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WHERE ARE TYPICAL C^1 FUNCTIONS ONE-TO-ONE?

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Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. Suppose $F \subset [0,1]$ is closed. Is it true that the typical (in the sense of Baire category) function in $C^1[0,1]$ is one-to-one on F? If $\underline{\dim}_B F < 1/2$ we show that the answer to this question is yes, though we construct an F with $\dim_B F = 1/2$ for which the answer is no. If C_α is the middle-α Cantor set we prove that the answer is yes if and only if $\dim(C_\alpha) \leqslant 1/2$. There are F's with Hausdorff dimension one for which the answer is still yes. Some other related results are also presented.

Keywords: typical function, box dimension, one-to-one function

MSC 2000: 26A15, 28A78, 28A80

1. Introduction

For the annual Miklós Schweitzer Competition organized by the János Bolyai Mathematical Society in 2004 the first listed author, having some generalizations in his mind as well, proposed the following problem:

Is it true that if the perfect set $F \subset [0,1]$ is of zero Lebesgue measure then those functions in $C^1[0,1]$ which are one-to-one on F form a dense subset of $C^1[0,1]$?

The answer to this question is negative. The winner of this Schweitzer Competition, the second listed author of this paper, found a particularly transparent solution to this problem and also suggested some generalizations. So the authors of this paper teamed up and wrote this paper.

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We do not interrupt this introduction with definitions and notation which can be found in Section 2 together with some further references.

If one considers the space $C[0,1]$ of the continuous functions equipped with the supremum norm, instead of the space $C^1[0,1]$ then the answer to the above problem is yes. In fact, much more is true. From Lemma 9 and the proof of Corollary 8 of [1] it follows that if $F \subset [0,1]$ is of first category then there exists a residual set $\mathcal{S} \subset C[0,1]$ such that for all $f \in \mathcal{S}$ the sets $f^n(F)$, $n = 0,1,\ldots$, are pairwise disjoint and f is one-to-one on each set $f^n(F)$, $n = 0,1,\ldots$, (where f^n denotes the nth iterate of f). This implies that for any nowhere dense perfect set $F \subset [0,1]$ if $\mathcal{F}$ denotes the set of those functions in $C[0,1]$ which are one-to-one on F then $\mathcal{F}$ is dense in $C[0,1]$.

In the space $C^1[0,1]$ the answer depends on F. In Theorem 5 we show that if the lower box dimension of F is less than $1/2$ then the typical $C^1[0,1]$ function is one-to-one on F. In Theorem 12 we construct a closed $F \subset [0,1]$ of box dimension $1/2$ such that the set of those $f \in C^1[0,1]$ for which $f|_F$ is one-to-one is not dense in $C^1[0,1]$. This shows that the value $1/2$ in Theorem 5 cannot be improved. The first natural idea to construct a closed set F for Theorem 12 would be by using a middle-α Cantor set, C_α or, more generally, by using a self similar set with the Open Set Condition. For these sets the Hausdorff and box dimension coincide and it is interesting that if the dimension of such sets equals $1/2$ then the typical $C^1[0,1]$ function is still one-to-one on them, see Theorems 7 and 10. In Theorem 10 we also show that for the Cantor sets C_α the typical $C^1[0,1]$ function is one-to-one on C_α if and only if $\dim(C_\alpha) \leqslant 1/2$.

Hausdorff dimension seems to be less appropriate since in Theorem 11 we construct a closed set F of Hausdorff dimension one such that the typical $C^1[0,1]$ function is one-to-one on F. Moreover, by Theorem 6 if the Hausdorff dimension of $F \times F$ is less than one then we can guarantee that a typical $C^1[0,1]$ function is one-to-one on F.

Several of the results in this paper depend on property $\mathcal{P}$, introduced in Section 2, which roughly says that the image of $F \times F$ is nowhere dense under projections in some "dense set of directions". In Theorem 2 we show that if the closed set $F \subset [0,1]$ has property $\mathcal{P}$ then the typical $C^1[0,1]$ function is one-to-one on F.

2. Notation and preliminary results

Recall that the usual metrics ϱ_0, and ϱ_1 on $C[0,1]$, and on $C^1[0,1]$, respectively, are given by

$$\varrho_0(f,g) = \max_{x\in[0,1]} |f(x) - g(x)| \quad \text{for } f,g \in C[0,1],$$

and

$$\varrho_1(f,g) = \varrho_0(f,g) + \varrho_0(f',g') \quad \text{for } f,g \in C^1[0,1].$$

It is well known that the metric spaces $(C[0,1],\varrho_0)$ and $(C^1[0,1],\varrho_1)$ are complete and hence Baire's category theorem holds in these spaces. We say that a typical $C[0,1]$, or $C^1[0,1]$ function has a certain property if the set of those functions which do not have this property is of first category in $C[0,1]$, or in $C^1[0,1]$. (Certain authors prefer using the term generic instead of typical.)

Let $F\subset\mathbb{R}$ be bounded. By $N_\delta(F)$ denote the minimum number of closed intervals of length δ that cover F. Then the lower and upper box dimensions of F are defined as

$$\underline{\dim}_B F=\liminf_{\delta\searrow 0}\frac{\log N_\delta(F)}{-\log\delta}\quad\text{and}\quad\overline{\dim}_B F=\limsup_{\delta\searrow 0}\frac{\log N_\delta(F)}{-\log\delta},$$

respectively. If $\underline{\dim}_B F=\overline{\dim}_B F$ then we call this number the box dimension of F and we denote it by $\dim_B F$. By equivalent definitions 3.1 on p. 41 of [4], instead of $N_\delta(F)$ several other expressions can be used in the definition of box dimension, for example, the number of those $[k\delta,(k+1)\delta]$, $k\in\mathbb{Z}$ grid intervals which intersect F.

Suppose $\varphi(0)=0$, $\varphi(x)>0$ for $x>0$, moreover φ is monotone increasing and continuous from the right.

For $A\subset\mathbb{R}$ we denote by $|A|$ the diameter of A. For $\delta>0$ set

$$\mathcal{H}^{\varphi}_{(\delta)}(A)=\inf\Big\{\sum_j\varphi(|A_j|)\colon\ A\subset\bigcup_j A_j,\ |A_j|<\delta\Big\},$$

and

$$\mathcal{H}^{\varphi}(A)=\lim_{\delta\searrow 0}\mathcal{H}^{\varphi}_{(\delta)}(A)=\sup_{\delta>0}\mathcal{H}^{\varphi}_{(\delta)}(A).$$

Then, (see Theorem 27 in [6], p. 50), $\mathcal{H}^{\varphi}$ is a regular Borel measure and each set of finite $\mathcal{H}^{\varphi}$ measure contains an F_σ set of the same measure.

If $\varphi(x)=x^s$ then we obtain the s-dimensional Hausdorff measure which will be denoted by $\mathcal{H}^s$. Set

$$\varphi_{1^-}(x)=\begin{cases}0, & \text{if } x=0;\\ -x\log x, & \text{if } 0<x<1/\mathrm{e};\\ x, & \text{if } 1/\mathrm{e}\leqslant x.\end{cases}$$

For ease of notation the measure $\mathcal{H}^{\varphi_{1^-}}$ will be denoted by $\mathcal{H}^{1^-}$. Since $\dim_H(A)$, the Hausdorff dimension of A equals $\inf\{s\colon\ \mathcal{H}^s(A)=0\}$ one can easily see that if $0<\mathcal{H}^{1^-}(A)<\infty$ then $\dim_H(A)=1$ and $\mathcal{H}^1(A)=\lambda(A)=0$, where λ denotes the Lebesgue measure.

Let F be a closed set in $[0,1]$. Consider the Cartesian product $F\times F$, and its projections in various directions. Let us denote by $\pi_{\beta/\alpha}$ the projection onto the line with tangent vector (α,β) of unit length, that is, $\alpha^2+\beta^2=1$ and $\pi_{\beta/\alpha}(x,y)=\alpha x+\beta y$. Note that β/α is the slope of the line with tangent vector (α,β).

We say that *property $\mathcal{P}$ holds* for the closed set $F\subset[0,1]$ if there exists a dense subset H of $\mathbb{R}$ for which $\pi_h(F\times F)\subset\mathbb{R}$ is nowhere dense for every $h\in H$. That is, the image of $F\times F$ is nowhere dense under projections in some "dense set of directions".

For the definition of iterated function systems and self-similar sets satisfying the Open Set Condition (OSC) we refer to Section 9 of [4]. We could not find an explicit reference to the next lemma so we outline its proof.

Lemma 1. *Let $F\subset\mathbb{R}$ be a self-similar set which satisfies OSC and which is of Hausdorff dimension s. Then the $2s$-dimensional Hausdorff measure of $F\times F$ is finite.*

Proof. Recall from Theorem 9.3 of [4] p. 118 that $\dim_H F=\dim_B F=s$ where $\sum\limits_{i=1}^{m} r_i^s=1$ and $F=\bigcup\limits_{i=1}^{m} S_i(F)$, with similarities S_i of contraction ratio $r_i<1$. By Corollary 7.4 of [4], $\dim_H(F\times F)=2\dim_H(F)$. Given $\delta>0$, as in the proof of Theorem 9.3 of [4], choose and fix a finite set Q of finite sequences $\underline{i}=(i_1,\dots,i_k)$ such that for every infinite sequence $(i_1,\dots)$ there is exactly one value of k with $\underline{i}\in Q$ and

$$(1)\qquad (\min_i r_i)\delta\leqslant r_{i_1}\dots r_{i_k}<\delta.$$

Considering the sets $F_{\underline{i}}=F_{i_1,\dots,i_k}\stackrel{\text{def}}{=}S_{i_1}\circ\dots\circ S_{i_k}(F)$ we obtain a covering $\{F_{\underline{i}}\colon\ \underline{i}\in Q\}$ of F such that (see the last paragraph of the proof of Theorem 9.3 in [4])

$$\sum_{\underline{i}\in Q}|F_{\underline{i}}|^s=|F|^s\sum_{\underline{i}\in Q}(r_{i_1}\dots r_{i_k})^s=|F|^s.$$

Moreover, from $\sum\limits_{\underline{i}\in Q}(r_{i_1}\dots r_{i_k})^s=1$ it also follows that Q contains at most $(\min\limits_i r_i)^{-s}\delta^{-s}=N^*(\delta)$ many sequences.

Now, $F\times F$ is covered by the sets $F_{\underline{i}}\times F_{\underline{j}}$, $(\underline{i},\underline{j})\in Q\times Q$ of diameter less than $\sqrt{2}\delta|F|$. Therefore,

$$\mathcal{H}^{2s}_{(\sqrt{2}\delta|F|)}(F\times F)<(N^*(\delta))^2(\sqrt{2}\delta|F|)^{2s}=(\min_i r_i)^{-2s}2^s|F|^{2s}.$$

This implies that $\mathcal{H}^{2s}(F\times F)<\infty$. □

3. Main results

Theorem 2. *Let $F \subset [0,1]$ be a closed set. If property $\mathcal{P}$ holds for F then the typical $C^1[0,1]$ function is one-to-one on F.*

To prove Theorem 2 we need Claim 3 and Lemma 4.

Claim 3. Let F be a closed subset of $[0,1]$ for which property $\mathcal{P}$ holds. Let N be a positive integer and m_i', c_i' be given real numbers ($i = 1, \ldots, N$). For any $\varepsilon > 0$ there exist real numbers $m_i \neq 0$, c_i for which $|m_i - m_i'| < \varepsilon, |c_i - c_i'| < \varepsilon$ and the sets $m_i F + c_i$ are pairwise disjoint.

Proof. We will prove the following slightly stronger statement, denoted by $\mathcal{S}_N$: *Suppose we are given real numbers $m_i \neq 0$, c_i ($i = 1, \ldots, N$), $\varepsilon > 0$, m' and c'. There exist real numbers $m \neq 0$ and c such that $|m - m'| < \varepsilon, |c - c'| < \varepsilon$ and for each $i = 1, \ldots, N$ the sets $m_i F + c_i$ are disjoint from $mF + c$.* From this, Claim 3 follows by induction, taking $m' = m'_{N+1}$ and $c' = c'_{N+1}$ and supposing that we have already found suitable m_i and c_i for $i = 1, \ldots, N$, by $\mathcal{S}_N$ we can find suitable m_{N+1} and c_{N+1}.

Statements $\mathcal{S}_N$ are also proved by induction on N. We start with $\mathcal{S}_1$. We need to choose m and c such that

$$(m_1 F + c_1) \cap (mF + c) = \emptyset.$$

Equivalently,

$$\Big(F + \frac{c_1}{m_1}\Big) \cap \Big(\frac{m}{m_1}F + \frac{c}{m_1}\Big) = \emptyset,\ F \cap \Big(\frac{m}{m_1}F + \frac{c - c_1}{m_1}\Big) = \emptyset,$$
$$0 \notin F - \Big(\frac{m}{m_1}F + \frac{c - c_1}{m_1}\Big),\ \frac{c - c_1}{m_1} \notin F + \Big(-\frac{m}{m_1}\Big)F.$$

By property $\mathcal{P}$ we have a dense subset H of $\mathbb{R}$, for which for every $h \in H$, $\pi_h(F \times F)$ is nowhere dense. If $\alpha \neq 0$ and β are such that $\beta/\alpha = h$ and $\alpha^2 + \beta^2 = 1$, then

$$\pi_h(F \times F) = \alpha F + \beta F = \alpha(F + hF). \tag{2}$$

Hence, for any $h \in H$ the set $F + hF$ is nowhere dense. Since H is dense we can choose $m \neq 0$ with $|m - m'| < \varepsilon$ such that $-m/m_1 \in H$. Thus $F + (-m/m_1)F$ is nowhere dense and we can choose c with $|c - c'| < \varepsilon$ such that $(c - c_1)/m_1 \notin F + (-m/m_1)F$. This proves statement S_1.

Suppose $N > 1$. By our induction hypothesis, $\mathcal{S}_{N-1}$ applied to the index set $i = 2, \ldots, N$, we can choose real numbers m'' and c'' such that $|m'' - m'| < \varepsilon/2$,

$|c''-c'|<\varepsilon/2$ and for each $i=2,\dots,N$ the sets m_iF+c_i are disjoint from $m''F+c''$. Since F is a compact set, $m''F+c''$ and m_iF+c_i are also compact sets, so they have a positive distance for each $i=2,\dots,N$. Hence, for a sufficiently small $\delta>0$, for any real numbers $m\in(m''-\delta,m''+\delta)$, $c\in(c''-\delta,c''+\delta)$ for each $i=2,\dots,N$ the sets $mF+c$ and m_iF+c_i are still disjoint. Now we use statement S_1 with m'' and c'' instead of m' and c' and with $\min(\varepsilon/2,\delta)$ instead of ε. We obtain $m\neq 0$ and c for which $|m-m''|<\min(\varepsilon/2,\delta)$, $|c-c''|<\min(\varepsilon/2,\delta)$ and $(mF+c)\cap(m_1F+c_1)=\emptyset$. By the choice of δ, $(mF+c)\cap(m_iF+c_i)=\emptyset$ also holds for every $i=2,\dots,N$. Since $|m''-m'|<\varepsilon/2$ and $|m-m''|<\min(\varepsilon/2,\delta)\leqslant\varepsilon/2$, we also have $|m-m'|<\varepsilon$. The same way we obtain $|c-c'|<\varepsilon$. This proves statement S_N. □

Lemma 4. *Suppose we have disjoint closed intervals $I_i=[x_i,y_i]$, $i=1,\dots,N$, in $[0,1]$ and a function $f\in C^1[0,1]$. For every $\varepsilon>0$ there exists $\gamma>0$ such that if the functions $h_i\in C^1[0,1]$ satisfy*

$$|h_i(x_i)-f(x_i)|\leqslant\gamma,\quad |h_i(y_i)-f(y_i)|\leqslant\gamma,\quad \max_{x\in I_i}|h_i(x)-f(x)|<\varepsilon,$$
$$\text{and}\quad \max_{x\in I_i}|h_i'(x)-f'(x)|<\varepsilon\quad (i=1,\dots,N),$$

then there exists a function $g\in C^1[0,1]$ for which $g|_{I_i}=h_i|_{I_i}$ $(i=1,\dots,N)$, and

$$\max_{x\in[0,1]}|g(x)-f(x)|<\varepsilon,\qquad \text{and}\qquad \max_{x\in[0,1]}|g'(x)-f'(x)|<\varepsilon,$$

which implies $\varrho_1(g,f)<2\varepsilon$.

P r o o f. The proof is straightforward and left to the reader. We only remark that γ is needed to handle the cases when x_i is too close to y_{i-1} to avoid that $(h_i(x_i)-h_{i-1}(y_{i-1}))/(x_i-y_{i-1})$ differs too much from $f'(x_i)$. □

P r o o f o f T h e o r e m 2. By property $\mathcal{P}$, the set F cannot contain any interval so it is nowhere dense and closed. Consider those functions in $C^1[0,1]$ which are one-to-one on F. We have to prove that these functions form a residual set in $C^1[0,1]$. First we will prove that this set is G_δ. Then we will prove that it is dense. This will prove the theorem.

Let

$$\mathcal{G}_n=\{f\in C^1[0,1]\colon\ \forall x,y\in F, |x-y|\geqslant 1/n\Longrightarrow f(x)\neq f(y)\}.$$

We claim that $\mathcal{G}_n$ is an open set in $C^1[0,1]$. Let $M=\{(x,y)\in\mathbb{R}^2\colon\ x,y\in F,\ |x-y|\geqslant 1/n\}$. The set M is clearly compact. Suppose that $f\in\mathcal{G}_n$. Let us define $f_0\colon\ M\to\mathbb{R}$ as $f_0((x,y))=f(x)-f(y)$. Then f_0 is continuous and nowhere zero on the compact set M. Hence, there exists an $\varepsilon>0$ for which $f_0(M)\cap(-\varepsilon,\varepsilon)=\emptyset$. Take a function

$g \in C^1[0,1]$ for which $\max\limits_{x\in[0,1]} |f(x) - g(x)| = \varrho_0(f,g) \leqslant \varrho_1(f,g) < \varepsilon/2$. Then the function $g_0\colon M \to \mathbb{R}$, $g_0((x,y)) = g(x) - g(y)$ is nowhere zero, therefore $g \in \mathcal{G}_n$. This proves that $\mathcal{G}_n$ is open. Put $\mathcal{G} = \bigcap\limits_{n=1}^{\infty} \mathcal{G}_n$. It is clear that $\mathcal{G}$ is a G_δ set and if $f \in \mathcal{G}$ then it is one-to-one on F.

Now we prove that $\mathcal{G}$ is dense. Take any $f \in C^1[0,1]$. Let $\varepsilon > 0$ be given. We will show that there exists a function $g \in C^1[0,1]$ which is one-to-one on F and $\varrho_1(f,g) < 6\varepsilon$.

Since $f' \in C[0,1]$ there exists $0 < \delta < 1$ such that for any $x, y \in [0,1]$ if $|x-y| \leqslant \delta$ then $|f'(x) - f'(y)| < \varepsilon$.

Let us cover the nowhere dense closed set F by disjoint intervals $I_i = [x_i, y_i]$, $i = 1, \ldots, N$ with $y_i - x_i < \delta$ for $i = 1, \ldots, N$, and $y_{i-1} < x_i$ for $i = 2, \ldots, N$.

Now choose real numbers m_i', c_i' such that $f(x_i) = m_i' x_i + c_i'$ and $f(y_i) = m_i' y_i + c_i'$ hold ($i = 1, \ldots, N$), that is, $y = g_i(x) \overset{\text{def}}{=} m_i' x + c_i'$ is the line passing through the points $(x_i, f(x_i))$ and $(y_i, f(y_i))$. Thus, $f(x_i) = g_i(x_i)$ and $f(y_i) = g_i(y_i)$. By the Mean Value Theorem there exists $z_i \in I_i$ for which $m_i' = f'(z_i)$. Since the length of I_i is at most δ we have

$$\text{(3)} \qquad \max_{x\in I_i} |f'(x) - m_i'| = \max_{x\in I_i} |f'(x) - g_i'(x)| = \max_{x\in I_i} |f'(x) - f'(z_i)| < \varepsilon.$$

Let $\gamma > 0$ be the constant we obtain from Lemma 4 applied with 3ε, for the function f and intervals I_i. We can suppose that $\gamma < \varepsilon$.

By applying Claim 3 we obtain real numbers $m_i \neq 0$, c_i for which $|m_i - m_i'| < \gamma/2$, $|c_i - c_i'| < \gamma/2$ and the sets $m_i F + c_i$ are pairwise disjoint ($i = 1, \ldots, N$). Let $h_i\colon [0,1] \to \mathbb{R}$ be the function $x \mapsto m_i x + c_i$. Then

$$|h_i(x_i) - f(x_i)| \leqslant |h_i(x_i) - g_i(x_i)| + |g_i(x_i) - f(x_i)| = |(m_i - m_i')x_i + c_i - c_i'| + 0 \leqslant \gamma,$$

and similarly we obtain $|h_i(y_i) - f(y_i)| \leqslant \gamma$. On the other hand by using (3)

$$\max_{x\in I_i} |h_i'(x) - f'(x)| \leqslant |m_i - m_i'| + \max_{x\in I_i} |m_i' - f'(x)| < 2\varepsilon,$$

and

$$\max_{x\in I_i} |h_i(x) - f(x)| \leqslant |h_i(x_i) - f(x_i)| + \max_{x\in I_i} \left| \int_{x_i}^{x} h_i'(t) - f'(t) \mathrm{d}t \right| \leqslant \gamma + \delta \cdot 2\varepsilon < 3\varepsilon.$$

Thus we can apply Lemma 4 for the functions f and h_i with intervals I_i. By the choice of γ we obtain a function $g \in C^1[0,1]$ such that $g|_{I_i} = h_i|_{I_i}$ ($i = 1, \ldots, N$), and $\varrho_1(f,g) < 6\varepsilon$.

Observe that $F \subset \bigcup\limits_{i=1}^{N} I_i$, $g(x) = h_i(x) = m_i x + c_i$ on I_i with $m_i \neq 0$, and the sets $m_i F + c_i$ are pairwise disjoint. This clearly implies that g is one-to-one on F. $\square$

Theorem 5. *If $F \subset [0,1]$ is closed and $\underline{\dim}_B F < 1/2$ then a typical $C^1[0,1]$ function is one-to-one on F.*

P r o o f. By Theorem 2 it is enough to show that property $\mathcal{P}$ holds for F. It is not too difficult to see that $\underline{\dim}_B(F\times F) \leqslant 2\cdot\underline{\dim}_B F$ which implies $\underline{\dim}_B(F\times F) < 1$. Projections cannot increase the lower box dimension, thus $\underline{\dim}_B \pi_h(F \times F) < 1$ for any $h \in \mathbb{R}$. Hence $\pi_h(F\times F)$ for any h cannot contain an interval and, being compact, it is nowhere dense. Therefore property $\mathcal{P}$ holds. □

Theorem 6. *Let F be a closed subset of $[0,1]$. If the Hausdorff dimension of $F \times F$ is less than one then a typical $C^1[0,1]$ function is one-to-one on F.*

P r o o f. An argument similar to the one in the proof Theorem 5 can be used, the details are left to the reader. □

Theorem 7. *Let $F \subset [0,1]$ be a self-similar set with OSC of dimension $\leqslant 1/2$. Then a typical $C^1[0,1]$ function is injective on F.*

P r o o f. If the dimension of F is smaller than $1/2$ then Theorem 5 implies the statement. Suppose that the dimension is $1/2$. Then from the product formulae and $\dim_B F = \dim_H F$ (Corollary 7.4 and Theorem 9.3 of [4]) one can obtain that the Hausdorff dimension of $F\times F$ is exactly one. Lemma 1 shows that its one dimensional Hausdorff measure is finite, so $F \times F$ is clearly an irregular 1-set (for the properties of irregular 1-sets we refer to Chapters 3 and 6 of [3] and Sections 5.2 and 6.2 of [4]). We obtain from the projection characterization of 1-sets that almost every projection of $F \times F$ has Lebesgue measure zero, and hence it is nowhere dense. This implies that property $\mathcal{P}$ holds for F. □

Definition 8. Suppose $0 < \alpha < 1$ and $t = (1-\alpha)/2$. The middle-α Cantor set, denoted by C_α, is the self-similar set generated by the similarities $\varphi_1\colon x \mapsto \frac{1}{2}(1-\alpha)x = tx$ and $\varphi_2\colon x \mapsto 1 + \frac{1}{2}(1-\alpha)(x-1) = (1-t) + tx$. When $\alpha = 1/3$ we obtain the usual triadic Cantor set.

Let Φ be the operator on compact subsets of $\mathbb{R}$ for which $\Phi(F) = \varphi_1(F) \cup \varphi_2(F)$. Put $F_n = \Phi^n([0,1])$, $(n = 0,1,\ldots)$, which is a union of 2^n intervals of length t^n. Then $C_\alpha = \bigcap\limits_{n=0}^{\infty} F_n$.

Set $h_0(x) = x/2$.

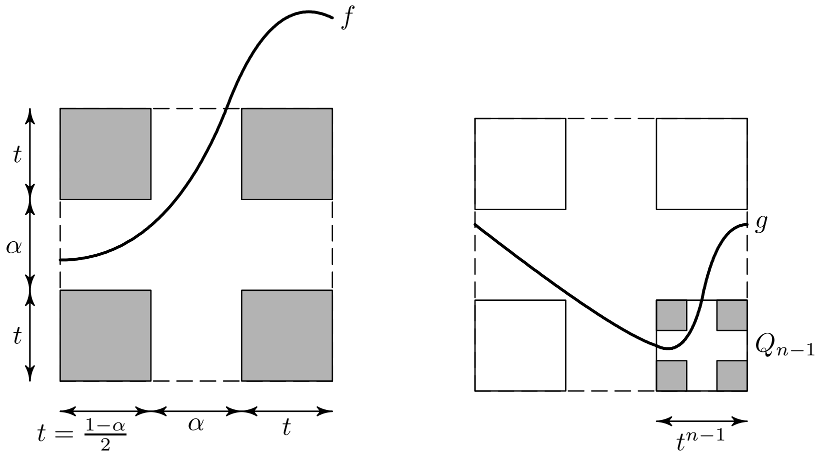

Figure 1. f and g

Lemma 9. *Suppose that $\alpha < 1/2$ and $g \in C^1[0,1]$ is such that $g([0,1]) \cap [0,1] \neq \emptyset$, and $g(C_\alpha) \cap C_\alpha = \emptyset$. Then, $\varrho_1(g, h_0) \geqslant \max\limits_{x\in[0,1]} |g'(x) - 1/2| > \delta$ for some $\delta > 0$ depending only on α.*

Proof. Suppose that $f \in C^1[0,1]$ is such that the graph of f, $\{(x, f(x)) \colon x \in [0,1]\}$, intersects the square $[0,1] \times [0,1]$, but does not intersect $F_1 \times F_1$ (which consists of four squares each of side length t, see the left side of Figure 1). By the location of these four squares and the Mean Value Theorem one can easily see that there exists $x \in [0,1]$ for which either $f'(x) < \alpha$ (when the graph of f has points on the two opposite vertical sides of $[0,1] \times [0,1]$), or $f'(x) > t/\alpha = (1-\alpha)/2\alpha > 1/2$ (when the graph of f has a point on one of the horizontal sides of $[0,1] \times [0,1]$, see the left side of Figure 1), hence $\max\limits_{x\in[0,1]} |f'(x) - 1/2| > \delta$ for some $\delta > 0$ depending only on α.

Suppose that $g \in C^1[0,1]$ is such that $g([0,1]) \cap [0,1] \neq \emptyset$ and $g(C_\alpha) \cap C_\alpha = \emptyset$. That is, the graph of g does not intersect $C_\alpha \times C_\alpha$. Since $C_\alpha \times C_\alpha = \bigcap\limits_{n=0}^{\infty} (F_n \times F_n)$, there exists a smallest $n \in \mathbb{N}$ for which the graph of g intersects $F_{n-1} \times F_{n-1}$ but does not intersect $F_n \times F_n$. Then there is a subsquare Q_{n-1} of $F_{n-1} \times F_{n-1}$ of side length t^{n-1} which contains points of the graph of g (see the right side of Figure 1). Since the graph of g does not intersect the four subsquares $Q_{n-1} \cap (F_n \times F_n)$ of side length t^n, an argument similar to the one stated above for f shows that there is an $x \in [0,1]$ for which $|g'(x) - 1/2| > \delta$. □

Theorem 10. *A typical $C^1[0,1]$ function is injective on C_α if and only if $\dim(C_\alpha) \leqslant 1/2$ (that is, $1/2 \leqslant \alpha < 1$).*

P r o o f. If $\alpha \geqslant 1/2$ then the dimension of C_α is at most $1/2$, so Theorem 7 yields the proof.

Suppose that $\alpha < 1/2$. Recall that $h_0(x) = x/2$ and set $h_1(x) = x$. Given α, by Lemma 9 choose $\delta > 0$. For any given functions $f_0, f_1 \in C^1[0,1]$ satisfying

$$(4) \qquad \varrho_1(f_0, h_0) < \delta/100 \quad \text{and} \quad \varrho_1(f_1, h_1) < \delta/100$$

the function $f(x) = f_1^{-1}(f_0(x))$ is such that $\varrho_1(f, h_0) < \delta$ and the graph of f intersects $[0,1] \times [0,1]$. Hence by Lemma 9, $f(C_\alpha) \cap C_\alpha \neq \emptyset$ and equivalently $f_0(C_\alpha) \cap f_1(C_\alpha) \neq \emptyset$.

Now suppose that $g_0(x) = x/2$ for $x \in [0,t]$, $g_0(x) = x + t - 1$ for $x \in [1-t, 1]$ and otherwise g_0 is defined so that $g_0 \in C^1[0,1]$. We claim that any $g \in C^1[0,1]$ satisfying

$$(5) \qquad \varrho_1(g, g_0) < \delta/400$$

is not one-to-one on C_α. Consider $f_0(x) = g(tx)/t$ and $f_1(x) = g(tx + 1 - t)/t$ for $x \in [0,1]$. Then (5) implies that f_0 and f_1 satisfy (4) and hence $f_0(C_\alpha) \cap f_1(C_\alpha) \neq \emptyset$, that is, there exists $x, y \in C_\alpha$ such that $g(tx)/t = g(ty + 1 - t)/t$. Since $tC_\alpha \cap (tC_\alpha + 1 - t) = \emptyset$ and $tC_\alpha \cup (tC_\alpha + (1-t)) = C_\alpha$, if we let $x' = tx$ and $y' = ty + (1-t)$ then $x', y' \in C_\alpha$, $x' \neq y'$ and $g(x') = g(y')$, showing that g is not one-to-one on C_α. □

Theorem 11. *There exists a closed set $F \subset [0,1]$ of Hausdorff dimension one such that a typical $C^1[0,1]$ function is one-to-one on F.*

P r o o f. The main idea of this proof is based on Lemma 1.3 of [2].

Choose a countable dense set $T = \{t_i \colon i \in \mathbb{N}^+\} \subset \mathbb{R}$.

Let $K \subset [0,1]$ be any compact 1^--set, that is, $0 < \mathcal{H}^{1^-}(K) < \infty$. Define a new measure $\mathcal{H}^{1^-}_K$ by $\mathcal{H}^{1^-}_K(H) = \mathcal{H}^{1^-}(H \cap K)$. Hence $\mathcal{H}^{1^-}_K$ is a finite Borel measure.

Fix an index i. Set $M = \{(x,y) \colon x + y \in t_i K\} \subset \mathbb{R}^2$. Clearly, M is a Borel set, and hence $\mathcal{H}^{1^-}_K \times \lambda$ measurable. We apply the Fubini theorem to the characteristic function of M. The vertical sections of M are of the form

$$\{y \in \mathbb{R} \colon x + y \in t_i K\} = \{y \in \mathbb{R} \colon y \in t_i K - x\} = t_i K - x.$$

Since K is a compact 1^- set, its Lebesgue measure is zero, hence all the vertical sections are of Lebesgue measure zero. By the Fubini theorem $(\mathcal{H}^{1^-}_K \times \lambda)(M) = 0$, and hence λ-almost every horizontal section of M is of $\mathcal{H}^{1^-}_K$-measure zero. A horizontal section of M is $\{x \in \mathbb{R} \colon x \in t_i K - y\} = t_i K - y$. From this it follows that for λ-almost every y we have $0 = \mathcal{H}^{1^-}_K(t_i K - y) = \mathcal{H}^{1^-}(K \cap (t_i K - y))$. Therefore, we can choose a countable dense set $D \subset \mathbb{R}$ such that $\mathcal{H}^{1^-}(K \cap (t_i K + d)) = 0$ for every $d \in D$.

Let $B = K \setminus \bigcup_{d \in D} (t_i K + d)$. Then B is a Borel set of the same $\mathcal{H}^{1^-}$-measure as K. We claim that

$$D \cap (B - t_i B) = \emptyset. \tag{6}$$

Suppose that for some $d \in D$ we have $d \in B - t_i B$, that is, $(d + t_i B) \cap B \neq \emptyset$. Since $B \subset K \setminus (t_i K + d)$ we have $(d + t_i B) \cap \big(K \setminus (t_i K + d)\big) \neq \emptyset$. Thus $(d + t_i K) \cap \big(K \setminus (t_i K + d)\big) \neq \emptyset$. But this is impossible.

Let F_i be a compact subset of B for which $\mathcal{H}^{1^-}(F_i) > \mathcal{H}^{1^-}(B) \cdot (1 - 4^{-i}) = \mathcal{H}^{1^-}(K) \cdot (1 - 4^{-i})$. By (6), $D \cap (F_i - t_i F_i) = \emptyset$. Using (2) with $h = -t_i$ one can see that $F_i - t_i F_i$ is similar to the set $\pi_{-t_i}(F_i \times F_i)$, that is, to the image of $F_i \times F_i$ under the projection onto the line with slope $-t_i$ passing through the origin. Since D is dense, the compact set $\pi_{-t_i}(F_i \times F_i)$ is nowhere dense.

Now consider the same construction for each index i. We obtain compact sets F_i with measures $\mathcal{H}^{1^-}(F_i) > \mathcal{H}^{1^-}(K) \cdot (1 - 4^{-i})$. Let $F = \bigcap_{i=1}^{\infty} F_i$. Then $0 < \mathcal{H}^{1^-}(F) < \infty$, that is, F is a 1^--set and for each index i the set $\pi_{-t_i}(F \times F)$ is nowhere dense. Since $T = \{t_i \colon i \in \mathbb{N}^+\}$ is dense in $\mathbb{R}$, property $\mathcal{P}$ holds for F. Thus, the typical $C^1[0,1]$ function is one-to-one on F. □

Theorem 12. *There exists a closed $F \subset [0,1]$ such that $\dim_B F = 1/2$ and the set of those $f \in C^1[0,1]$ for which $f|_F$ is one-to-one is not dense in $C^1[0,1]$.*

Proof. Set $F_{1,1} = [1/2, 1]$, $l_{1,1} = 1/2 = \lambda(F_{1,1})$, $r_{1,0} = 1$, $r_{1,j} = 4^{-2j+1}$ for $j = 1, 2, \ldots$, and $l_{1,j} = l_{1,j-1} r_{1,j-1}^2 = l_{1,1}(r_{1,1} \ldots r_{1,j-1})^2 = \frac{1}{2}(4^{-1} \ldots 4^{-2j+3})^2$ for $j = 2, 3, \ldots$.

We also put $F_{2,1} = [0, 1/4]$, $l_{2,1} = 1/4 = \lambda(F_{1,2})$, $r_{2,0} = 1$, $r_{2,j} = 4^{-2j} = r_{1,j}/4$ for $j = 1, 2, \ldots$, and $l_{2,j} = l_{2,j-1} r_{2,j-1}^2 = l_{2,1}(r_{2,1} \ldots r_{2,j-1})^2 = \frac{1}{4}(4^{-2} \ldots 4^{-2j+2})^2$ for $j = 2, 3, \ldots$.

Direct computation shows that

$$2 r_{1,k} l_{1,k} = l_{2,k} \quad \text{and} \quad 2 r_{2,k} l_{2,k} = l_{1,k+1}. \tag{7}$$

Suppose that $j \geqslant 1$, $F_{1,j}$ and $F_{2,j}$ are defined and they consist of the union of systems of disjoint closed intervals $\mathcal{I}_{1,j}$ and $\mathcal{I}_{2,j}$, respectively. We also suppose that each interval I belonging to $\mathcal{I}_{i,j}$, $(i = 1, 2)$ is of length $l_{i,j}$ and

$$\#\mathcal{I}_{i,j} = (r_{i,0} \ldots r_{i,j-1})^{-1} \quad \text{for} \quad i = 1, 2. \tag{8}$$

Next we define $F_{i,j+1} \subset F_{i,j}$ and $\mathcal{I}_{i,j+1}$. Suppose that $I = [a,b] \in \mathcal{I}_{i,j}$. Then $b = a + l_{i,j}$. For $m = 1, \dots, r_{i,j}^{-1}$ we consider the intervals $I_m = [a+(m-1)l_{i,j}r_{i,j}, a+(m-1)l_{i,j}r_{i,j}+l_{i,j}r_{i,j}^2]$, that is, we divide I into $r_{i,j}^{-1}$ many equal subintervals of length $r_{i,j}l_{i,j}$ and select in each piece the "first" closed sub-subinterval of length $r_{i,j}^2 l_{i,j}$. Then $\lambda(I_m) = l_{i,j+1}$ for all m. We will define $F_{i,j+1}$ so that $F_{i,j+1} \cap I = \bigcup_{m=1}^{r_{i,j}^{-1}} I_m$. We repeat this procedure in all $I \in \mathcal{I}_{i,j}$. Let $\mathcal{I}_{i,j+1}$ be the set of the intervals I_m ($m = 1, \dots, r_{i,j}^{-1}$) for all the intervals $I \in \mathcal{I}_{i,j}$ and $F_{i,j+1} = \bigcup \mathcal{I}_{i,j+1}$. It is clear that $\#\mathcal{I}_{i,j+1} = r_{i,j}^{-1}\#\mathcal{I}_{i,j}$, showing that (8) holds for $j+1$.

We set $F_i = \bigcap_{j=1}^{\infty} F_{i,j}$, $(i = 1,2)$.

Suppose $f_0(x) = x$ on $[0,1/4]$ and $f_0(x) = x - \frac{1}{2}$ on $[1/2,1]$ and otherwise f_0 is defined so that $f_0 \in C^1[0,1]$.

Set $\varepsilon_0 = 1/1000$ and suppose $\varrho_1(f, f_0) < \varepsilon_0$ for some $f \in C^1[0,1]$. We show that f is not one-to-one on $F = F_1 \cup F_2$ by finding $x \in F_1$ and $y \in F_2$ such that $f(x) = f(y)$.

From $\varrho_1(f, f_0) < \varepsilon_0$ it follows that for $x \in [0,1/4] \cup [1/2,1]$

$$(9) \qquad 1 - \varepsilon_0 < f'(x) < 1 + \varepsilon_0 \quad \text{and} \quad |f(0)|, |f(1/2)| < \varepsilon_0.$$

Set $I_{1,1} = F_{1,1} = [1/2,1]$ and $I_{2,1} = F_{2,1} = [0,1/4]$. Observe that by (7), $\lambda(I_{2,1}) = l_{2,1} = 2r_{1,1}l_{1,1}$. Recall that during the definition of $F_{1,2}$ we subdivide $I_{1,1}$ into subintervals of length $r_{1,1}l_{1,1}$ and in each such subinterval we keep the first sub-subinterval of length $r_{1,1}^2 l_{1,1}$. By (9) and the Mean Value Theorem we can select an interval $I_{1,2} \in \mathcal{I}_{1,2}$ such that $f(I_{1,2}) \subset f(I_{2,1})$, $I_{1,2} \subset I_{1,1}$. Now, by (7), $\lambda(I_{1,2}) = l_{1,2} = 2r_{2,1}l_{2,1}$. During the definition of $F_{2,2}$ we subdivide $I_{2,1}$ into subintervals of length $r_{2,1}l_{2,1}$ and in each such subinterval we keep the first sub-subinterval of length $r_{2,1}^2 l_{2,1}$. By (9) and the Mean Value Theorem we can select an interval $I_{2,2} \in \mathcal{I}_{2,2}$ such that $f(I_{2,2}) \subset f(I_{1,2})$, $I_{2,2} \subset I_{2,1}$.

Repeating the above steps one can select sequences of intervals $I_{1,1}, I_{1,2}, \dots$ and $I_{2,1}, I_{2,2}, \dots$ such that $f(I_{2,1}) \supset f(I_{1,2}) \supset f(I_{2,2}) \supset f(I_{1,3}) \supset f(I_{2,3}) \supset \dots$, $I_{1,1} \supset I_{1,2} \supset I_{1,3} \supset \dots$, $I_{2,1} \supset I_{2,2} \supset I_{2,3} \supset \dots$, and $I_{i,j} \in \mathcal{I}_{i,j}$, $(i = 1,2,\ j = 1,2,3,\dots)$. Then for $x = \bigcap_{j=1}^{\infty} I_{1,j} \in F_1$ and $y = \bigcap_{j=1}^{\infty} I_{2,j} \in F_2$ we have $f(x) = f(y)$.

By Theorem 5, $\underline{\dim}_B F \geqslant 1/2$. So we need to show that $\overline{\dim}_B F \leqslant 1/2$.

For a given δ, $(0 < \delta < l_{1,1})$ choose k such that

$$(10) \qquad l_{1,1}(r_{1,0} \dots r_{1,k-1})^2 \geqslant \delta > l_{1,1}(r_{1,0} \dots r_{1,k-1}r_{1,k})^2,$$

that is, $l_{1,k} \geqslant \delta > l_{1,k+1}$. By (7), $l_{2,k+1} < l_{1,k+1}$. Since $F_{i,k+1}$ consists of $(r_{i,0} \dots r_{i,k})^{-1}$ many intervals of length $l_{i,k+1} < \delta$, clearly $N_\delta(F_{i,k+1}) \leqslant (r_{i,0} \dots$

$r_{i,k})^{-1}$ $(i=1,2)$. Hence from $F_1 \subset F_{1,k+1}$, we obtain

$$\log N_\delta(F_1) \leqslant \log N_\delta(F_{1,k+1}) \leqslant (1+3+\ldots+(2k-1))\log 4,$$

and from $F_2 \subset F_{2,k+1}$ we obtain

$$\log N_\delta(F_2) \leqslant \log N_\delta(F_{2,k+1}) \leqslant (2+4+\ldots+2k)\log 4.$$

By an elementary calculation

$$\log N_\delta(F_i) \leqslant (k^2+k)\log 4, \quad (i=1,2). \tag{11}$$

From (10) we obtain $\log\delta \leqslant \log(l_{1,1}(r_{1,0}\ldots r_{1,k-1})^2) = \log l_{1,1} - 2(\log 4)(1+3+\ldots+(2k-3)) = \log l_{1,1} - 2(k^2-2k+1)\log 4$. Using (11)

$$\begin{aligned}\overline{\dim}_B(F_i) &= \limsup_{\delta\searrow 0} \frac{\log N_\delta(F_i)}{-\log\delta} \\ &\leqslant \limsup_{k\to\infty} \frac{(k^2+k)\log 4}{-\log l_{1,1} + 2(k^2-2k+1)\log 4} = \frac{1}{2}.\end{aligned}$$

Therefore the upper box dimension of $F = F_1 \cup F_2$ is also at most $1/2$, which proves the theorem. □

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