Barbara Glanc; Aleksander Misiak; Maria Szmuksta-Zawadzka

Equivariant mappings from vector product into G-spaces of φ-scalars with $G = O(n, 1, \mathbb{R})$

Mathematica Bohemica, Vol. 132 (2007), No. 3, 325–332

Persistent URL: `http://dml.cz/dmlcz/134120`

EQUIVARIANT MAPPINGS FROM VECTOR PRODUCT INTO G-SPACES OF φ-SCALARS WITH $G = O\,(n, 1, \mathbb{R})$

Barbara Glanc, Aleksander Misiak,
Maria Szmuksta-Zawadzka, Szczecin

(Received April 5, 2006)

Abstract. There are four kinds of scalars in the n-dimensional pseudo-Euclidean geometry of index one. In this note, we determine all scalars as concomitants of a system of $m \leqslant n$ linearly independent contravariant vectors of two so far missing types. The problem is resolved by finding the general solution of the functional equation $F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = \varphi\,(A) \cdot F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$ using two homomorphisms φ from a group G into the group of real numbers $\mathbb{R}_0 = (\mathbb{R} \setminus \{0\}\,, \cdot)$.

Keywords: G-space, equivariant map, pseudo-Euclidean geometry

MSC 2000: 53A55

1. Introduction

For $n \geqslant 2$ consider the matrix $E_1 = [e_{ij}] \in GL\,(n, \mathbb{R})$ where

$$e_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ +1 & \text{for } i = j \neq n, \\ -1 & \text{for } i = j = n. \end{cases}$$

Definition 1. A pseudo-orthogonal group of index one is a subgroup of the group $GL\,(n, \mathbb{R})$ satisfying the condition

$$G = O\,(n, 1, \mathbb{R}) = \left\{A\colon\ A \in GL\,(n, \mathbb{R}) \wedge A^T \cdot E_1 \cdot A = E_1\right\}.$$

The class of G-spaces (M_α, G, f_α) where f_α is an action of G on the space M_α constitutes a category if we take as morphisms equivariant maps $F_{\alpha\beta}\colon\ M_\alpha \longrightarrow M_\beta$, i.e. the maps which satisfy the condition

$$\bigwedge_{\alpha,\beta}\bigwedge_{x\in M_\alpha}\bigwedge_{A\in G} F_{\alpha\beta}\left(f_\alpha\left(x,A\right)\right)=f_\beta\left(F_{\alpha\beta}\left(x\right),A\right). \tag{1}$$

This category is called a geometry of the group G.

If we denote $A=\left[A^i_j\right]^n_1\in G$ then there exist exactly four homomorpisms of the group G in the group $\mathbb{R}_0$, namely $1\left(A\right)=1, \varepsilon\left(A\right)=\operatorname{sign}\left(\det A\right), \eta\left(A\right)=\operatorname{sign}\left(A^n_n\right)$ and $\varepsilon\left(A\right)\cdot\eta\left(A\right)$ (see [3]). In the pseudo-Euclidean geometry of index one there are the G-space of contravariant vectors

$$\left(\mathbb{R}^n,G,f\right),\quad \text{where}\quad \bigwedge_{u\in\mathbb{R}^n}\bigwedge_{A\in G} f\left(u,A\right)=A\cdot u, \tag{2}$$

and four G-spaces of objects with one component and a linear transformation rule

$$\left(\mathbb{R},G,h\right),\quad \text{where}\quad \bigwedge_{x\in\mathbb{R}}\bigwedge_{A\in G} h\left(x,A\right)=\begin{cases} 1\cdot x & \text{for 1-scalars,}\\ \varepsilon\left(A\right)\cdot x & \text{for } \varepsilon\text{-scalars,}\\ \eta\left(A\right)\cdot x & \text{for } \eta\text{-scalars,}\\ \varepsilon\left(A\right)\cdot\eta\left(A\right)\cdot x & \text{for } \varepsilon\eta\text{-scalars.}\end{cases} \tag{3}$$

Every equivariant map F of a system of linearly independent vectors $\underset{1}{u},\underset{2}{u},\ldots,\underset{m}{u}$ with $m\leqslant n$ into $\mathbb{R}$ satisfies the equality (1), which by applying the transformations rules (2) and (3) may be rewritten in the form

$$\bigwedge_{A\in G} F(A\underset{1}{u},A\underset{2}{u},\ldots,A\underset{m}{u})=1\cdot F(\underset{1}{u},\underset{2}{u},\ldots,\underset{m}{u})\qquad\text{for 1-scalars,} \tag{4}$$

$$\bigwedge_{A\in G} F(A\underset{1}{u},A\underset{2}{u},\ldots,A\underset{m}{u})=\varepsilon(A)\cdot F(\underset{1}{u},\underset{2}{u},\ldots,\underset{m}{u})\qquad\text{for } \varepsilon\text{-scalars,} \tag{5}$$

$$\bigwedge_{A\in G} F(A\underset{1}{u},A\underset{2}{u},\ldots,A\underset{m}{u})=\eta(A)\cdot F(\underset{1}{u},\underset{2}{u},\ldots,\underset{m}{u})\qquad\text{for } \eta\text{-scalars,} \tag{6}$$

$$\bigwedge_{A\in G} F(A\underset{1}{u},A\underset{2}{u},\ldots,A\underset{m}{u})=\varepsilon(A)\cdot\eta(A)\cdot F(\underset{1}{u},\underset{2}{u},\ldots,\underset{m}{u})\qquad\text{for } \varepsilon\eta\text{-scalars.} \tag{7}$$

For the pair $\underset{1}{u},\underset{2}{u}$ of contravariant vectors the mapping $p(\underset{1}{u},\underset{2}{u})=\underset{1}{u}^T\cdot E_1\cdot\underset{2}{u}$ satisfies (4), namely

$$p(A\underset{1}{u},A\underset{2}{u})=(A\underset{1}{u})^T\cdot E_1\cdot(A\underset{2}{u})=\underset{1}{u}^T(A^T\cdot E_1\cdot A)\underset{2}{u}=\underset{1}{u}^T\cdot E_1\cdot\underset{2}{u}=p(\underset{1}{u},\underset{2}{u}).$$

In [7] it was proved that the general solution of the equation (4) is of the form

$$F(\underset{1}{u},\underset{2}{u},\ldots,\underset{m}{u})=\Theta(p(\underset{i}{u},\underset{j}{u}))=\Theta(p_{ij}) \tag{8}$$

for $i \leqslant j = 1, 2, \ldots, m \leqslant n$, where Θ is an arbitrary function of $\frac{1}{2}m(m+1)$ variables p_{ij}.

The invariant p allows us to decompose the space of contravariant vectors (2) into invariant subsets

$$\mathbb{R}^n = \overset{+}{V} \cup (\overset{-}{V} \cup \overset{0}{V}) \cup \{0\} = \overset{+}{V} \cup \overset{*}{V} \cup \{0\}$$

where

$\overset{+}{V} = \{u\colon\ u \in \mathbb{R}^n \wedge p(u,u) > 0\}$ is the set of Euclidean vectors,

$\overset{-}{V} = \{u\colon\ u \in \mathbb{R}^n \wedge p(u,u) < 0\}$ is the set of pseudo-Euclidean vectors,

$\overset{0}{V} = \{u\colon\ u \in \mathbb{R}^n \wedge p(u,u) = 0 \wedge u \neq 0\}$ is the set of isotropic vectors,

$\overset{*}{V} = \overset{-}{V} \cup \overset{0}{V}$ is the set of non-Euclidean vectors.

The isotropic cone $\overset{0}{V}$ is, moreover, a transitive set. Let K^{n-1} denote a ball in the hyperplane $q^n = 1$, namely

$$K^{n-1} = \left\{ q = \left[q^1, q^2, \ldots, q^{n-1}, 1\right]^T :\ \sum_{i=1}^{n-1} \left(q^i\right)^2 \leqslant 1 = q^n \right\}.$$

Lemma 1. *For every point $q \in K^{n-1}$ and arbitrary matrix $A \in G$ we have*

$$\operatorname{sign}\left(w\left(q, A\right)\right) = \operatorname{sign}\left(\sum_{i=1}^{n} A_i^n q^i \right) = \operatorname{sign}\left(A_n^n\right) = \eta\left(A\right). \tag{9}$$

P r o o f. From the condition $A^T \cdot E_1 \cdot A = E_1$ and the Cauchy-Schwarz inequality we get

$$A_n^n - \sqrt{\left[(A_n^n)^2 - 1\right] \sum_{i=1}^{n-1} \left(q^i\right)^2} \leqslant w\left(q, A\right) \leqslant A_n^n + \sqrt{\left[(A_n^n)^2 - 1\right] \sum_{i=1}^{n-1} \left(q^i\right)^2},$$

which proves the lemma. □

From $u \in \overset{*}{V}$ it follows that $u^n \neq 0$ and for an arbitrary matrix $A \in G$ we have also $A \cdot u \in \overset{*}{V}$. Then

$$\begin{aligned} u = \left[u^1, u^2, \ldots, u^n\right]^T &= u^n \left[\frac{u^1}{u^n}, \frac{u^2}{u^n}, \ldots, \frac{u^{n-1}}{u^n}, 1\right]^T \\ &= u^n \left[q^1, q^2, \ldots, q^{n-1}, 1\right]^T = u^n \cdot q, \end{aligned}$$

where $q \in K^{n-1}$. The point q is called the direction of the non-Euclidean vector u. The last coordinate of the vector $A \cdot u$ is equal to

$$\left(Au\right)^n = \sum_{i=1}^{n} A_i^n u^i = u^n \left(\sum_{i=1}^{n} A_i^n q^i \right) = u^n \cdot w\left(q, A\right). \tag{10}$$

In accordance with (9) and (10) we get

$$\bigwedge_{u \in \overset{*}{V}} \bigwedge_{A \in G} \operatorname{sign}\left[(Au)^n\right] = \eta(A) \cdot \operatorname{sign}(u^n). \tag{11}$$

Now, let an arbitrary system of linearly independent vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}$ be given. Let $L_m = L(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$ denote the linear subspace generated by the vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}$ and $p|L_m$ the restriction of the form p to the subspace L_m.

Definition 2. The subspace L_m is called:

(i) a Euclidean subspace if the form $p|L_m$ is positive definite,
(ii) a pseudo-Euclidean subspace if the form $p|L_m$ is regular and indefinite,
(iii) a singular subspace if the form $p|L_m$ is singular.

If we denote

$$P(m) = P(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = \det\left[p_{ij}\right]_1^m$$

then the above three cases are equivalent to $P(m) > 0, P(m) < 0$ and $P(m) = 0$, respectively. Let $\overset{m}{P}_{ij}$ denote the cofactor of the element p_{ij} of the matrix $[p_{ij}]_1^m$ and let $\overset{1}{P}_{11} = 1, P(0) = 1$ by definition.

If $m = n-1$ and $P(n-1) = 0$ then the singular subspace $L(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n-1}{u})$ determines exactly one isotropic direction $q \in K^{n-1}$ whose representative, if $P(n-2) \neq 0$, is of the form

$$v = \frac{1}{P(n-2)} \sum_{i=1}^{n-1} \overset{n-1}{P}_{n-1,i} \cdot \underset{i}{u} = v^n \cdot \left[q^1, q^2, \ldots, q^{n-1}, 1\right]^T \in \overset{0}{V} \cap L_{n-1}. \tag{12}$$

From $p(\underset{i}{u}, v) = 0$ for $i = 1, 2, \ldots, n-1$ it follows that each vector $\underset{i}{u}$ is of the form

$$\underset{i}{u} = \left[\underset{i}{u}^1, \underset{i}{u}^2, \ldots, \underset{i}{u}^{n-1}, \sum_{k=1}^{n-1} \underset{i}{u}^k q^k\right] \quad \text{where} \quad \Delta = \det[\underset{i}{u}^j]_1^{n-1} \neq 0. \tag{13}$$

Let us denote

$$B_r(\underset{1}{u}, \ldots, \underset{r}{u}, \ldots, \underset{n-1}{u}) = \begin{vmatrix} \underset{1}{u}^1 & \underset{1}{u}^2 & \ldots & \underset{1}{u}^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ \underset{r-1}{u}^1 & \underset{r-1}{u}^2 & \ldots & \underset{r-1}{u}^{n-1} \\ q^1 & q^2 & \ldots & q^{n-1} \\ \underset{r+1}{u}^1 & \underset{r+1}{u}^2 & \ldots & \underset{r+1}{u}^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ \underset{n-1}{u}^1 & \underset{n-1}{u}^2 & \ldots & \underset{n-1}{u}^{n-1} \end{vmatrix} \quad \text{for} \quad r = 1, 2, \ldots, n-1. \tag{14}$$

It is easy to prove that $B_r \cdot B_k = \overset{n-1}{P}_{rk}$, so at least one of B_r is different from zero. In [5] it was proved that

$$\bigwedge_{A \in G} B_r(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{n-1}{u}) = \varepsilon\,(A) \cdot B_r(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n-1}{u}) \tag{15}$$

and the general solution of the functional equation (5) was given, namely

$$F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}) = \begin{cases} 0 \quad \text{if } m < n-1 \text{ or } m = n-1 \text{ and } P\,(m) \neq 0, \\ \Theta\,(p_{ij}) \cdot B(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n-1}{u}) \text{ if } m = n-1 \text{ and } P\,(m) = 0, \\ \Theta\,(p_{ij}) \cdot \det(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}) \text{ if } m = n, \end{cases} \tag{16}$$

where Θ is an arbitrary function of $\frac{1}{2}m\,(m+1)$ variables and B is an arbitrary nonzero ε-scalar among $B_1, B_2, \dots, B_{n-1}$. In this paper we determine the general solution of the functional equations (6) and (7).

2. General solution of the functional equation (6)

Definition 3. We say that a system of vectors $\underset{1}{e}, \underset{2}{e}, \dots, \underset{n}{e}$ constitutes a pseudo-orthonormal base if

$$[p(\underset{i}{e}, \underset{j}{e})]_1^n = E_1.$$

To each vector $\underset{i}{e}$ of the pseudo-orthonormal base we assign the covector $\underset{i}{\overset{*}{e}} = \underset{i}{e}^T \cdot E_1$ and then $\underset{i}{\overset{*}{e}} \cdot \underset{j}{u} = p(\underset{i}{e}, \underset{j}{u})$.

Definition 4. We say that a pseudo-orthogonal matrix A whose rows consist of coordinates of covectors $\underset{1}{\overset{*}{e}}, \underset{2}{\overset{*}{e}}, \dots, \underset{n}{\overset{*}{e}}$ corresponds to the pseudo-orthonormal base $\underset{1}{e}, \underset{2}{e}, \dots, \underset{n}{e}$.

Let a sequence of linearly idependent vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}, \dots, \underset{n}{u}$ be given. Let us denote $\varepsilon_m = \operatorname{sign} P\,(m)$. Apparently $\varepsilon_n = -1$ and from the definition $\varepsilon_0 = +1$.

Definition 5. The sequence $(+1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_m, \dots, \varepsilon_{n-1}, -1)$ will be called the signature of the sequence of subspaces $L(\underset{1}{u}), L(\underset{1}{u}, \underset{2}{u}), \dots, L_m, \dots, L_n$.

In [7] it was proved that the only restriction is $\varepsilon_{i+1} \leqslant \varepsilon_i$.

Lemma 2. *The functional equation*

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{m}{u}) = \eta(A)\cdot F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u})$$

in the case $P\,(m) > 0$ has only the trivial solution $F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}) \equiv 0$.

P r o o f. Evidently $m \in \{1, 2, \dots, n-1\}$ and the partial signature up to ε_m is of the form $(+1, +1, \dots + 1)$. The vectors

$$\underset{k}{e} = \frac{1}{\sqrt{P\,(k-1)\cdot P\,(k)}}\cdot \sum_{i=1}^{k} \overset{k}{P}_{ki}\underset{i}{u} \quad \text{for } k = 1, 2, \dots m \tag{17}$$

constitute in the Euclidean subspace $L(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u})$ the first m vectors of a pseudo-orthonormal base. We have

$$\underset{k}{e} = \underset{k}{e}(\underset{1}{u}, \underset{2}{u}, \dots, \underset{k}{u}) \text{ and } p(\underset{k}{e}, \underset{r}{u}) = \begin{cases} 0 & \text{for } r < k, \\ \Theta\,(p_{ij}) & \text{for } k \leqslant r \leqslant m. \end{cases} \tag{18}$$

The other vectors $\underset{m+1}{e}, \underset{m+2}{e}, \dots, \underset{n}{e}$ of the pseudo-orthonormal base is chosen from the orthogonal complement $L_m^{\perp}$. Inserting the matrix A corresponding to this base into equation (6) we get

$$F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}) = \eta\,(A)\cdot F(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{m}{u}) = \eta\,(A)\cdot \Theta_1\,(p_{ij})\,.$$

For the matrix C corresponding to the base $\underset{1}{e}, \underset{2}{e}, \dots, \underset{n-1}{e}, -\underset{n}{e}$ we get

$$F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}) = \eta\,(C)\cdot F(C\underset{1}{u}, C\underset{2}{u}, \dots, C\underset{m}{u}) = -\eta\,(A)\cdot \Theta_1\,(p_{ij})\,.$$

Because in both cases the scalar $\Theta_1\,(p_{ij})$ is the same we conclude that $F \equiv 0$. □

In the case $P\,(m) \leqslant 0$ the partial signature up to ε_m is either of the form
$(*)$ $(+1, \dots, +1, -1 = \varepsilon_s, -1, \dots, -1)$ for $s = 1, 2, \dots, m \leqslant n$ or
$(**)$ $(+1, \dots, +1, 0 = \varepsilon_s, \varepsilon_{s+1}, \dots, \varepsilon_m)$ for $s = 1, 2, \dots, \min\{m, n-1\}$.

In both the cases we have $P\,(s-1) > 0$ and $P\,(s) \leqslant 0$. Using the vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}$ we get

$$v = v(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}) = \sum_{i=1}^{s} \overset{s}{P}_{si}\underset{i}{u} \in \overset{*}{V} \tag{19}$$

because $v \neq 0$ and $p\,(v, v) = P\,(s-1)\cdot P\,(s) \leqslant 0$.

Let us assume that $F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u})$ is a general solution of the functional equation (6) in the case $P\,(m) \leqslant 0$. We conclude that in this case the function $\operatorname{sign} v^n F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u})$ is the general solution of the functional equation (4). Using (8) we get

Theorem 1. *The general solution of the functional equation*

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = \eta\,(A) \cdot F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$$

is of the form

$$F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = \begin{cases} 0 \quad \textit{if} \quad m = 1, 2, \ldots, n-1 \textit{ and } P\,(m) > 0, \\ \Theta\,(p_{ij}) \cdot \operatorname{sign}(v^n) \quad \textit{if} \quad m = 1, 2, \ldots, n \textit{ and } P\,(m) \leqslant 0, \end{cases} \tag{20}$$

where Θ is an arbitrary function of $\frac{1}{2}m(m+1)$ variables and v is an arbitrary vector belonging to $L_m \cap \overset{*}{V}$.

3. General solution of the functional equation (7)

The homomorphisms $1, \varepsilon, \eta, \varepsilon \cdot \eta$ constitute a Klein group and the relations (16) and (20) enable us to determine the general solution of the functional equation (7).

Lemma 3. *The general solution of the functional equation* (7) *is of the form*

$$F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = \begin{cases} 0 \quad \textit{if } m < n-1 \textit{ or } m = n-1 \textit{ and } P\,(m) \neq 0, \\ \Theta\,(p_{ij}) \cdot B \cdot \operatorname{sign}(v^n) \ \textit{ if } m = n-1 \textit{ and } P\,(m) = 0, \\ \Theta\,(p_{ij}) \cdot \operatorname{sign}(v^n) \cdot \det(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n}{u}) \textit{ if } m = n, \end{cases} \tag{21}$$

where Θ is an arbitrary function of $\frac{1}{2}m(m+1)$ variables, v is an arbitrary non-Euclidean vector belonging to the subspace L_m and B is an arbitrary nonzero ε-scalar amongst $B_1, B_2, \ldots, B_{n-1}$.

In the formulas (20) and (21) we can use in particular the vector v described by (19) whereas the ε-scalar B in (21) is not determined. However, there exist relations between v^n in (12), B_{n-1} in (14) and Δ in (13). If $P\,(n-2) \neq 0$ we have the formula (12) and $B_{n-1} \neq 0$. In what follows

$$\begin{aligned} v^n \cdot B_{n-1} &= \frac{B_{n-1}}{P\,(n-2)} \sum_{i=1}^{n-1} \overset{n-1}{P}{}_{n-1,i} \underset{i}{u}^n = \frac{1}{B_{n-1}} \sum_{i=1}^{n-1} B_{n-1} \cdot B_i \Big(\sum_{j=1}^{n-1} \underset{i}{u}^j q^j \Big) \\ &= \sum_{j=1}^{n-1} q^j \Big(\sum_{i=1}^{n-1} B_i \underset{i}{u}^j \Big) = \sum_{j=1}^{n-1} q^j \left(\Delta \cdot q^j\right) = \Delta(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n-1}{u}). \end{aligned}$$

We conclude that

$$\bigwedge_{A\in G} \Delta(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{n-1}{u}) = \varepsilon\,(A) \cdot w\,(q, A) \cdot \Delta(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n-1}{u}). \tag{22}$$

We give a more convenient version of this result.

Theorem 2. *The general solution of the functional equation*

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = \varepsilon(A)\cdot\eta(A)\cdot F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$$

is of the form

$$\text{(23)}\quad F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = \begin{cases} 0 \quad \textit{if} \;\; m < n-1 \;\; \textit{or} \;\; m = n-1 \;\; \textit{and} \;\; P(m) \neq 0, \\ \Theta(p_{ij})\cdot \operatorname{sign}\Delta(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n-1}{u}) \;\; \textit{if} \;\; m = n-1 \;\; \textit{and} \;\; P(m) = 0, \\ \Theta(p_{ij})\cdot \operatorname{sign}(v^n)\cdot \det(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n}{u}) \;\; \textit{if} \;\; m = n, \end{cases}$$

where Θ is an arbitrary function of $\frac{1}{2}m(m+1)$ variables and v is an arbitrary non-Euclidean vector.

References

[1] *J. Aczél, S. Gołąb*: Functionalgleichungen der Theorie der geometrischen Objekte. Panstwowe Wydawnietvo Naukove, Warszawa, 1960. zbl

[2] *L. Bieszk, E. Stasiak*: Sur deux formes équivalents de la notion de (r, s)-orientation de la géométrie de Klein. Publ. Math. Debrecen *35* (1988), 43–50.

[3] *E. Kasparek*: The homomorphisms of the pseudo-orthogonal group of index one into an abelian group. Demonstratio Math. *22* (1989), 763–771. zbl

[4] *M. Kucharzewski*: Über die Grundlagen der Kleinschen Geometrie. Period. Math. Hungar. *8* (1977), 83–89. zbl

[5] *A. Misiak, E. Stasiak*: Equivariant maps between certain G-spaces with $G = O(n-1, 1)$. Math. Bohem. *126* (2001), 555–560. zbl

[6] *E. Stasiak*: O pewnym działaniu grupy pseudoortogonalnej o indeksie jeden $O(n, 1, \mathbb{R})$ na sferze S^{n-2}. Prace Naukowe P.S. *485* (1993).

[7] *E. Stasiak*: Scalar concomitants of a system of vectors in pseudo-Euclidean geometry of index 1. Publ. Math. Debrecen *57* (2000), 55–69. zbl

Author's address: *Barbara Glanc*, *Aleksander Misiak*, *Maria Szmuksta-Zawadzka*, Instytut Matematyki, Politechnika Szczecińska, Al. Piastów 17, 70-310 Szczecin, Poland, e-mail: `misiak@ps.pl`.