SOME INITIAL BOUNDARY PROBLEMS IN ELECTRODYNAMICS FOR CANONICAL DOMAINS IN QUATERNIONS

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Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. The initial boundary-transmission problems for electromagnetic fields in homogeneous and anisotropic media for canonical semi-infinite domains, like halfspaces, wedges and the exterior of half- and quarter-plane obstacles are formulated with the use of complex quaternions. The time-harmonic case was studied by A. Passow in his Darmstadt thesis 1998 in which he treated also the case of an homogeneous and isotropic layer in free space and above an ideally conducting plane. For thin layers and free space on the top a series of generalized vectorial Leontovich boundary value conditions were deduced and systems of Wiener-Hopf pseudo-differential equations for the tangential components of the electric and magnetic field vectors and their jumps across the screens were formulated as equivalent unknowns in certain anisotropic boundary Sobolev spaces. Now these results may be formulated with alternating differential forms in Lorentz spaces or with complex quaternions.

Keywords: electromagnetic fields by complex quaternions, initial boundary transmission problems for semi-infinite domains, reduction to Wiener-Hopf pseudo-differential systems, anisotropic Leontovitch boundary conditions

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1. Introduction

Since more than one hundred years ago, Maxwell's theory of electromagnetic wave propagation has attracted a growing interest of scientists and engineers. The main results of existence and uniqueness of solutions to boundary and transmission value problems for piecewise smooth scattering objects have been understood and efficient numerical schemes have been developed. Lipschitz continuous boundaries for Maxwell's equations could be of different types: a wedge, a cone, the exterior of a half-plane, etc. The semi-infinite domains, so-called *canonical obstacles*, and the corresponding scattering problems for acoustic, electromagnetic, and elastodynamic wave fields have been treated by many authors mainly under the assumptions of homogeneous and isotropic media which allow to make use of integral transformations and the Wiener-Hopf technique or the *Maliuzhinets method* [9]. Bounded domains and their unbounded complements were considered in [7], etc. A vast list of references on this subject is given in [15].

At the same time in many areas of electronic microwave technique (antennas, printed chips) there exists a demand for a handy theory for materials consisting of dielectrics with very thin conductors on the top (like strips, or screens, or "needles"). High frequency approximations, e.g. the geometric optics' method, have been developed basing on *Leontovich boundary conditions* [8]. The time-harmonic cases for acoustic, elastodynamic and electrodynamic wave fields for Sommerfeld-halfplanes and wedges have been treated in particular by members of the Darmstadt research group [4], [10], [12], [16]. A. Passow [15] got interesting results for time-harmonic electrodynamic waves for the canonical case of a Sommerfeld-halfplane with anisotropic Leontovich boundary conditions on both sides of this semi-infinite obstacle. He transformed the boundary-transmission problem into equivalent systems of Wiener-Hopf equations for the jumps of the vector fields. A general scheme for hyperbolic systems was developed by K. Rottbrand [16]. He solved aperiodic problems with very general time-profiles of the incident waves leading to boundary value problems on thin screens, strips, wedges generalizing the explicitly solved canonical ones [13].

The great success in electro- and magnetostatics on the background of potential theory, and particularly on complex function theory, initiated the studies in a quaternion approach already at the beginning of the last century [18]. Now the fast development of Clifford analysis leads to a geometric algebra view of electrodynamics ([1], [6], [11] and others). We use this approach to some open problems for canonical semi-infinite domains concerning the asymptotic behaviour of the scattered waves near singularities of the obstacles, for short and long times, and when a time-harmonic plane wave, which hits the obstacle, has a small wave-length compared with the diameter of the domain.

In the present paper we use quaternions for the impedance type initial boundary value problems. A short introduction into quaternion algebra and calculus is given. Then we consider Maxwell's equations for inhomogeneous and anisotropic materials by generalized Dirac operators. Introducing the four-potential for divergence-free fields we get the field equations as generalized d'Alembert's wave equations. Application of Fourier-Laplace transformation reduces the problem to parameter dependent ordinary differential equations.

Real quaternions. We start with a short review of quaternions' properties basing on [2], [5], [14], [17]. Let $\mathcal{H}$ denote the set of real quaternions

$$A = a_0\mathbf{1} + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}, \tag{1}$$

where a_0, a_1, a_2, a_3 are real numbers (or functions) and $\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the basis elements of $\mathcal{H}$ with the multiplication rules

$$\mathbf{1}^2 = 1, \quad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \tag{2}$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}. \tag{3}$$

Using (1) we can consider a quaternion A as a 4D real column-vector $\boldsymbol{v}_A$ with components a_0, a_1, a_2, a_3, i.e. $\mathcal{H} \cong \mathbb{R}^4$, or as a symbolic sum of a real number a_0 and a 3D real column-vector $\boldsymbol{a} = (a_1, a_2, a_3)^T$, that is $A \cong a_0 + \boldsymbol{a}$. As usual, "$T$" means transposition. $\mathrm{Sc}(A) = a_0$ is the *scalar part* of A and the sum of all other terms is the *vector part* $\mathrm{Vec}(A)$.[1] If $\mathrm{Sc}(A) = 0$ the quaternion A is called *pure*. The set of pure quaternions is isomorphic to the set of real 3D vectors. Starting from now we will not make difference between 3D vectors and the corresponding pure quaternions. More about relations between quaternions and vectors see in [14].

According to (2)–(3), the multiplication of real quaternions A and B gives

$$AB = a_0b_0 - (\boldsymbol{a}, \boldsymbol{b}) + a_0\boldsymbol{b} + b_0\boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b} \tag{4}$$

where $(\boldsymbol{a}, \boldsymbol{b})$ is the scalar and $\boldsymbol{a} \times \boldsymbol{b}$ is the vector product of vectors. In the case of pure quaternions we have $AB = -(\boldsymbol{a}, \boldsymbol{b}) + \boldsymbol{a} \times \boldsymbol{b}$. Clearly, $2(\boldsymbol{a} \times \boldsymbol{b}) = AB - BA$ and $2(\boldsymbol{a}, \boldsymbol{b}) = -(AB + BA)$.

The quaternion $A^* = a_0 - \boldsymbol{a}$ is called the *conjugate* quaternion. Multiplication of A and A^* leads to the *norm of A squared*

$$AA^* = A^*A = a_0^2 + a_1^2 + a_2^2 + a_3^2 = \|A\|^2, \tag{5}$$

if the corresponding $\mathbb{R}^4$ is endowed with the usual Euclidean metric.

Quaternions may be isomorphically represented by real (4×4) matrices of a special structure

$$A \cong G_1(A) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} a_0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & a_0 I_3 + K(\boldsymbol{a}) \end{pmatrix} \tag{6}$$

[1] $\mathrm{Sc}(A)$ and $\mathrm{Vec}(A)$ are also called the *real* ($\mathrm{Re}\,A$) and *imaginary* ($\mathrm{Im}\,A$) parts of A. We will not use this notation to avoid ambiguity in the case of complex quaternions.

where I_3 is the (3×3) unit matrix and $K(\boldsymbol{a})$ is the skew-symmetric matrix

$$K(\boldsymbol{a}) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}. \tag{7}$$

Clearly, $K(\boldsymbol{a})\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$ and $K(\boldsymbol{a})\boldsymbol{a} = 0$, $K^T(\boldsymbol{a}) = -K(\boldsymbol{a})$, $K(\boldsymbol{a})\boldsymbol{b} = -K(\boldsymbol{b})\boldsymbol{a}$.

The product of real quaternions A and B can be written in the matrix form

$$AB \cong \boldsymbol{v}_{AB} = G_1(A)\boldsymbol{v}_B \tag{8}$$

where $\boldsymbol{v}_{AB}, \boldsymbol{v}_B \in \mathbb{R}^4$. The matrix $G_1(A)$ can be split additively: $G_1(A) = a_0 I_4 + |\boldsymbol{a}|^2 S_A$ with the (4×4) unit matrix I_4 and a skew symmetric orthogonal matrix S_A. The linear transformation $y = G_1(A)x$, $y, x \in \mathbb{R}^4$ describes a similarity transformation in $\mathbb{R}^4$, i.e. a rotation in $\mathbb{R}^4$ followed by an isotropic stretching by $\|A\|^2$.

For $A \neq 0$ there exists a uniquely defined, both-sided inverse $A^{-1} = A^*/\|A\|^2$.

Complex quaternions. Let $\mathcal{M}$ be the complex extension of real quaternions $\mathcal{M} = \{M = A + \mathrm{i}B;\ A, B \in \mathcal{H},\ \mathrm{i} = \sqrt{-1}\}$. We emphasize that i is different from the basis element $\boldsymbol{i}$ of $\mathcal{H}$ and assume that i commutes with the basis elements of $\mathcal{H}$.

Evidently, $\mathcal{M} = \mathcal{H} \oplus \mathrm{i}\mathcal{H} \widehat{=} \mathcal{H}^2$ with addition and multiplication obeying the common rules of complexified real vector spaces. The multiplication rules for the coefficients are taken from (2)–(3). It is possible to prove that $\mathcal{M}$ is an algebra (sometimes it is called Pauli algebra).

For complex quaternions $M = A + \mathrm{i}B$ we consider the Clifford involution $M^* = A^* + \mathrm{i}B^*$ and the complex involution $\overline{M} = A - \mathrm{i}B$. In the cases of real quaternions and complex numbers these coincide with the usual definitions. It is clear that $(\overline{M^*}) = (\overline{M})^*$ and for complex quaternions M and N one has $(MN)^* = N^* M^*$ and $\overline{MN} = \overline{M}\,\overline{N}$.

Quaternion-valued functions. Dirac operator. To represent Maxwell's field equations we need complex quaternion-valued functions $W(X)$ of real quaternion variables $X = x_0\mathbf{1} + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$. The variables $\boldsymbol{x} = (x_1, x_2, x_3)^T$ are identified with the three space variables while x_0 corresponds to time.

Let $W(X)$ be a complex quaternion-valued function with the variable X in a time cylinder domain $\Omega_T = (0, T) \times \Omega$, with $\Omega \subset \mathbb{R}^3$:

$$W(X) = w_0(x_0, \boldsymbol{x})\mathbf{1} + w_1(x_0, \boldsymbol{x})\boldsymbol{i} + w_2(x_0, \boldsymbol{x})\boldsymbol{j} + w_3(x_0, \boldsymbol{x})\boldsymbol{k} \tag{9}$$

The Dirac operator $\mathcal{D}$ acting on the complex quaternion-valued functions $W(X)$ is determined as follows. Using the "quaternion" notation

$$\mathcal{D} = \boldsymbol{i}\partial_{x1} + \boldsymbol{j}\partial_{x2} + \boldsymbol{k}\partial_{x3} \tag{10}$$

we can write the left Dirac operator as the "product" of $\mathcal{D}$ and W

$$\mathcal{D}W = -\operatorname{div} \boldsymbol{w} + \operatorname{grad} w_0 + \operatorname{rot} \boldsymbol{w} \tag{11}$$

and the right Dirac operator

$$W\mathcal{D} = -\operatorname{div} \boldsymbol{w} + \operatorname{grad} w_0 - \operatorname{rot} \boldsymbol{w} \tag{12}$$

where, as usual, $\operatorname{grad} w_0 = (\partial_{x1} w_0, \partial_{x2} w_0, \partial_{x3} w_0)^T$, $\operatorname{div} \boldsymbol{w} = \partial_{x1} w_1 + \partial_{x2} w_2 + \partial_{x3} w_3$, and $\operatorname{rot} \boldsymbol{w} = (\partial_{x1}, \partial_{x2}, \partial_{x3})^T \times \boldsymbol{w}$.

It follows from (11) and (12) that for pure quaternion-valued functions W one gets

$$\operatorname{div} W = -\frac{\mathcal{D}W + W\mathcal{D}}{2}; \quad \operatorname{rot} W = \frac{\mathcal{D}W - W\mathcal{D}}{2}. \tag{13}$$

Using the matrix representation (6) we can rewrite (11) in the *quaternion matrix differential operator form*

$$\mathcal{D}W \cong G_1(\mathcal{D})W = \begin{pmatrix} 0 & -\operatorname{div} \\ \operatorname{grad} & \operatorname{rot} \end{pmatrix} \begin{pmatrix} w_0 \\ \boldsymbol{w} \end{pmatrix}. \tag{14}$$

Differentiable quaternion-valued functions $W(X)$ in a 4D-domain are called *left (*or *right) monogenic*, if they belong to the kernel of the left (or right) operator $\mathcal{D}$. Hence, for left monogenic complex quaternion-valued functions we have

$$\begin{aligned} -\partial_{x1} w_1 - \partial_{x2} w_2 - \partial_{x3} w_3 = 0; \quad & \partial_{x1} w_0 - \partial_{x3} w_2 + \partial_{x2} w_3 = 0; \\ \partial_{x2} w_0 + \partial_{x3} w_1 - \partial_{x1} w_3 = 0; \quad & \partial_{x3} w_0 - \partial_{x2} w_1 + \partial_{x1} w_2 = 0. \end{aligned} \tag{15}$$

This is a generalization of the well-known Cauchy-Riemann system for complex-valued holomorphic functions.

Basing on (10) it is easy to get $\mathcal{D}\mathcal{D}^* = \mathcal{D}^*\mathcal{D} = \|\mathcal{D}\|^2 = \triangle_3$, where $\triangle_3$ is the Laplacian operator $\triangle_3 = \partial_{x1}^2 + \partial_{x2}^2 + \partial_{x3}^2$.

We also consider the Dirac-type operator $\mathcal{D}[t] = (1/\mathrm{i}c)\partial_t + \mathcal{D}$, where c is the speed of light in vacuum. If $\mathcal{D}^*[t] = (1/\mathrm{i}c)\partial_t - \mathcal{D}$ is the Clifford involution, then $\mathcal{D}[t]\mathcal{D}^*[t] = \mathcal{D}^*[t]\mathcal{D}[t] = \triangle_3 - (1/c^2)\partial_{tt}^2$ is d'Alembert's wave operator.

General form of Maxwell's equations. Electromagnetic wave-propagation is governed by Maxwell's equations holding in a time cylinder domain Ω_T with $0 < T \leqslant \infty$. The general form of Maxwell's equations for inhomogeneous and anisotropic media is

$$\operatorname{rot} \boldsymbol{E} + \partial_t \boldsymbol{B} = \boldsymbol{J}', \tag{16}$$

$$\operatorname{rot} \boldsymbol{H} - \partial_t \boldsymbol{D} = \boldsymbol{J}, \tag{17}$$

$$\operatorname{div} \boldsymbol{B} = \varrho', \tag{18}$$

$$\operatorname{div} \boldsymbol{D} = \varrho \tag{19}$$

with the electric vector field $\boldsymbol{E}$, the magnetic vector field $\boldsymbol{H}$, the bivectors[2] of magnetic induction $\boldsymbol{B}$, and dielectric displacement $\boldsymbol{D}$. The right-hand sides of (16)–(19) are the given current density bivector field $\boldsymbol{J}$, the space-charge density ϱ, the magnetic density bivector field $\boldsymbol{J}'$, and the magnetic density ϱ'. The last two are usually assumed to vanish.

For inhomogeneous and anisotropic media we have

$$\boldsymbol{B} = \boldsymbol{\mu} \boldsymbol{H}; \qquad \boldsymbol{D} = \boldsymbol{\varepsilon} \boldsymbol{E} \tag{20}$$

where the Hermitean (3×3) matrices correspond to the permitivity ($\boldsymbol{\varepsilon}$) and to the permeability ($\boldsymbol{\mu}$).

In order to solve the system (16)–(19) one has to add initial conditions at $t = 0$ and boundary conditions on $\partial\Omega_T = [0, T) \times \partial\Omega$, or transmission conditions on interfaces $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$ of subdomains $\Omega_j \subset \Omega$ with boundaries $\partial\Omega_j$. The most common boundary conditions for ideally conducting surfaces $\partial\Omega$ are

$$\boldsymbol{n} \times \boldsymbol{E}_{\mathrm{tot}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega \tag{21}$$

and additionally

$$(\boldsymbol{n}, \boldsymbol{D}_{\mathrm{tot}}) = \tau_n \quad \text{on} \quad (0, T) \times \partial\Omega \tag{22}$$

where $\boldsymbol{E}_{\mathrm{tot}}$ and $\boldsymbol{D}_{\mathrm{tot}}$ are the corresponding total fields, $\boldsymbol{n}$ is the outer normal to $\partial\Omega$, and τ_n is a prescribed surface charge density.

[2] The difference between *vectors* and *bivectors* see [1].

At interfaces $\Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k \neq 0$, $j \neq k$, being parts of joint boundaries the integral forms of Maxwell's equations and the corresponding linear constitutive equations lead to a system of equations corresponding to the case of approaching boundary points from the interior parts of Ω_j. Sophisticated investigations have been done [11] for electromagnetic fields in the time-harmonic case for Lipschitz domains Ω_j and L^p-integrable traces on $\partial\Omega_j$, $1 < p < \infty$.

The general Maxwell's equations can be written in the complex form

$$\operatorname{div}(\boldsymbol{\varepsilon}\boldsymbol{E} + \mathrm{i}\,\boldsymbol{\mu}\boldsymbol{H}) = \varrho + \mathrm{i}\,\varrho', \tag{23}$$

$$\frac{1}{\mathrm{i}}\frac{\partial}{\partial t}(\boldsymbol{\varepsilon}\boldsymbol{E} + \mathrm{i}\,\boldsymbol{\mu}\boldsymbol{H}) + \operatorname{rot}(\boldsymbol{E} + \mathrm{i}\,\boldsymbol{H}) = \boldsymbol{J}' + \mathrm{i}\,\boldsymbol{J}. \tag{24}$$

Homogeneous isotropic media. In the special case of a homogeneous isotropic medium $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ simplify to εI_3 and μI_3, with the unit (3×3) matrix I_3 and scalar constants ε and μ. In the case of vacuum we denote these by ε_0, μ_0 and have $c^2 = (\varepsilon_0\mu_0)^{-1}$ as the velocity of light propagation.

Introducing field vectors $\sqrt{\varepsilon_0}\,\boldsymbol{E}$ and $\sqrt{\mu_0}\,\boldsymbol{H}$ and a complex *Faraday bivector* $\boldsymbol{F} = \sqrt{\varepsilon_0}\boldsymbol{E} + \mathrm{i}\sqrt{\mu_0}\boldsymbol{H}$, after pulling out $\sqrt{\varepsilon_0\mu_0}$ the system (24) becomes

$$\operatorname{div}\boldsymbol{F} = \frac{\varrho}{\sqrt{\varepsilon_0}} + \mathrm{i}\frac{\varrho'}{\sqrt{\mu_0}}. \tag{25}$$

$$\frac{1}{\mathrm{i}\,c}\frac{\partial}{\partial t}\boldsymbol{F} + \operatorname{rot}\boldsymbol{F} = \sqrt{\varepsilon_0}\,\boldsymbol{J}' + \mathrm{i}\sqrt{\mu_0}\,\boldsymbol{J}. \tag{26}$$

In quaternions this yields

$$\mathcal{D}[t]F = R \tag{27}$$

where F, R are complex quaternions corresponding to the unknown vector $\boldsymbol{F}$ and to the given right-hand side of (25)–(26), respectively.

If F is twice differentiable (may be only in the weak sense) and R is once differentiable, we get after applying $\mathcal{D}^*[t]$

$$\mathcal{D}^*[t]\mathcal{D}[t]F = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}F + \triangle_3 F = \mathcal{D}^*[t]R. \tag{28}$$

This result is well-known from electrodynamics [3].

Inhomogeneous anisotropic media. In the case of inhomogeneous and anisotropic media we cannot pull $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ out of the space equation (24).

Matrices $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are Hermitean conjugate with positive eigenvalues (in general depending on the points $\boldsymbol{x} \in \Omega$) and one is able to define uniquely functions $\boldsymbol{\varepsilon}^{-1}$, $\sqrt{\boldsymbol{\varepsilon}}$, etc.

Let us introduce operators acting on quaternion-valued functions $W = w_0 + \boldsymbol{w}$ as follows: the weighted time operator $\mathcal{T}_{\boldsymbol{\sigma}}$

$$
\mathcal{T}_{\boldsymbol{\sigma}} W = \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} w_0 \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} \partial_t w_0 \\ \partial_t \boldsymbol{\sigma} \boldsymbol{w} \end{pmatrix} \tag{29}
$$

with a (3×3) non-degenerate matrix $\boldsymbol{\sigma}$, and the generalized Dirac operator $\mathcal{D}_{\boldsymbol{\sigma}}$

$$
\mathcal{D}_{\boldsymbol{\sigma}} W = \begin{pmatrix} 0 & -\operatorname{div} \boldsymbol{\sigma} \\ \operatorname{grad} & \operatorname{rot} \end{pmatrix} \begin{pmatrix} w_0 \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} -\operatorname{div}(\boldsymbol{\sigma} \boldsymbol{w}) \\ \operatorname{grad} w_0 + \operatorname{rot}(\boldsymbol{w}) \end{pmatrix}. \tag{30}
$$

Then we can rewrite Maxwell's equations (16)–(19) in quaternions with respect to unknown pure quaternions E and H corresponding to the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$
\begin{aligned}
&\mathcal{D}_{\boldsymbol{\varepsilon}} E + \mathcal{T}_{\boldsymbol{\mu}} H = R_1, \\
&\mathcal{D}_{\boldsymbol{\mu}} H - \mathcal{T}_{\boldsymbol{\varepsilon}} E = R_2
\end{aligned} \tag{31}
$$

where quaternions $R_1 = \varrho + \boldsymbol{J}'$ and $R_2 = \varrho' + \boldsymbol{J}$ are given.

If we apply the time operator $\mathcal{T}_{\boldsymbol{\varepsilon}}$ to the first equation in (31) and $\mathcal{T}_{\boldsymbol{\mu}}$ to the second equation, we get

$$
\begin{aligned}
\mathcal{T}_{\boldsymbol{\varepsilon}} \mathcal{T}_{\boldsymbol{\mu}} H &= -\mathcal{T}_{\boldsymbol{\varepsilon}} \mathcal{D}_{\boldsymbol{\varepsilon}} E + \mathcal{T}_{\boldsymbol{\varepsilon}} R_1, \\
\mathcal{T}_{\boldsymbol{\mu}} \mathcal{T}_{\boldsymbol{\varepsilon}} E &= \mathcal{T}_{\boldsymbol{\mu}} \mathcal{D}_{\boldsymbol{\mu}} H + \mathcal{T}_{\boldsymbol{\mu}} R_2.
\end{aligned} \tag{32}
$$

Applying the operator $\mathcal{D}_{\boldsymbol{\mu}}$ to the first equation in (31) and $\mathcal{D}_{\boldsymbol{\varepsilon}}$ to the second equation, we get

$$
\begin{aligned}
\mathcal{D}_{\boldsymbol{\mu}} \mathcal{D}_{\boldsymbol{\varepsilon}} E &= -\mathcal{D}_{\boldsymbol{\mu}} \mathcal{T}_{\boldsymbol{\mu}} H + \mathcal{D}_{\boldsymbol{\mu}} R_1, \\
\mathcal{D}_{\boldsymbol{\varepsilon}} \mathcal{D}_{\boldsymbol{\mu}} H &= \mathcal{D}_{\boldsymbol{\varepsilon}} \mathcal{T}_{\boldsymbol{\varepsilon}} E + \mathcal{D}_{\boldsymbol{\varepsilon}} R_2.
\end{aligned} \tag{33}
$$

Since $\mathcal{T}_{\boldsymbol{\sigma}_1}$ and $\mathcal{D}_{\boldsymbol{\sigma}_2}$ are commuting for arbitrary non-degenerate matrices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ not depending on time, we finally have

$$
\begin{aligned}
(\mathcal{T}_{\boldsymbol{\mu}} \mathcal{T}_{\boldsymbol{\varepsilon}} + \mathcal{D}_{\boldsymbol{\mu}} \mathcal{D}_{\boldsymbol{\varepsilon}}) E &= \mathcal{D}_{\boldsymbol{\mu}} R_1 + \mathcal{T}_{\boldsymbol{\mu}} R_2, \\
(\mathcal{T}_{\boldsymbol{\varepsilon}} \mathcal{T}_{\boldsymbol{\mu}} + \mathcal{D}_{\boldsymbol{\varepsilon}} \mathcal{D}_{\boldsymbol{\mu}}) H &= \mathcal{T}_{\boldsymbol{\varepsilon}} R_1 + \mathcal{D}_{\boldsymbol{\varepsilon}} R_2.
\end{aligned} \tag{34}
$$

We call this system *generalized inhomogeneous anisotropic wave equations.*

Formulation of Maxwell's equations by quaternion-valued potentials. Under the condition of a source-free magnetic density field, i.e. $\varrho' = 0$ in Ω_T, we may introduce a vector potential $\boldsymbol{A}$, uniquely defined in simply connected domains $\Omega \subset \mathbb{R}^3$ up to a smooth gradient field ψ by

$$\boldsymbol{B} = \boldsymbol{\mu}\boldsymbol{H} = \operatorname{rot} \boldsymbol{A} = \operatorname{rot}(\boldsymbol{A}' + \operatorname{grad} \psi) \tag{35}$$

and obtain from the first Maxwell's equation (16)

$$\boldsymbol{E} = -\partial_t \boldsymbol{A} - \operatorname{grad} V \tag{36}$$

with a scalar potential $V\colon \Omega_T \to \mathbb{R}$. It is well-known that by an appropriate choice of $\boldsymbol{A}'$ and ψ the *dynamical Lorenz condition* holds in Ω_T:

$$\operatorname{div} \boldsymbol{A} + \partial_t V = 0. \tag{37}$$

Using the grad of this equation one gets

$$\operatorname{grad} \operatorname{div} \boldsymbol{A} + \partial_t \operatorname{grad} V = 0. \tag{38}$$

The second Maxwell's equation (17) can be written as

$$\operatorname{rot}(\boldsymbol{\mu}^{-1} \operatorname{rot} \boldsymbol{A}) + \partial_t(\boldsymbol{\varepsilon}(\partial_t \boldsymbol{A} + \operatorname{grad} V)) = \boldsymbol{J}. \tag{39}$$

Multiplying (39) by $\boldsymbol{\mu}$, assuming the symmetric matrices $\boldsymbol{\mu}, \boldsymbol{\varepsilon}$ to be time-independent, and using (38) one gets

$$\boldsymbol{\mu} \operatorname{rot}(\boldsymbol{\mu}^{-1} \operatorname{rot} \boldsymbol{A}) + \boldsymbol{\mu}\boldsymbol{\varepsilon}\partial_t^2 \boldsymbol{A} - \boldsymbol{\mu}\boldsymbol{\varepsilon} \operatorname{grad} \operatorname{div} \boldsymbol{A} = \boldsymbol{\mu}\boldsymbol{J}. \tag{40}$$

The equation (40) is a *generalized anisotropic and inhomogeneous vectorial d'Alembert's wave equation.* This equation is a vector-potential variant of (31).

Introducing $\boldsymbol{E}$ from (36) into the charge density equation leads to

$$-\operatorname{div}(\boldsymbol{\varepsilon}(\partial_t \boldsymbol{A} + \operatorname{grad} V)) = \varrho. \tag{41}$$

Applying div to (39) gives

$$-\partial_t \varrho = \operatorname{div}(\boldsymbol{\varepsilon}(\partial_t^2 \boldsymbol{A} + \partial_t \operatorname{grad} V)) = \operatorname{div} \boldsymbol{J}. \tag{42}$$

It follows from (38) that

$$\operatorname{div}(\boldsymbol{\varepsilon}\partial_t \operatorname{grad} V) = -\operatorname{div}(\boldsymbol{\varepsilon} \operatorname{grad}\operatorname{div} \boldsymbol{A}). \tag{43}$$

Introducing it into (42) one gets the relationship

$$\operatorname{div}(\boldsymbol{\varepsilon}(\partial_t^2 \boldsymbol{A} - \operatorname{grad}\operatorname{div} \boldsymbol{A})) = \operatorname{div} \boldsymbol{J}. \tag{44}$$

This gives a second equation depending only on $\boldsymbol{A}$.

Formulation of boundary conditions by quaternion-valued potentials. Now we are going to formulate some initial boundary problems for electromagnetic wave-fields in half-spaces $\mathbb{R}^3_{\pm}$ and a slab $\mathbb{R}^3_h := \{(x_1, x_2, x_3) \in \mathbb{R}^3; x' := (x_1, x_2) \in \mathbb{R}^2; 0 < x_3 < h\}$. We assume the total fields to be split according to the incoming and scattered parts $\boldsymbol{E}_{\text{tot}} = \boldsymbol{E}_{\text{inc}} + \boldsymbol{E}_{\text{sc}}$, $\boldsymbol{H}_{\text{tot}} = \boldsymbol{H}_{\text{inc}} + \boldsymbol{H}_{\text{sc}}$.

Introducing the vector and scalar potentials from (36) the boundary conditions (21)–(22) become

$$\boldsymbol{n} \times (\partial_t \boldsymbol{A} + \operatorname{grad} V) = \boldsymbol{n} \times \boldsymbol{E}_{\text{inc}}, \tag{45}$$

$$(\boldsymbol{n},\ \boldsymbol{\varepsilon}(\partial_t \boldsymbol{A} + \operatorname{grad} V)) = (\boldsymbol{n},\ \boldsymbol{D}_{\text{inc}}) + 2\tau_n. \tag{46}$$

We can assume without any loss of generality that the outer normal $\boldsymbol{n}$ has coordinates $(0, 0, 1)$ and the corresponding quaternion is denoted by N. Clearly, $N^2 = -1$. Then, (45) and (46) in quaternions give

$$-(NE_{\text{sc}} - E_{\text{sc}}N) = NE_{\text{inc}} - E_{\text{inc}}N, \tag{47}$$

$$-(ND_{\text{sc}} + D_{\text{sc}}N) = ND_{\text{inc}} + D_{\text{inc}}N + \tau_n. \tag{48}$$

We may now add up (47) and (48) multiplied by N from the right:

$$\begin{aligned} &NE_{\text{sc}} - E_{\text{sc}}N + (ND_{\text{sc}} + D_{\text{sc}}N)N \\ &\qquad = -(NE_{\text{inc}} - E_{\text{inc}}N) - (ND_{\text{inc}} + D_{\text{inc}}N)N + \tau_n N \end{aligned} \tag{49}$$

with a given right-hand side.

Correspondingly, for the magnetic field we have

$$\boldsymbol{n} \times \boldsymbol{\mu}^{-1} \operatorname{rot} \boldsymbol{A} = -\boldsymbol{n} \times \boldsymbol{\mu}^{-1} \boldsymbol{B}_{\text{inc}}, \tag{50}$$

$$(\boldsymbol{n},\ \operatorname{rot} \boldsymbol{A}) = -(\boldsymbol{n},\ \boldsymbol{B}_{\text{inc}}) + \tau_n' \tag{51}$$

or in quaternions

$$-(NH_{\mathrm{sc}} - H_{\mathrm{sc}}N) = NH_{\mathrm{inc}} - H_{\mathrm{inc}}N, \tag{52}$$

$$-(NB_{\mathrm{sc}} + B_{\mathrm{sc}}N) = NB_{\mathrm{inc}} + B_{\mathrm{inc}}N + 2\tau_n{}'. \tag{53}$$

More general boundary conditions of the *anisotropic Leontovich type* for the time-harmonic case ($\sim \mathrm{e}^{\mathrm{i}\omega t}$) were studied in [15] and partly published in [12]. These conditions are

$$-\boldsymbol{n} \times \boldsymbol{E}_{\mathrm{sc}} - \Xi(\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{H}_{\mathrm{sc}})) = \boldsymbol{T}_{\mathrm{tang}} \tag{54}$$

with a given tangential field $\boldsymbol{T}_{\mathrm{tang}}$ on the plane boundary $\partial\mathbb{R}^3_{\pm}$ arising from the incoming wave-fields. The (3×3) matrix

$$\Xi = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{55}$$

is called the *impedance matrix*. It is assumed to be constant (or piecewise constant) on $\partial\mathbb{R}^3_{\pm}$.

In potentials (54) becomes

$$\boldsymbol{n} \times (\partial_t \boldsymbol{A} + \operatorname{grad} V) + \Xi(\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\mu}^{-1} \operatorname{rot} \boldsymbol{A})) = -\boldsymbol{T}_{\mathrm{tang}} \tag{56}$$

or in quaternions

$$-(NE_{\mathrm{sc}} - E_{\mathrm{sc}}N) + \Xi(H_{\mathrm{sc}} + NH_{\mathrm{sc}}N) = 2T_{\mathrm{tang}} \tag{57}$$

where T_{tang} is the quaternion corresponding to the vector $\boldsymbol{T}_{\mathrm{tang}}$. We see that (57) is similar to (49). The condition in the $\boldsymbol{n}$-direction is obtained from (41) by multiplying it by the quaternion N from the right:

$$\varepsilon_{31}(\partial_t A_1 + \partial_{x1} V)N + \varepsilon_{32}(\partial_t A_2 + \partial_{x2} V)N + \varepsilon_{33}(\partial_t A_3 + \partial_{x3} V)N = -\varrho N. \tag{58}$$

The right-hand sides in (57) and (58) are known, and the left-hand sides contain first order partial differential operators acting on four dimensional fields arising in a linear way from the four-potential $(V, \boldsymbol{A})$ as real quaternion-valued functions.

Application of Fourier and Laplace transformations. To solve the equations for the above mentioned *canonical domains*, like the two half-spaces $\mathbb{R}^3_{\pm}$ or the exterior to a Sommerfeld half-plane or a right-angle wedge, it is efficient to use a 2D Fourier transformation $\mathcal{F}$ with respect to $x' = (x_1, x_2) \to \xi' = (\xi_1, \xi_2)$:

$$\mathcal{F}[u(\boldsymbol{x}, t)] = \hat{u}(\xi', x_3, t) = \int_{\mathbb{R}^2}\int e^{\mathrm{i}\,(x',\xi')} u(x', x_3, t)\mathrm{d}x_1\mathrm{d}x_2, \tag{59}$$

and a unilateral Laplace transformation $\mathcal{L}$, denoted by

$$\mathcal{L}[\hat{u}(\xi', x_3, t)] = \int_0^\infty e^{-st}\hat{u}(\xi', x_3, t)\mathrm{d}t. \tag{60}$$

The representation of the tangential and normal components of the electrical and magnetic complex vector fields is described in detail in [12], [15]. Here we have to replace $\mathrm{i}\omega$ by $-s$ and ∂_{x1} by $-\mathrm{i}\xi_1$, ∂_{x2} by $-\mathrm{i}\xi_2$, leaving the derivative ∂_{x3}.

Let us introduce for shortness the vector $\boldsymbol{\zeta} = (-\mathrm{i}\xi_1,\, -\mathrm{i}\xi_2,\, \partial_{x3})^T$. Applying the 2D Fourier and unilateral Laplace transformations to (36) and (37) yields

$$\mathcal{L}[\widehat{\boldsymbol{E}}(\xi', x_3, t)] = -s\mathcal{L}[\widehat{\boldsymbol{A}}(\xi', x_3, t)] - \boldsymbol{\zeta}\mathcal{L}[\widehat{V}(\xi', x_3, t)] + \widehat{\boldsymbol{A}}(\xi', x_3, +0), \tag{61}$$

$$\boldsymbol{\zeta}^T\mathcal{L}[\widehat{\boldsymbol{A}}(\xi', x_3, t)] + s\mathcal{L}[\widehat{V}(\xi', x_3, t)] - \widehat{V}(\xi', x_3, +0) = 0. \tag{62}$$

The equations (40) and (41) after applying the 2D Fourier and unilateral Laplace transformations become (with the matrix K from (7))

$$\begin{aligned}&\{\boldsymbol{\mu}K(\boldsymbol{\zeta})\boldsymbol{\mu}^{-1}K(\boldsymbol{\zeta}) + \boldsymbol{\mu\varepsilon}s^2 + \boldsymbol{\mu\varepsilon}(\boldsymbol{\zeta\zeta}^T)\}\mathcal{L}[\widehat{\boldsymbol{A}}(\xi', x_3, t)]\\ &\qquad = \boldsymbol{\mu\varepsilon}[s\widehat{\boldsymbol{A}}(\xi', x_3, +0) + \partial_t\widehat{\boldsymbol{A}}(\xi', x_3, +0)] + \boldsymbol{\mu}\mathcal{L}[\widehat{\boldsymbol{J}}(\xi', x_3, t)],\end{aligned} \tag{63}$$

$$-\boldsymbol{\zeta}^T\boldsymbol{\varepsilon}\{s\mathcal{L}[\widehat{\boldsymbol{A}}(\xi', x_3, t)] + \boldsymbol{\zeta}^T\mathcal{L}[\widehat{V}(\xi', x_3, t)]\} = \mathcal{L}[\widehat{\varrho}(\xi', x_3, t)] - \boldsymbol{\zeta}^T\boldsymbol{\varepsilon}\widehat{\boldsymbol{A}}(\xi', x_3, +0). \tag{64}$$

These equations form a 4×4 system of ordinary differential equations in the half-spaces $x_3 < 0$ or $x_3 > 0$. If $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ are symmetric and piecewise constant for $0 < x_3 < h$ and $x_3 > h$ and independent of t the usual methods for constant coefficients-here parameter dependent $(\xi_1,\ \xi_2, s)$-ODE systems can be applied. These lead to a complex eigenvalue problem which is quite complicated in the general anisotropic case. An explicit solution (four parameter dependent eigenvalues and the corresponding eigenvectors) results from the time-harmonic case (with $-s$ instead of $\mathrm{i}\omega$) which was treated thoroughly in [15].

The normal components may be calculated from Maxwell's equations and the boundary conditions on $\mathbb{R}^2$ or on the Sommerfeld half-plane are given as Leontovich boundary conditions. The authors could show the equivalence of the boundary-transmission problem for $\overline{\sum} \cup \overline{\sum}' = \mathbb{R}^2$ (where $\overline{\sum}$ is a closed screen and $\overline{\sum}'$ is a complementary closed screen) for $\mathbb{R}^3_{\pm}$ with a system of Wiener-Hopf equations acting on anisotropic Sobolev spaces with respect to x'. Existence and uniqueness has been proved in smoother Sobolev spaces than the usual trace spaces $[H_{-1/2,1/2}(\mathbb{R}^2) \times H_{1/2,-1/2}(\mathbb{R}^2)]^2$ for the tangential components $u_{\mathrm{elec}}(x', \pm 0)$, and $u_{\mathrm{magn}}(x', \pm 0)$. In [15] a detailed investigation of the invertibility of the Wiener-Hopf operators and the explicit representation of the scattered fields are displayed in Chapter 3 including Neumann-series under additional conditions for the impedance matrix Ξ. The aperiodic initial boundary condition problem can be treated analogously by including inhomogeneous initial values for the fields. Details still have to be worked out, in particular the formulation of a Riemann boundary value problem in complex Fourier transformed quaternions.

5. Conclusions

It was shown that general linear initial boundary problems for electrodynamical fields can be reformulated as such for quaternion-valued functions and reduced by Fourier and Laplace transformations to equivalent Wiener-Hopf systems of the unknown traces of the potentials and their jumps across the boundaries of the screen. The crucial Fourier symbol's factorization of the boundary data for the Wiener-Hopf equation can be performed in some partially anisotropic cases. Problems in elasto-, thermoelasto-, and viscoelastodynamics lead to new parameter dependent Wiener-Hopf equations where the equations have to be investigated in time-weighted (anisotropic) Sobolev spaces parallel to the plane boundaries. The authors hope to publish an extended version of the paper in near future.

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