedure described in the previous paragraph, we obtain a criterion for the existence of fixed points as follows:

We set

$$F(t, x) = \varphi^t(x) - G(x) \tag{13}$$

and solve

$$F(t, x) = 0. \tag{14}$$

If $(p, \hat{x}_1)$ is a solution of the equation (14), then the point $\hat{x}_1$ is a fixed point of the mapping H. So let x_0 be a fixed point of the diffeomorphism G. Then $(0, x_0)$ is a solution of the equation (14), since $F(0, x_0) = \varphi^0(x_0) - G(x_0) = x_0 - x_0 = 0$. If $D_x F(0, x_0)$ is a regular matrix, then according to the Implicit Function Theorem, there is an $\varepsilon > 0$ such that for every $p \in (-\varepsilon, \varepsilon)$ there exists $\hat{x}(p)$ satisfying $F(p, \hat{x}(p)) = 0$.

The matrix

$$D_x F(0, x_0) = \frac{\partial \varphi^0}{\partial x}(x_0) - D_x G(x_0) = E - D_x G(x_0)$$
$$(E - \text{ identity matrix})$$

is regular if and only if 1 is not an eigenvalue of $D_x G(x_0)$, which is exactly when x_0 is an isolated fixed point of G. So we have proved

Theorem 3. *Let x_0 be an isolated fixed point of G. Then there is $\varepsilon > 0$ such that the system (4) has a $2p$-periodic solution for each $p \in (0, \varepsilon)$, or equivalently, the period map P has a fixed point.*

R e m a r k 2. When we find a fixed point $\hat{x}$ of H (or P) for some small value of the switching period, we can obtain the fixed points of P for bigger values of p by continuation, during which changes of stability of the fixed point $x(p)$ may occur and consequently various bifurcations may take place.

(For instance, in the system investigated in [2], the fixed point $\hat{x}(p)$ of H loses stability and becomes a 2-periodic orbit $\{\hat{x}_1, \hat{x}_2\}$ of H.)

Let us remark that the stability of fixed points of the mapping $H = G \circ \varphi^p$ is determined by the eigenvalues of the matrix $D_x H(\hat{x}(p))$. We have

$$D_x H(\hat{x}) = D_x(G \circ \varphi^p)(\hat{x}) = D_x G(\varphi^p(\hat{x})) \cdot \frac{\partial \varphi^p}{\partial x}(\hat{x}) = D_x G(G(\hat{x})) \cdot U_{\hat{x}}(p),$$

where $U_{\hat{x}}(p)$ is a fundamental matrix of the variational equations

$$\dot{y} = \frac{\partial \vec{v}(\varphi^t(\hat{x}))}{\partial x} y.$$

4. Measure of invariant sets

Theorem 4. *Let G be a linear diffeomorphism and*

$$\operatorname{div} \vec{v} < 0$$

on $\mathbf{R}^n$. Then every bounded invariant set of the period map P is of zero Lebesgue measure.

P r o o f. First we show that we can infer from the assumptions that

$$\operatorname{div} \vec{f} < 0 \tag{15}$$

on $\mathbf{R} \times \mathbf{R}^n$. To achieve this we need to show that $\operatorname{div} \vec{w} < 0$ on $\mathbf{R}^n$.

Differentiating the defining relation for $\vec{w} = G_*(\vec{v})$, i.e. $\vec{w}(x) = G\vec{v}(G(x))$, gives

$$\frac{\partial \vec{w}}{\partial x}(x) = G \frac{\partial \vec{v}}{\partial x}(G(x))G, \tag{16}$$

which shows that the matrices $\frac{\partial \vec{w}}{\partial x}(x)$, $\frac{\partial \vec{v}}{\partial x}(G(x))$ are similar and hence their traces are equal, which means that

$$\operatorname{div} \vec{w}(x) = \operatorname{div} \vec{v}(G(x))$$

for all $x \in \mathbf{R}^n$.

Thus the inequality (15) holds for the right hand side of the system (4).

The assertion now follows from Theorem 1.9 in [4], page 77. We should note that Theorem 1.9 is formulated for a periodic system with a continuous right hand side, but it is readily verified that its proof can be repeated in the case of the system (4). Let us remark that the matrix

$$J(t) = \frac{\partial \Phi(t; 0, x)}{\partial x}$$

is a continuous function of t (see [5], p. 329) and its derivative with respect to t exists for all $t \in \mathbf{R} \setminus D$, where $D = \{kp, k \in \mathbf{Z}\}$. The matrix $J(t)$ is a fundamental matrix of the variational equations

$$\dot{y} = \frac{\partial \vec{f}(t, \Phi(t; 0, x))}{\partial x} y$$

and the Liouville formula holds:

$$\det J(t) = \det J(t_0) \exp \left(\int_{t_0}^{t} \mathrm{Tr} \left(\frac{\partial \vec{f}}{\partial x} \right) \mathrm{d}t \right).$$

$\square$

R e m a r k 3. Let us conclude this section by one "pathological" property of the system (4). Let us suppose that the flow φ^t has a G-invariant closed orbit γ with a period q. Then the curve γ is an invariant curve of the period map P. If $x_0 \in \gamma$, then $P(x_0) = (G \circ \varphi^p \circ G \circ \varphi^p)(x_0) \in \gamma$, as can be easily seen. So $P|\gamma \colon \gamma \longrightarrow \gamma$ is a diffeomorphism. If its rotation number is rational, then there is $x_0 \in \gamma$ and $k \in \mathcal{N}$ so that $P^k(x_0) = x_0$ and the system (4) has a periodic solution with the period $k \cdot 2p$.

If the rotation number of P is irrational and $\vec{v}|\gamma = \vec{w}|\gamma$, then the system (4) has a periodic solution with a period q which is incommensurable to the period of the system (4) (cf. [4], p. 74, Theorem 1.8 or [6]).

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