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A MIXED FINITE ELEMENT METHOD FOR PLATE BENDING WITH A UNILATERAL INNER OBSTACLE

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Summary. A unilateral problem of an elastic plate above a rigid interior obstacle is solved on the basis of a mixed variational inequality formulation. Using the saddle point theory and the Herrmann-Johnson scheme for a simultaneous computation of deflections and moments, an iterative procedure is proposed, each step of which consists in a linear plate problem. The existence, uniqueness and some convergence analysis is presented.

Keywords: unilateral plate problem, inner obstacle, mixed finite elements, Herrmann-Johnson mixed model, fourth order variational inequality

AMS classification: 65N30, 73K10, 49D29

INTRODUCTION

In technical applications of elastic plates, the bending moments appear to be the most required quantities very often. For instance, in optimal design, such as minimum weight problems a frequent constraint function is an integral of a quadratic form of the moment tensor. Therefore, mixed variational methods have been developed, in which both the deflections and the moments are computed simultaneously.

One of the most effective mixed models is the so-called Herrmann-Johnson finite element scheme, a thorough analysis of which was given by C. Johnson and then by Brezzi and Raviart in [2]. The advantage of the latter method is the fact, that only standard C^0-elements are needed for the approximation of deflections, whereas piecewise constant elements can be used to approximate the moment field. Comodi [7] employed the mixed finite element model of Herrmann-Johnson to the plate bending with unilateral displacements on the boundary.

The aim of the present paper is to extend the method and the analysis to inner obstacle problems, i.e., to variational inequalities of the fourth order.

Following some ideas of Glowinski, Lions and Trémolières ([4], Section 4.2.2, Example 3) we introduce a saddle point problem and study its existence and uniqueness in Section 1. We present a discretization of the previous problem by finite elements in Section 2. Its unique solvability is proved and an algorithm of Uzawa's type proposed for the search of the discrete saddle point. In Section 3 we investigate the distance between the discrete and original saddle points. Under some particular assumptions, we prove the convergence of approximate bending moments in L^2 and displacements in H^1, as the mesh size tends to zero.

1. A SADDLE POINT PROBLEM

Let a $p > 2$ be chosen. Assume that Ω is a bounded domain with polygonal boundary and $\mathcal{T}_h$ is a triangulation of Ω. Denote by h the maximal side in $\mathcal{T}_h$.

We introduce the following function spaces on the domain Ω.

$$\begin{aligned} S &= \left\{ \{\tau_{ij}\}_{i,j=1,2} \mid \tau_{ij} \in L^2(\Omega),\ \tau_{12} = \tau_{21} \right\}, \\ Q(\mathcal{T}_h) &= \left\{ q \in S \mid q_{ij}|_T \in H^1(T)\ \forall T \in \mathcal{T}_h,\ i,j = 1,2, \right. \\ &\qquad \left. M_n(q) \text{ continuous at each interelement edge} \right\}, \end{aligned}$$

where

$$M_n(q) = q_{ij}\nu_i\nu_j,\ \nu_i \text{ are components of the unit normal to the edge.}$$

The norm in $Q(\mathcal{T}_h)$ is defined by

$$\|q\|_Q = \left(\sum_{T\in\mathcal{T}_h} \sum_{i,j=1,2} \|q_{ij}\|_{1,T}^2 \right)^{1/2}.$$

Moreover, we introduce

$$\begin{aligned} \mathcal{X} &= W_0^{1,p}(\Omega), \\ \Lambda &= \left\{ \mu \in \left[L^\infty(\Omega)\right]' \mid \langle \mu, v\rangle_\infty \geqslant 0 \quad \forall v \in L^\infty(\Omega), v \geqslant 0 \right\}, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_\infty$ denotes the bilinear form of the duality between $\left[L^\infty(\Omega)\right]'$ and $L^\infty(\Omega)$.

Assume that $f \in \mathcal{X}'$ and $\varphi \in L^\infty(\Omega)$ is given, such that $\varphi < 0$ in a neighborhood of the boundary $\partial\Omega$.

Let as define the following functional

$$\mathcal{H}(v,\mu) = \frac{1}{2}\int_\Omega e^3 C^0_{ijkm} v_{'ij} v_{'km} \mathrm{d}x - \langle f, Ev\rangle + \langle \mu, \varphi - I_0 v\rangle_\infty$$

for $v \in H_0^2(\Omega)$ and $\mu \in \Lambda$, where we denote by $\langle .\,,.\rangle$ the duality between Z' and Z,

$$E \text{ embedding } H_0^2(\Omega) \hookrightarrow \mathscr{Z} \text{ and}$$
$$I_0 \text{ embedding } H_0^2(\Omega) \hookrightarrow L^\infty(\Omega),$$

e —is the thickness of the plate (possibly variable),

C^0_{ijkm} are coefficients of the moment - curvature relations

$$q_{ij} = e^3 C^0_{ijkm} w_{'km},\ i,j = 1,2,$$

f is the transversal loading and φ the interior obstacle.

The repeated index implies summation within the range $1,2$; $v_{'ij} = \partial^2 v/\partial x_i \partial x_j$.

Assume that positive constants $e_{\min}$, $e_{\max}$, c_0 exist such that

$$e_{\max} \geqslant e(x) \geqslant e_{\min},\ C^0_{ijkm}\tau_{ij}\tau_{km} \geqslant c_0 \tau_{ij}\tau_{ij} \quad \forall\{\tau_{ij}\}_{i,j=1,2},\ \tau_{12} = \tau_{21},$$
$$C^0_{ijkm} = C^0_{jikm} = C^0_{kmij} \in L^\infty(\Omega).$$

We define the following bilinear forms

$$a(p,q) = \int_\Omega e^{-3} B_{ijkm} p_{ij} q_{km} \mathrm{d}x,\ p,q \in Q(\mathscr{T}_h),$$

where $B = (C^0)^{-1}$;

$$b(p,z) = \sum_{T\in\mathscr{T}_h} \left(\int_T p_{ij'j} z_{'i} \mathrm{d}x - \int_{\partial T} M_{nt}(p) \frac{\partial z}{\partial t} \mathrm{d}s \right), p \in Q(\mathscr{T}_h),\ z \in \mathscr{Z},$$

where

$M_{nt}(p) = p_{ij}\nu_i t_j$ and t_j are components of the unit tangential vector to ∂T.

The last integral can be interpreted as the duality pairing between the Sobolev spaces $H^{1/p}(\partial T)$ and $H^{-1/p}(\partial T)$ (cf. [2]). (Note that $z \in W^{1-1/p,p}(\partial T)$ for $z \in W^{1,p}(T)$.)

The form a is continuous on $[Q(\mathscr{T}_h)]^2$ and positive definite on S. The bilinear form b is continuous on $Q(\mathscr{T}_h) \times \mathscr{Z}$. Another property of b will be given later in Lemma 1.1.

Theorem 1.1. *There exists a unique solution w of the problem*

$$\text{(P)} \qquad w = \arg\min_{v\in K_0} \left\{ \frac{1}{2} \int_\Omega e^3 C^0_{ijkm} v_{'ij} v_{'km} - \langle f, Ev\rangle \right\}$$

where

$$K_0 = \{v \in H_0^2(\Omega) \mid I_0 v \geqslant \varphi\}.$$

There exists $\lambda^0 \in \Lambda$ such that $\{w, \lambda^0\}$ is a saddle point of $\mathscr{H}$ on $H_0^2(\Omega) \times \Lambda$ and

$$\langle \lambda^0, I_0 w - \varphi \rangle_\infty = 0, \tag{1}$$

$$I_0^* \lambda^0 = (e^3 C'^0_{ijkm} w_{'km})_{'ij} - f \tag{2}$$

holds, where I_0^* denotes the mapping adjoint to I_0, $I_0^* \colon [L^\infty(\Omega)]' \to H^{-2}(\Omega)$.

Proof. The first assertion follows from the fact that K_0 is a nonempty, convex and closed subset of $H_0^2(\Omega)$ and the functional is strictly convex, continuous and coercive on $H_0^2(\Omega)$. The second assertion is a consequence of Theorem 5.1 of [3], (p. 66). □

Corollary 1.1. *If we denote*

$$\bar{q}_{ij} = e^3 C'^0_{ijkm} w_{'km}, \; i, j = 1, 2,$$

then

$$\int_\Omega \bar{q}_{ij} v_{'ij} \mathrm{d}x = \langle f, Ev \rangle + \langle \lambda^0, I_0 v \rangle_\infty \quad \forall v \in H_0^2(\Omega) \tag{3}$$

holds.

Proof is a direct consequence of the following condition of the saddle point

$$\mathscr{H}(w, \lambda^0) \leqslant \mathscr{H}(v, \lambda^0) \quad \forall v \in H_0^2(\Omega).$$

□

Lemma 1.1. (cf. [2]). *For any $z \in \mathscr{Z}$ we have*

$$\sup_{p \in Q(\mathscr{T}_h)} \frac{b(p, z)}{\|p\|_Q} \geqslant C \|z\|_{1,\Omega}$$

with some positive constant C.

Proof. If we choose $p^0_{ij} = z\delta_{ij}$, then $p^0 \in Q(\mathscr{T}_h)$ (since $z \in C(\bar{\Omega}) \cap H_0^1(\Omega)$ by Sobolev embedding theorem) and we obtain

$$\frac{b(p^0, z)}{\|p^0\|_Q} = \frac{\|\nabla z\|^2_{0,\Omega}}{\|z\|_{1,\Omega}\sqrt{2}} \geqslant C \|z\|_{1,\Omega}$$

using the Friedrichs inequality.

Let us introduce the following subspace of $\mathscr{X} \times Q$.

$$W(\mathscr{T}_h) = \{[z, q] \in \mathscr{X} \times Q(\mathscr{T}_h) \mid a(q, p) + b(p, z) = 0 \quad \forall p \in Q(\mathscr{T}_h)\}$$

and denote by

$$I: \mathscr{X} \hookrightarrow L^\infty(\Omega), \quad I^*: [L^\infty(\Omega)]' \to \mathscr{X}'$$

the above mentioned embedding and its adjoint mapping. □

Theorem 1.2. *Assume that*

$$\{e^3 C^0_{ijkm} w_{'km}\}_{i,j=1,2} \in Q(\mathscr{T}_h), \tag{4}$$

(where w is the solution of the problem (P)).

Then there exists a saddle-point $\{[\bar{z}, \bar{q}], \bar{\lambda}\}$ of the following functional

$$\mathscr{L}([z, q], \lambda) = \frac{1}{2} a(q, q) - \langle f, z\rangle + \langle \lambda, \varphi - Iz\rangle_\infty$$

on $W(\mathscr{T}_h) \times \Lambda$, i.e.,

$$\mathscr{L}([\bar{z}, \bar{q}], \lambda) \leqslant \mathscr{L}([\bar{z}, \bar{q}], \bar{\lambda}) \leqslant \mathscr{L}([z, q], \bar{\lambda}) \tag{5}$$

holds for all $[z, q] \in W(\mathscr{T}_h)$ and $\lambda \in \Lambda$.

Moreover, we have

$$\bar{z} = Ew, \quad \bar{q}_{ij} = e^3 C^0_{ijkm} w_{'km}, \quad I^*\bar{\lambda} = (\bar{q}_{ij'ij}) - f, \tag{6}$$

where $(\bar{q}_{ij'ij}) \in \mathscr{X}'$ is the extension of the functional $\bar{q}_{ij'ij} \in H^{-2}$ by continuity, and the following optimality condition

$$\langle \bar{\lambda}, I\bar{z} - \varphi\rangle_\infty = 0 \tag{7}$$

holds for any saddle point. The first component $[\bar{z}, \bar{q}]$ is uniquely determined, whereas the second components of the saddle points may differ by elements $\mu \in [L^\infty(\Omega)]'$ such that $\bar{\lambda} + \mu \in \Lambda$ and

$$\langle \mu, \varphi - Iz\rangle_\infty = 0 \quad \forall z \in \mathscr{X}. \tag{8}$$

Proof. Obviously, $\bar{z} = Ew \in \mathscr{X}$ and $\bar{q} \in Q(\mathscr{T}_h)$ by assumption. Let us set $\bar{\lambda} = \lambda^0$, where λ^0 is the saddle-point component of $\mathscr{H}$ from Theorem 1.1. Using (6), (2) and the decomposition $I_0 = IE$, we may write

$$\langle I_0^* \lambda^0, v\rangle_{H_0^2} = \langle \lambda^0, IEv\rangle_\infty = \langle E^* I^* \lambda^0, v\rangle_{H_0^2} = \langle I^* \lambda^0, Ev\rangle_{\mathscr{X}}.$$

Hence

$$\langle I^*\lambda^0, Ez\rangle_{\mathscr{X}} = \langle \bar{q}_{ij'ij} - f, z\rangle_{H_0^2} \quad \forall z \in H_0^2(\Omega)$$

holds and

$$I^*\bar{\lambda} = I^*\lambda^0 = (\bar{q}_{ij'ij}) - f$$

follows.

Let us verify that $[\bar{z}, \bar{q}] \in W(\mathscr{T}_h)$. Indeed, for any $p \in Q(\mathscr{T}_h)$ we have

$$\begin{aligned} a(\bar{q}, p) + b(p, \bar{z}) &= \int_\Omega w_{'ij} p_{ij} \, \mathrm{d}x + \sum_T \left(\int_T p_{ij'j} w_{'i} \, \mathrm{d}x - \int_{\partial T} M_{nt}(p) \partial w / \partial t \, \mathrm{d}s \right) \\ &= \sum_T \int_{\partial T} (p_{ij} \nu_j w_{'i} - M_{nt}(p) \partial w / \partial t) \, \mathrm{d}s = 0, \end{aligned}$$

by virtue of the continuity of $M_n(p)$ on the interelement boundaries.

The inequalities (5) imply that a point $\{[\hat{z}, \hat{q}], \hat{\lambda}\} \in W(\mathscr{T}_h) \times \Lambda$ is a saddle point, if and only if the following two conditions are satisfied:

$$0 = D\mathscr{L}([\hat{z}, \hat{q}], \hat{\lambda}; [z, q]) \quad \forall [z, q] \in W(\mathscr{T}_h) \tag{9}$$

which is equivalent with

$$\begin{aligned} 0 &= a(\hat{q}, q) - \langle f, z\rangle + \langle \hat{\lambda}, -Iz\rangle_\infty \\ &= -b(\hat{q}, z) - \langle f + I^*\hat{\lambda}, z\rangle \quad \forall z \in \mathscr{X} \end{aligned}$$

and

$$0 \geqslant D\mathscr{L}([\hat{z}, \hat{q}], \hat{\lambda}; \lambda - \hat{\lambda}) = \langle \lambda - \hat{\lambda}, \varphi - I\hat{z}\rangle_\infty \quad \forall \lambda \in \Lambda. \tag{10}$$

Let us show that $[\bar{z}, \bar{q}]$ and $\bar{\lambda}$ defined by the formulas (6), satisfy (9) and (10). In fact, using Corollary 1.1, we obtain

$$-b(\bar{q}, v) = \langle f, Ev\rangle + \langle \bar{\lambda}, I_0 v\rangle_\infty \quad \forall v \in H_0^2(\Omega), \tag{11}$$

since

$$\int_\Omega \bar{q}_{ij} v_{'ij} \, \mathrm{d}x = -b(\bar{q}, v).$$

The set $EH_0^2(\Omega)$ is dense in the space $\mathscr{X}$ and the mapping $v \longmapsto b(\bar{q}, v)$ is continuous in $\mathscr{X}$ (cf. [1]). Consequently, the equation (11) yields that

$$b(\bar{q}, z) + \langle f, z\rangle + \langle \bar{\lambda}, Iz\rangle_\infty = 0 \quad \forall z \in \mathscr{X}.$$

Thus we obtain that the condition (9) is satisfied.

Since $I_0 w = IEw = I\bar{z}$, Theorem 1.1 implies that

$$\left\langle \bar{\lambda}, I\bar{z} - \varphi \right\rangle_\infty = 0,$$

i.e., (7) holds. As $I_0 w \geqslant \varphi$, we obtain $I\bar{z} \geqslant \varphi$ and

$$\left\langle \lambda - \bar{\lambda}, \varphi - I\bar{z} \right\rangle_\infty = \left\langle \lambda, \varphi - I\bar{z} \right\rangle_\infty \leqslant 0 \ \forall \lambda \in \Lambda$$

follows. Consequently, also (10) is satisfied.

Next let us prove the uniqueness of the first component $[\bar{z}, \bar{q}]$. Assume that $\{[\hat{z}, \hat{q}], \hat{\lambda}\}$ is another saddle point. Inserting $z := \bar{z} - \hat{z}$, $q := \bar{q} - \hat{q}$ into the condition (9), we obtain

$$a(\hat{q}, \bar{q} - \hat{q}) - \langle f, \bar{z} - \hat{z} \rangle + \left\langle \hat{\lambda}, -I(\bar{z} - \hat{z}) \right\rangle_\infty = 0.$$

Changing the role of the two saddle points, we arrive at

$$a(\bar{q}, \hat{q} - \bar{q}) - \langle f, \hat{z} - \bar{z} \rangle + \left\langle \bar{\lambda}, -I(\hat{z} - \bar{z}) \right\rangle_\infty = 0.$$

By addition we obtain that

$$a(\hat{q} - \bar{q}, \bar{q} - \hat{q}) + \left\langle \hat{\lambda} - \bar{\lambda}, I\hat{z} - I\bar{z} \right\rangle_\infty = 0. \tag{12}$$

From the condition (10) we derive that

$$\left\langle \bar{\lambda} - \hat{\lambda}, \varphi - I\hat{z} \right\rangle_\infty \leqslant 0 \tag{13}$$

and changing the role of the saddle points we obtain

$$\left\langle \hat{\lambda} - \bar{\lambda}, \varphi - I\bar{z} \right\rangle_\infty \leqslant 0, \tag{14}$$

so that

$$\left\langle \hat{\lambda} - \bar{\lambda}, I(\hat{z} - \bar{z}) \right\rangle_\infty \leqslant 0, \tag{15}$$

follows. Inserting (15) into (12) and using also positive definiteness of a, we arrive at

$$a_0 \|\hat{q} - \bar{q}\|_S^2 \leqslant a(\hat{q} - \bar{q}, \hat{q} - \bar{q}) \leqslant 0.$$

Consequently, $\hat{q} = \bar{q}$ a.e. in Ω.

Since $[\hat{z}-\bar{z},\hat{q}-\bar{q}] \in W$, we have

$$0 = a(0,p) + b(p,\hat{z}-\bar{z}) \quad \forall p \in Q.$$

From Lemma 1.1 we conclude that

$$\|\hat{z}-\bar{z}\|_{1,\Omega} = 0.$$

Consequently, the component $[\bar{z},\bar{q}]$ is unique. Combining the inequalities (13) and (14), we obtain

$$(16) \qquad \left\langle \bar{\lambda}-\hat{\lambda}, \varphi - I\bar{z}\right\rangle_{\infty} = 0,$$

i.e., (7) is a necessary condition for any saddle point. Let us denote $\hat{\lambda} = \bar{\lambda} + \mu$. Let us show that (8) is a necessary condition for the second component of the saddle point. Indeed, since (9) is necessary condition and $\bar{\lambda}$ satisfies it, we have

$$(17) \qquad 0 = b(\bar{q},z) + \langle f,z\rangle + \left\langle \bar{\lambda}+\mu, Iz\right\rangle_{\infty} = \langle \mu, Iz\rangle_{\infty} \ \forall z \in \mathscr{Z}.$$

From (16) we conclude that

$$0 = \langle \mu, \varphi - I\bar{z}\rangle_{\infty} = \langle \mu, \varphi\rangle_{\infty}$$

and (8) follows.

On the other hand, if (8) holds and $\bar{\lambda}+\mu \in \Lambda$, then

$$0 = \langle \mu, \varphi\rangle_{\infty} = \langle \mu, Iz\rangle_{\infty} \quad \forall z \in \mathscr{Z}$$

so that (17) and (9) is fulfilled. Moreover,

$$\left\langle \lambda - (\bar{\lambda}+\mu), \varphi - I\bar{z}\right\rangle_{\infty} = \left\langle \lambda - \bar{\lambda}, \varphi - I\bar{z}\right\rangle_{\infty} \leqslant 0 \quad \forall \lambda \in \Lambda$$

holds, which implies that (10) is fulfilled. Consequently, $\bar{\lambda}+\mu \in \Lambda$ and (8) is sufficient for $\bar{\lambda}+\mu$ to be a second component. □

Remark 1.1. Note that the set $I\mathscr{Z}$ is not dense in $L^{\infty}(\Omega)$, due to the zero boundary values of any $z \in \mathscr{Z}$. Hence (8) does not imply $\mu = 0$.

2. Approximate saddle point problem

Let us introduce the following approximate spaces of finite elements:

$$\begin{aligned} \mathscr{X}_h &= \{ z_h \in \mathscr{X} \mid z_h|_T \in P_1(T) \quad \forall T \in \mathscr{T}_h \} \\ Q_h &= \{ q_h \in Q(\mathscr{T}_h) \mid q_h|_T \in [P_0(T)]^4 \quad \forall T \in \mathscr{T}_h \}, \\ W_h &= \{ [z_h, q_h] \in \mathscr{X}_h \times Q_h \mid a(q_h, p_h) + b(p_h, z_h) = 0 \quad \forall p_h \in Q_h \}, \\ K_h &= \{ [z_h, q_h] \in W_h \mid z_h(P) \geqslant \varphi(P) \quad \forall P \in \Sigma_h^0 \}, \end{aligned}$$

where Σ_h^0 denotes the set of all vertices of $\mathscr{T}_h$ inside Ω. Here we assume that φ is defined everywhere in Ω.

Next we introduce the following function

$$J_0([z_h, q_h]) = \frac{1}{2} a(q_h, q_h) - \langle f, z_h \rangle \quad \text{on } \mathscr{X}_h \times Q_h$$

and an *auxiliary problem*

$$(\mathscr{P}_h) \qquad [\bar{z}_h, \bar{q}_h] = \operatorname*{arg\,min}_{[z_h, q_h] \in K_h} J_0([z_h, q_h]).$$

In order to prove the unique solvability of $(\mathscr{P}_h)$ we need the following

Lemma 2.1. *There exists a linear continuous mapping*

$$G_h : \mathscr{X}_h \to S$$

such that

$$(2.1) \qquad q_h = G_h z_h \iff [z_h, q_h] \in W_h,$$

$$(2.2) \qquad C\|[z_h, q_h]\|_{\mathscr{X} \times Q} \leqslant \|z_h\|_{\mathscr{X}} \leqslant \tilde{C} h^{-\alpha} \|G_h z_h\|_S, \quad \alpha = \frac{p-2}{p},$$

with some positive constants C, $\tilde{C}$, independent of h.

Proof. Since $q_h \in Q_h$ is piecewise constant, we may write

$$\begin{aligned} a(q_h, q_h) &\geqslant a_0 \|q_h\|_S^2 \quad \forall q_h \in Q_h, \\ |b(p_h, z_h)| &\leqslant C \|p_h\|_Q \|z_h\|_{\mathscr{X}} = C \|p_h\|_S \|z_h\|_{\mathscr{X}}. \end{aligned}$$

Then for any given $z_h \in \mathscr{X}_h$ there exists a unique element $q_h \in Q_h$, such that

$$a(q_h, p_h) = -b(p_h, z_h) \quad \forall p_h \in Q_h.$$

If we denote $q_h = G_h z_h$, G_h is a linear mapping from $\mathscr{Z}_h$ in S and

$$\|G_h z_h\|_S \leqslant a_0^{-1} C \|z_h\|_{\mathscr{Z}} \tag{2.3}$$

holds. Consequently, we have

$$[z_h, q_h] \in W_h \Longleftrightarrow z_h \in \mathscr{Z}_h, q_h = G_h z_h. \tag{2.4}$$

Choosing $p_h = \Pi_h p^0$, where $p^0_{ij} = z_h \delta_{ij}$ and Π_h is the mapping from Lemma 4 of [2], the following result can be proven (see also the proof of Lemma 1.1):

$$\sup_{p_h \in Q_h} \big(b(p_h, z_h)/\|p_h\|_Q\big) \geqslant \beta \|z_h\|_{1,\Omega} \tag{2.5}$$

holds for any $z_h \in \mathscr{Z}_h$ with some positive constant β.

If $[z_h, q_h] \in W_h$, we may write

$$\begin{aligned} \beta\|z_h\|_{\mathscr{Z}} &\leqslant h^{-\alpha} C\beta \|z_h\|_{1,\Omega} \leqslant C h^{-\alpha} \sup_{p_h \in Q_h} \big(- a(G_h z_h, p_h)/\|p_h\|_Q\big) \\ &\leqslant \tilde{C} h^{-\alpha} \|G_h z_h\|_S, \end{aligned} \tag{2.6}$$

since the interpolation theory yields

$$\|z_h\|_{1,p} \leqslant C h^{-\alpha} \|z_h\|_{1,2} \quad \forall z_h \in \mathscr{Z}_h$$

and

$$|a(p,q)| \leqslant C \|p\|_S \|q\|_Q \quad \forall p \in S,\ q \in Q.$$

Using (2.4), (2.3) and (2.6), we obtain

$$\|G_h z_h\|_Q = \|G_h z_h\|_S \leqslant \tilde{C} \|z_h\|_Z \leqslant \hat{C} h^{-\alpha} \|G_h z_h\|_S,$$

$$\|[z_h, q_h\|_{\mathscr{Z}\times Q} \leqslant \|z_h\|_{\mathscr{Z}} + \|G_h z_h\|_Q \leqslant (1 + \tilde{C})\|z_h\|_{\mathscr{Z}} \leqslant C h^{-\alpha} \|G_h z_h\|_S.$$

□

Theorem 2.1. *The problem ($\mathscr{P}_h$) has a unique solution.*

Proof. Obviously, the set K_h is convex and closed in $\mathscr{Z}_h \times Q_h$. The function J_0 is continuous on W_h. From Lemma 2.1 we easily derive that J_0 is coercive on W_h. Indeed, we have

$$\begin{aligned} J_0([z_h, q_h]) &= \frac{1}{2} a(G_h z_h, G_h z_h) - \langle f, z_h \rangle \\ &\geqslant \frac{1}{2} a_0 \|G_h z_h\|_S^2 - C\|z_h\|_{\mathscr{Z}} \geqslant \hat{C}(h) \|[z_h, q_h\|^2_{\mathscr{Z}\times Q} - \hat{C}(h) \|[z_h, q_h]\|_{\mathscr{Z}\times Q}. \end{aligned}$$

Consequently, a minimizer $[\bar{z}_h, \bar{q}_h]$ of J_0 exists on K_h.

Next let $[\hat{z}_h, \hat{q}_h]$ be another solution of $(\mathscr{P}_h)$. Then we may write (dropping the subscripts "h")

$$a(\bar{q}, \hat{q} - \bar{q}) - \langle f, \hat{z} - \bar{z}\rangle \geqslant 0$$

$$a(\hat{q}, \bar{q} - \hat{q}) - \langle f, \bar{z} - \hat{z}\rangle \geqslant 0.$$

By addition we obtain

$$a(\bar{q} - \hat{q}, \hat{q} - \bar{q}) \geqslant 0,$$

so that

$$\|G(\hat{z} - \bar{z})\|_S^2 \leqslant 0$$

and

$$\hat{z}_h - \bar{z}_h = 0,\ \hat{q}_h - \bar{q}_h = G_h(\hat{z}_h - \bar{z}_h) = 0$$

follows from Lemma 2.1. □

Remark 2.1. The uniqueness is also a consequence of the convexity of K_h and strict convexity of J_0 on W_h.

Let us define

$$\Lambda_h = \{\lambda_h \in \mathscr{Z}_h \mid \lambda_h \geqslant 0 \text{ in } \Omega\}$$

and

$$\langle X, Y\rangle_h = \frac{1}{3}\sum_{P\in\Sigma_h^0} A(P)X(P)Y(P) \tag{2.7}$$

for all functions X, $Y : \Sigma_h^0 \to \mathbf{R}^{m_h}$, (where m_h denotes the number of vertices in Σ_h^0) with $A(P)$ denoting the sum of the areas of the triangles in $\mathscr{T}_h$, which admit P as common vertex.

By the bilinear form (2.7) a scalar product in $\mathbf{R}^{m_h}$ is defined. The convex cone Λ_h is isomorphic with the cone $\mathbf{R}_+^{m_h}$ of the vectors with non-negative coordinates, if the nodal values $\lambda_h(P)$, $P \in \Sigma_h^0$ are taken into consideration.

Let us introduce the following Lagrangian

$$\mathscr{L}_h([z_h, q_h], \lambda_h) = \frac{1}{2}\, a(q_h, q_h) - \langle f, z_h\rangle + \langle \lambda_h, \varphi - z_h\rangle_h$$

for $[z_h, q_h] \in W_h$ and $\lambda_h \in \mathbf{R}_+^{m_h}$.

Theorem 2.2. *There exists a unique saddle point $\{[\bar{z}_h, \bar{q}_h], \bar{\lambda}_h\}$ of $\mathscr{L}_h$ on $W_h \times \mathbf{R}_+^{m_h}$. The first component $[\bar{z}_h, \bar{q}_h]$ coincides with the solution of the problem $(\mathscr{P}_h)$.*

Moreover, the following "optimality condition" holds

$$\langle \bar{\lambda}_h, \varphi - \bar{z}_h \rangle_h = 0. \tag{2.8}$$

Proof. The existence of a saddle point $\{[\bar{z}_h, \bar{q}_h], \bar{\lambda}_h\}$, the first component of which is the solution of $(\mathscr{P}_h)$, can be proved on the basis of Theorem 5.1 of [3] (p. 66), if we use Theorem 2.1.

Thus it remains to verify the uniqueness.

For brevity, let us drop the subscripts "h" in what follows. Let $\{[\hat{z}, \hat{q}], \hat{\lambda}\}$ be another saddle point. Arguing in the same way as in the proof of Theorem 1.2 (see (12)–(15)), we obtain

$$a(\hat{q} - \bar{q}, \hat{q} - \bar{q}) = \langle \bar{\lambda} - \hat{\lambda}, \bar{z} - \hat{z} \rangle_h$$

and

$$\langle \bar{\lambda} - \hat{\lambda}, \bar{z} - \hat{z} \rangle_h \leqslant 0.$$

Making use of Lemma 2.1, we may write

$$0 \geqslant a(\hat{q} - \bar{q}, \hat{q} - \bar{q}) \geqslant a_0 \|G(\hat{z} - \bar{z})\|_S^2 \geqslant C \|\hat{z} - \bar{z}\|_{\mathscr{Z}}^2.$$

Consequently,

$$\hat{z} - \bar{z} = 0, \ \hat{q} - \bar{q} = 0.$$

Let us denote $\hat{\lambda} = \bar{\lambda} + \mu$. Since

$$\mathscr{L}_h([\bar{z}, \bar{q}], \bar{\lambda} + \mu) \leqslant \mathscr{L}_h([z, q], \bar{\lambda} + \mu) \quad \forall [z, q] \in W_h,$$

we have

$$0 = a(\bar{q}, q) - \langle f, z \rangle - \langle \bar{\lambda} + \mu, z \rangle_h = -\langle \mu, z \rangle_h \tag{2.8'}$$

for all $z \in \mathbf{R}^{m_h}$. (Note that $\mathscr{Z}_h$ is isomorphic with $\mathbf{R}^{m_h}$.) As $\mu \in \mathbf{R}^{m_h}$, $\mu = 0$ follows. □

To find the saddle point we employ the following *algorithm of Uzawa's type*:

Let us choose $\lambda_h^0 \equiv 0$, $\varrho \in \mathbf{R}$.

If λ_h^n is known, $n = 0, 1, 2, \ldots$, we calculate $[z_h^n, q_h^n] \in W_h$ and $\lambda_h^{n+1} \in \mathbf{R}_+^{m_h}$ as follows:

$$a(q_h^n, p_h) + b(p_h, z_h^n) = 0 \quad \forall p_h \in Q_h \tag{2.9}$$

$$(2.10) \qquad b(q_h^n, z_h) = -\langle f, z_h\rangle - \langle \lambda_h^n, z_h\rangle_h \quad \forall z_h \in \mathscr{Z}_h,$$

$$(2.11) \qquad \lambda_h^{n+1}(P) = \left[\lambda_h^n(P) + \varrho\big(\varphi(P) - z_h^n(P)\big)\right]^+ \ \forall P \in \Sigma_h^0.$$

Theorem 2.3. *For any $n = 0, 1, \ldots$, there exists a unique element $[z_h^n, q_h^n]$, satisfying* (2.9), (2.10).

If ϱ is positive and sufficiently small, then

$$\lim_{n\to\infty} \left(\|q_h^n - \bar{q}_h\|_Q + \|z_h^n - \bar{z}_h\|_Z + \|\lambda_h^n - \bar{\lambda}_h\|_h\right) = 0,$$

where $\{[\bar{z}_h, \bar{q}_h], \bar{\lambda}_h\}$ is the saddle point of $\mathscr{L}_h$ on $W_h \times \mathbf{R}_+^{m_h}$.

Proof. For brevity, we drop the subscripts "h" in what follows. The conditions (2.9), (2.10) are equivalent with $[z^n, q^n] \in W_h$ and

$$(2.12) \qquad a(q^n, p) - \langle f, z\rangle - \langle \lambda^n, z\rangle_h = 0 \quad \forall [z, p] \in W_h.$$

The both conditions are satisfied, if

$$(2.13) \qquad [z^n, q^n] = \arg \min_{[z,q]\in W_h} \left\{\frac{1}{2}a(q, q) - \langle f, z\rangle - \langle \lambda^n, z\rangle_h\right\} = \arg \min_{W_h} \mathscr{L}_h\big([z, q], \lambda^n\big).$$

We can easily show that the problem (2.13) has a unique solution. In fact, W_h is a closed subspace of $\mathscr{Z}_h \times Q_h$, the function $\mathscr{L}_h(., \lambda^n)$ is continuous and coercive on W_h (see the proof of Theorem 2.1). Since the latter function is also strictly convex on W_h, (2.13) is uniquely solvable.

If we insert $p := \bar{q} - q^n$, $z := \bar{z} - z^n$ into (2.12), we obtain

$$a(q^n, \bar{q} - q^n) - \langle f, \bar{z} - z^n\rangle - \langle \lambda^n, \bar{z} - z^n\rangle_h = 0.$$

For the saddle point $\{[\bar{z}, \bar{q}], \bar{\lambda}\}$ of $\mathscr{L}_h$ (see the proof of Theorem 2.2) we can derive that

$$a(\bar{q}, q^n - \bar{q}) - \langle f, z^n - \bar{z}\rangle - \langle \bar{\lambda}, z^n - \bar{z}\rangle_h = 0.$$

Adding the two conditions we are led to the equation

$$(2.14) \qquad a(q^n - \bar{q}, q^n - \bar{q}) + \langle \lambda^n - \bar{\lambda}, \bar{z} - z^n\rangle_h = 0.$$

The second inequality, characterizing the saddle point of $\mathscr{L}_h$ yields that

$$(2.15) \qquad \langle \lambda - \bar{\lambda}, \varphi - \bar{z}\rangle_h \leqslant 0 \quad \forall \lambda \in \mathbf{R}_+^{m_h}.$$

Introducing the projection $P_\Lambda : \mathbf{R}^{m_h} \to \mathbf{R}_+^{m_h}$ with respect to the scalar product (2.7), we may write

$$\bar{\lambda} = P_\Lambda(\bar{\lambda} + \varrho(\varphi - \bar{z})), \quad \varrho > 0 \tag{2.16}$$

on the basis of (2.15). Moreover, we have

$$[P_\Lambda \lambda](P) = [\lambda(P)]^+ \quad \forall P \in \Sigma_h^0, \quad \forall \lambda \in \mathbf{R}^{m_h}. \tag{2.17}$$

If we denote $r^n = \lambda^n - \bar{\lambda} \in \mathbf{R}^{m_h}$, then using (2.11), (2.16) and (2.14) we obtain

$$\begin{aligned} \|r^{n+1}\|_h^2 &= \|\lambda^{n+1} - \bar{\lambda}\|_h^2 = \|P_\Lambda(\lambda^n + \varrho(\varphi - z^n)) - P_\Lambda(\bar{\lambda} + \varrho(\varphi - \bar{z}))\|_h^2 \\ &\leqslant \|\lambda^n - \bar{\lambda} + \varrho(\bar{z} - z^n)\|_h^2 = \|r^n\|_h^2 + \varrho^2\|\bar{z} - z^n\|_h^2 + 2\varrho\langle r^n, \bar{z} - z^n\rangle_h \\ &\leqslant \|r^n\|_h^2 + \varrho^2\|z^n - \bar{z}\|_h^2 - 2\varrho a(q^n - \bar{q}, q^n - \bar{q}). \end{aligned}$$

From Lemma 2.1 we derive the following estimate

$$a(q^n - \bar{q}, q^n - \bar{q}) \geqslant a_0\|G(z^n - \bar{z})\|_S^2 \geqslant C\|z^n - \bar{z}\|_Z^2 \geqslant \hat{C}\|z^n - \bar{z}\|_h^2.$$

Hence we may write

$$\|r^{n+1}\|_h^2 \leqslant \|r^n\|_h^2 + (\varrho^2 - 2\varrho\hat{C})\|z^n - \bar{z}\|_h^2.$$

If $\varrho \in (0, 2\hat{C})$, the coefficient $(\varrho^2 - 2\varrho\hat{C})$ is negative, the sequence $\{\|r^n\|_h\}$ is non-increasing and therefore converging. Consequently,

$$\lim_{n\to\infty} \|z^n - \bar{z}\|_h = 0.$$

Using again Lemma 2.1 and the well-known equivalence of norms, we arrive at

$$\begin{aligned} \|q^n - \bar{q}\|_Q = \|G(z^n - \bar{z})\|_Q \leqslant C\|G(z^n - \bar{z})\|_S \leqslant \tilde{C}\|z^n - \bar{z}\|_Z \\ \leqslant \hat{C}\|z^n - \bar{z}\|_h \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.18}$$

Since the sequence $\{\|r^n\|_h\}_{n=1}^\infty$ is non-increasing, the sequence $\{\lambda^n\}$ is bounded in $\mathbf{R}^{m_h}$. There exists a vector $\hat{\lambda} \in \mathbf{R}_+^{m_h}$ and a subsequence $\{\lambda^{n_k}\}_{k=1}^\infty$, such that

$$\lambda^{n_k} \to \hat{\lambda} \text{ as } k \to \infty.$$

From (2.10) we get

$$b(q^{n_k}, z) = -\langle f, z\rangle - \langle \lambda^{n_k}, z\rangle_h \quad \forall z \in \mathscr{Z}_h.$$

Passing to the limit with $k \to \infty$ and using the continuity of the form b (cf. [1]) together with (2.18), we arrive at

$$b(\bar{q}, z) = -\langle f, z\rangle - \langle \hat{\lambda}, z\rangle_h \quad \forall z \in \mathscr{Z}_h.$$

Since the saddle point of $\mathscr{L}_h$ fulfils an analogous condition, namely

$$b(\bar{q}, z) = -\langle f, z\rangle - \langle \bar{\lambda}, z\rangle_h \quad \forall z \in \mathscr{Z}_h.$$

(cf. (2.8′)) we obtain

$$\langle \hat{\lambda} - \bar{\lambda}, z\rangle_h = 0 \quad \forall z \in \mathbf{R}^{m_h}$$

and therefore

$$\hat{\lambda} - \bar{\lambda} = 0.$$

The uniqueness of $\bar{\lambda}$ (see Theorem 2.2) implies that the whole sequence $\{\lambda^n\}$ converges to $\bar{\lambda}$. □

3. Some convergence analysis

A natural question arises, whether the approximate saddle points $\{[\bar{z}_h, \bar{q}_h], \bar{\lambda}_h\}$ tend to the "exact" one, if h tends to zero.

In the present section we give a partial answer to the latter question. Namely, we prove an a priori error estimate for $\bar{q} - \bar{q}_h$ in $[L^2(\Omega)]^4$ and $\bar{z} - \bar{z}_h$ in H^1, provided the obstacle is represented by a continuous piecewise linear function. We employ some ideas of Cornodi [7].

Let us introduce the standard space of linear finite elements

$$Y_h = \{z_h \in C(\bar{\Omega}) \mid z_h|_T \in P_1(T) \quad \forall T \in \mathscr{T}_h\}$$

If the following, we consider a regular family $\{\mathscr{T}_h\}$, $h \to 0_+$, of triangulations, which *refine* an initial triangulation $\mathscr{T}_{h_0}$.

First we introduce two auxiliary lemmas.

Lemma 3.1. *Assume that $\varphi \in C(\bar{\Omega})$ and $\varphi < 0$ on the boundary $\partial\Omega$. Given any function $z \in \mathscr{Z}$ such that $z \geqslant \varphi$ in Ω, there exists a sequence $\{v_n\}$ such that $v_n \in C_0^\infty(\Omega)$, $v_n \geqslant \varphi$ in Ω and*

$$\|v_n - z\|_{1,p} \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Proof. Let us consider a sequence $\{z_n\}$, $z_n \in C_0^\infty(\Omega)$,

$$\|z_n - z\|_{1,p} \to 0 \quad \text{as } n \to 0. \tag{3.2}$$

If

$$\min_{x\in\bar{\Omega}} \big(z(x) - \varphi(x)\big) > 0,$$

then the assertion follows for $v_n = z_n$. Henceforth, let us assume that

$$\min_{\bar{\Omega}}(z - \varphi) = 0.$$

There exists a subset $G \subset \Omega$ and a positive integer n_0 such that $\overline{G} \subset \Omega$ and

$$\varphi(x) > z_n(x) \implies x \in G \quad \forall n \geqslant n_0. \tag{3.3}$$

In fact, introducing

$$\bar{\varphi} = \frac{1}{2}\,\big|\max_{s\in\partial\Omega} \varphi(s)\big|,$$

we can find n_0 such that for $n \geqslant n_0$ and for all $x \in \Omega$

$$z_n(x) \geqslant z(x) - \bar{\varphi}.$$

Let us define

$$\Omega^* = \big\{x \in \Omega \mid z(x) - \bar{\varphi} - \varphi(x) \geqslant 0\big\},$$
$$G = \Omega \mathbin{\dot{-}} \Omega^*.$$

If $\varphi(x) > z_n(x)$, $n \geqslant n_0$, then

$$\varphi(x) > z(x) - \bar{\varphi}$$

so that $x \in G$, which yields (3.3).

Since $\bar{\varphi} > 0$ and z vanishes on the boundary, $\overline{G} \subset \Omega$ holds.

Next let us introduce the following number

$$c_n = \max\Big\{0,\ \max_{y\in\bar{\Omega}} \big(z(y) - z_n(y)\big)\Big\}.$$

There exists a function $\psi \in C_0^\infty(\Omega)$ such that

$$\psi \geqslant 0 \text{ in } \Omega \text{ and } \psi = 1 \text{ for all } x \in G.$$

We define

$$v_n = z_n + c_n \psi$$

and show that $v_n \geqslant \varphi$ in Ω. To this end, first let us consider any point $x \in \Omega$ such that $\varphi(x) > z_n(x)$. Using (3.3), we may write $x \in G$, $\psi(x) = 1$,

$$v_n(x) = z_n(x) + c_n \geqslant z_n(x) + \max_{y \in \bar{G}} (z(y) - z_n(y)) \geqslant z(x) \geqslant \varphi(x).$$

Second, let $x \in \Omega$ be any point such that $\varphi(x) \leqslant z_n(x)$. Then we have

$$v_n(x) \geqslant z_n(x) \geqslant \varphi(x)$$

by definition of c_n and ψ.

Finally, we have

$$|c_n| \leqslant \|z_n - z\|_{0,\infty} \longrightarrow 0$$

and therefore

$$\|v_n - z\|_{1,p} \leqslant \|z_n - z\|_{1,p} + |c_n| \, \|\psi\|_{1,p} \longrightarrow 0 \qquad \text{as } n \longrightarrow 0.$$

□

Lemma 3.2. *The saddle point component* $[\bar{z}, \bar{q}]$ *from Theorem* 1.2 *satisfies the following conditions*

$$a(\bar{q}, p) + b(p, \bar{z}) = 0 \quad \forall p \in Q(\mathscr{T}_h), \tag{3.4}$$

$$-b(\bar{q}, z - \bar{z}) \geqslant \langle f, z - \bar{z} \rangle \quad \forall z \in \mathscr{Z} \text{ s.t. } z \geqslant \varphi \text{ in } \Omega. \tag{3.5}$$

P r o o f. Since $[\bar{z}, \bar{q}] \in W(\mathscr{T}_h)$, (3.4) is fulfilled. To verify (3.5), we first realize that $\bar{z} = Ew$ (cf. Theorem 1.2) and

$$\langle \bar{q}_{ij}, (v - w)_{'ij} \rangle_0 \geqslant \langle f, v - \bar{z} \rangle \quad \forall v \in K_0$$

follows from Theorem 1.1 and the definition of the problem ($\mathscr{P}$). Here we denote by $\langle u, v \rangle_0$ the integral $\int_\Omega uv \, dx$.

Using the assumption $\bar{q} \in Q(\mathscr{T}_h)$, we easily derive that

$$\langle \bar{q}_{ij}, (v - w)_{'ij} \rangle_0 = -b(\bar{q}, Ev - Ew).$$

Inserting into the previous inequality, we obtain

$$-b(\bar{q}, Ev - Ew) \geqslant \langle f, Ev - Ew \rangle \quad \forall v \in K_0. \tag{3.6}$$

Next we consider any function $z \in \mathscr{Z}$ such that $z \geqslant \varphi$ in Ω and apply Lemma 3.1. We may insert $v := v_n \in K_0$ into (3.6) and pass to the limit with $n \longrightarrow \infty$. Using (3.1) and the continuity of b with respect to the second argument, we arrive at the condition (3.5). □

Theorem 3.1. *Assume that $\varphi \in Y_{h_0}$ and*

$$\bar{q} = \{e^3 C^0_{ijkm} w'_{km}\}_{i,j=1,2} \in Q(\mathscr{T}_{h_0}) \text{ for some } \mathscr{T}_{h_0}.$$

Then

$$\|\bar{z} - \bar{z}_h\|_{1,\Omega} + \|\bar{q} - \bar{q}_h\|_S \leqslant C(\bar{z}, \bar{q}, \bar{\lambda}) h^{1/2}. \tag{3.7}$$

holds for all $h < h_0$ with some constant $C(\bar{z}, \bar{q}, \bar{\lambda})$ independent of h.

Proof. Let us realize that

$$\bar{q} \in Q(\mathscr{T}_h)$$

holds for all triangulations $\mathscr{T}_h$, refining $\mathscr{T}_{h_0}$.

Using the mapping Π_h from Lemma 4 of [2] and the definitions of $\bar{q}$, $\bar{q}_h$, $W(\mathscr{T}_h)$, we may write

$$\begin{aligned} a_0 \|\bar{q} - \bar{q}_h\|_S^2 &\leqslant a(\bar{q} - \bar{q}_h, \bar{q} - \bar{q}_h) \\ &= a(\bar{q} - \bar{q}_h, \bar{q} - \Pi_h \bar{q}) - b(\Pi_h \bar{q} - \bar{q}_h, \bar{z} - \bar{z}_h). \end{aligned} \tag{3.9}$$

On the other hand, for the linear interpolate $I_h \bar{z} \in \mathscr{Z}_h$ we have

$$\begin{aligned} b(\Pi_h \bar{q} - \bar{q}_h, \bar{z} - \bar{z}_h) &= b(\Pi_h \bar{q} - \bar{q}, \bar{z} - \bar{z}_h) + b(\bar{q} - \bar{q}_h, \bar{z} - \bar{z}_h) \\ &= b(\Pi_h \bar{q} - \bar{q}, \bar{z} - I_h \bar{z}) \\ &\quad + b(\Pi_h \bar{q} - \bar{q}, I_h \bar{z} - \bar{z}_h) + b(\bar{q} - \bar{q}_h, \bar{z} - \bar{z}_h) \\ &= -b(\bar{q}, \bar{z} - I_h \bar{z}) + b(\bar{q} - \bar{q}_h, \bar{z} - \bar{z}_h), \end{aligned} \tag{3.10}$$

since

$$b(p_h, \bar{z} - I_h \bar{z}) = 0 \quad \forall p_h \in Q_h, \tag{3.11}$$

(see [7], Lemma 5.3) and

$$b(p - \Pi_h p, z_h) = 0 \quad \forall p \in Q(\mathscr{T}_h), \; z_h \in \mathscr{Z}_h. \tag{3.12}$$

(see [2], Lemma 4).

Using (3.5) and (3.11), we obtain

$$\begin{aligned} -b(\bar{q} - \bar{q}_h, \bar{z} - \bar{z}_h) &= -b(\bar{q}, \bar{z} - \bar{z}_h) + b(\bar{q}_h, \bar{z} - I_h \bar{z}) \\ &\quad + b(\bar{q}_h, I_h \bar{z} - \bar{z}_h) \\ &\leqslant \langle f, \bar{z} - \bar{z}_h \rangle - \langle f, I_h \bar{z} - \bar{z}_h \rangle \\ &= \langle f, \bar{z} - I_h \bar{z} \rangle. \end{aligned} \tag{3.13}$$

In fact,

$$\bar{z}_h \in \mathscr{Z}_h \subset \mathscr{Z} \quad \text{and} \quad \bar{z}_h \geqslant \varphi \text{ in } \Omega$$

by virtue of the assumption $\varphi \in Y_{h_0}$; moreover,

$$-b(\bar{q}_h, z_h - \bar{z}_h) \geqslant \langle f, z_h - \bar{z}_h \rangle$$

for all $z_h \in \mathscr{Z}_h$ such that $z_h(P) \geqslant \varphi(P)$ at the vertices $P \in \Sigma_h^0$, follows from the definition of the problem $(\mathscr{P}_h)$.

Combining (3.9), (3.10), (3.13), we arive the inequality

$$a_0\|\bar{q} - \bar{q}_h\|_S^2 \leqslant a(\bar{q} - \bar{q}_h, \bar{q} - \Pi_h \bar{q}) + b(\bar{q}, \bar{z} - I_h \bar{z}) + \langle f, \bar{z} - I_h \bar{z} \rangle \,. \tag{3.14}$$

In the proof of Theorem 1.2 we had the following relation (cf. (9))

$$b(\bar{q}, z) + \langle f, z \rangle = -\left\langle I^* \bar{\lambda}, z \right\rangle = -\left\langle \bar{\lambda}, I(z) \right\rangle_\infty \quad \forall z \in \mathscr{Z}.$$

Consequently, using also the interpolation theory, we obtain

$$b(\bar{q}, \bar{z} - I_h \bar{z}) + \langle f, \bar{z} - I_h \bar{z} \rangle \leqslant \hat{C}(\bar{\lambda})\|\bar{z} - I_h \bar{z}\|_{0,\infty} \leqslant \hat{C}(\bar{\lambda})\tilde{C}h|\bar{z}|_{2,2}. \tag{3.15}$$

A slight modification of Lemma 4 in [2] yields

$$\|\bar{q} - \Pi_h \bar{q}\|_S \leqslant Ch\|\bar{q}\|_{Q(\mathscr{T}_h)}.$$

Thus we may write

$$\begin{aligned} a(\bar{q} - \bar{q}_h, \bar{q} - \Pi_h \bar{q}) &\leqslant \frac{1}{2}a_0\|\bar{q} - \bar{q}_h\|_S^2 + C\|\bar{q} - \Pi_h \bar{q}\|_S^2 \\ &\leqslant \frac{1}{2}a_0\|\bar{q} - \bar{q}_h\|_S^2 + C_1 h^2\|\bar{q}\|_Q^2. \end{aligned} \tag{3.16}$$

Then (3.14), (3.15) and (3.16) imply

$$\frac{1}{2}a_0\|\bar{q} - \bar{q}_h\|_S^2 \leqslant C_1 h^2\|\bar{q}\|_Q^2 + C(\bar{\lambda})h|\bar{z}|_{2,2} \leqslant C(\bar{z}, \bar{q}, \bar{\lambda})h \tag{3.17}$$

For any $p_h \in Q(\mathscr{T}_h)$, we may write

$$a(\bar{q}_h - \bar{q}, p_h) = -b(p_h, \bar{z}_h - \bar{z}) = -b(p_h, \bar{z} - I_h \bar{z})$$

if we use (3.11). From the inequality (2.5) and (3.17) we obtain the following estimate

$$\begin{aligned} \beta\|\bar{z}_h - I_h \bar{z}\|_1 &\leqslant \sup_{p_h \in Q_h} b(p_h, \bar{z}_h - I_h \bar{z})/\|p_h\|_Q \\ &= \sup_{p_h \in Q_h} a(\bar{q}_h - \bar{q}, p_h)/\|p_h\|_S \\ &\leqslant C\|\bar{q}_h - \bar{q}\|_S \leqslant C_1 h^{1/2}. \end{aligned} \tag{3.18}$$

Finally, the standard interpolation estimate

$$\|I_h \bar{z} - \bar{z}\|_1 \leqslant Ch|\bar{z}|_2,$$

(3.18) and the triangle inequality yield the $O(h^{1/2})$ estimate for $\|\bar{z} - \bar{z}_h\|_1$. □

Remark 3.1. If a higher regularity of the solution $\bar{z}$ is assumed, the estimate (3.7) can be improved, as follows from (3.15). Thus if e.g.

$$\bar{z}\big|_T \in W^{3,2}(T) \quad \forall T \in \mathscr{T}_{h_0},$$

then the embedding $W^{3,2} \hookrightarrow W^{2,q}$ with any $q > 1$ and the interpolation theory [8] imply that

$$\|\bar{z} - I_h \bar{z}\|_{0,\infty} \leqslant Ch^{2-2/q} \max_{T \in \mathscr{T}_{h_0}} |\bar{z}|_{2,q,T}.$$

Using this in (3.15), the estimate (3.7) can be changed to $O(h^{1-\varepsilon})$ with any positive ε.

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