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ON OPTIMAL DECAY RATES FOR WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^n$

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Dedicated to Professor Jindřich Nečas on his 70th birthday

Abstract. This paper is concerned with optimal lower bounds of decay rates for solutions to the Navier-Stokes equations in $\mathbb{R}^n$. Necessary and sufficient conditions are given such that the corresponding Navier-Stokes solutions are shown to satisfy the algebraic bound

$$\|u(t)\| \geqslant (t+1)^{-\frac{n+4}{2}}.$$

Keywords: decay rates, Navier-Stokes equations

MSC 2000: 35Q10

1. Introduction and the results

Consider the Navier-Stokes equations in $\mathbb{R}^n$, $n \geqslant 2$, which will be treated in this paper in the form of the integral equation

$$\text{(NS)} \qquad u(t) = \mathrm{e}^{-tA}a - \int_0^t \nabla \cdot \mathrm{e}^{-(t-s)A} P(u \otimes u)(s)\,\mathrm{d}s,$$

for prescribed initial velocity $a(x) = (a_1(x), \ldots, a_n(x))$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and unknown velocity $u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$. Here, $A = -\Delta$ is the Laplacian on $\mathbb{R}^n$; $\{\mathrm{e}^{-tA}\}_{t\geqslant 0}$ is the heat semigroup; $P = (P_{jk})$ is the bounded projection onto divergence-free vector fields; $u \otimes v$ is the matrix with entries $(u \otimes v)_{jk} = u_j v_k$; $\nabla = (\partial_1, \ldots, \partial_n)$ with $\partial_j = \partial/\partial x_j$; and

$$(\nabla \cdot \mathrm{e}^{-tA} P(u \otimes u))_j = \sum_{k,\ell=1}^{n} \partial_\ell \mathrm{e}^{-tA} P_{jk}(u_\ell u_k), \quad j = 1, \ldots, n.$$

It is well known that for each $a \in \boldsymbol{L}^2$ with $\nabla \cdot a = 0$, (NS) has a weak solution u defined for all $t \geqslant 0$, satisfying the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 \,\mathrm{d}s \leqslant \|a\|_2^2 \quad \text{for all } t \geqslant 0.$$

Hereafter $\|\cdot\|_r$ denotes the L^r-norm.

As shown in [10], there exists a weak solution u such that

$$\|u(t)\|_2 \leqslant C(1+t)^{-\frac{n+2}{4}}, \tag{1.1}$$

whenever

$$a \in \boldsymbol{L}^2, \quad \nabla \cdot a = 0 \quad \text{and} \quad \int (1+|y|)|a(y)| \,\mathrm{d}y < \infty. \tag{1.2}$$

Assumption (1.2) implies $a \in \boldsymbol{L}^1$; so the divergence-free condition gives (see [4])

$$\int a(y) \,\mathrm{d}y = 0. \tag{1.3}$$

Furthermore, it is shown in [2] that in this case the solution u satisfies

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{n+2}{4}} \Big\| u_j(t) + (\partial_k E_t)(\cdot) \int y_k a_j(y) \,\mathrm{d}y \\ + F_{\ell,jk}(\cdot,t) \int_0^\infty \int (u_\ell u_k)(y,s) \,\mathrm{d}y \,\mathrm{d}s \Big\|_2 = 0\end{aligned} \tag{1.4}$$

for $j = 1, \ldots, n$, where

$$E_t(x) = (4\pi t)^{-n/2} \mathrm{e}^{-|x|^2/4t}, \quad F_{\ell,jk}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_t^\infty \partial_\ell \partial_j \partial_k E_s(x) \,\mathrm{d}s.$$

(Hereafter, we use the summation convention). Equation (NS) is then written in the form

$$u_j(x,t) = \int E_t(x-y) a_j(y) \,\mathrm{d}y - \int_0^t \int F_{\ell,jk}(x-y,t-s)(u_\ell u_k)(y,s) \,\mathrm{d}y \,\mathrm{d}s, \; j = 1, \ldots, n,$$

as proved in [2]; and the integrals in (1.4) are finite, due to (1.1) and (1.2). Assertion (1.4) was first proved in [1] for smooth solutions when $n = 3$, and then extended in [2] to the case of weak solutions in all space dimensions by applying the spectral method as given in [3, 5].

The argument of [10] suggests that the decay property (1.1) will be optimal in general. So we are interested in finding a class of weak solutions u satisfying the reverse estimate

$$\|u(t)\|_2 \geqslant Ct^{-\frac{n+2}{4}} \quad \text{at least for large } t.$$

In this paper we discuss this kind of *lower bound problem*.

Theorem A. *Under the assumption* (1.2), *let*

$$b_{k\ell} = \int y_\ell a_k(y)\,\mathrm{d}y, \qquad c_{k\ell} = \int_0^\infty \int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

(i) *We have*

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \|u(t)\|_2 = 0 \tag{1.5}$$

if and only if $(b_{k\ell}) = 0$ *and* $(c_{k\ell}) = (c\delta_{k\ell})$ *for some constant* $c \geqslant 0$.

(ii) *There exists* $c' > 0$ *such that*

$$\|u(t)\|_2 \geqslant c' t^{-\frac{n+2}{4}} \quad \textit{for large } t > 0, \tag{1.6}$$

if and only if $(b_{k\ell}) \neq 0$ *or* $(c_{k\ell}) \neq (c\delta_{k\ell})$. *In particular,* u *satisfies* (1.6) *whenever* $(b_{k\ell}) \neq 0$.

Remark. Theorem A (i) implies only that

$$\limsup_{t\to\infty} t^{\frac{n+2}{4}} \|u(t)\|_2 > 0 \tag{1.5$'$}$$

if and only if $(b_{k\ell}) \neq 0$ or $(c_{k\ell}) \neq (c\delta_{k\ell})$. Note, however, that our second assertion (1.6) is more stringent than (1.5′). Moreover, (1.6) holds for *all* large $t > 0$ and for *all* space dimensions, although $\|u(t)\|_2$ is only known to be lower semicontinuous when $n \geqslant 3$. We know nothing about the characterization of solutions satisfying $(c_{k\ell}) = (c\delta_{k\ell})$.

We next consider weak solutions u satisfying

$$\|u(t)\|_2 \leqslant C(1+t)^{-\frac{n}{4}}. \tag{1.7}$$

As shown in [3, 6, 10], such solutions exist for all $a \in \boldsymbol{L}^2$ satisfying

$$\nabla \cdot a = 0, \quad \|\mathrm{e}^{-tA} a\|_2 \leqslant C(1+t)^{-\frac{n}{4}}. \tag{1.8}$$

Theorem B. *Suppose a satisfies* (1.8) *and let u be a weak solution satisfying* (1.7). *Then*

$$\|u(t)\|_2 \geqslant ct^{-\frac{n}{4}} \quad \text{for large } t > 0, \tag{1.9}$$

if and only if

$$\|\mathrm{e}^{-tA}a\|_2 \geqslant ct^{-\frac{n}{4}} \quad \text{for large } t > 0. \tag{1.10}$$

The lemma below gives simple examples of a satisfying (1.10).

Lemma. *Let $a \in \boldsymbol{L}^2$, $\nabla \cdot a = 0$, and suppose that*

$$\int_{S^{n-1}} |\hat{a}(r,\omega)|^2 \,\mathrm{d}\omega \in L^\infty(\mathbb{R}_+), \quad \liminf_{r\to 0} \int_{S^{n-1}} |\hat{a}(r,\omega)|^2 \,\mathrm{d}\omega > 0, \tag{1.11}$$

where the Fourier transform $\hat{a}$ is defined by

$$\hat{a}(\xi) = \int \mathrm{e}^{-\mathrm{i}x\cdot\xi} a(x)\,\mathrm{d}x, \quad \mathrm{i} = \sqrt{-1},$$

S^{n-1} is the unit sphere of $\mathbb{R}^n$, and $\xi = (r,\omega)$ in polar coordinates. Then,

$$\|\mathrm{e}^{-tA}a\|_2 \leqslant C(1+t)^{-\frac{n}{4}} \text{ for all } t > 0; \quad \|\mathrm{e}^{-tA}a\|_2 \geqslant c't^{-\frac{n}{4}} \text{ for large } t > 0, \tag{1.12}$$

with constants $C > 0$ and $c' > 0$ independent of t.

P r o o f. Parseval's relation gives

$$\|\mathrm{e}^{-tA}a\|_2^2 = (2\pi)^{-n} \int \mathrm{e}^{-2t|\xi|^2} |\hat{a}(\xi)|^2 \,\mathrm{d}\xi = (8\pi^2 t)^{-\frac{n}{2}} \int \mathrm{e}^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}})|^2 \,\mathrm{d}\eta$$

so that

$$(8\pi^2 t)^{\frac{n}{2}} \|\mathrm{e}^{-tA}a\|_2^2 = \int \mathrm{e}^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}})|^2 \,\mathrm{d}\eta.$$

The assumption and Fatou's lemma together imply

$$\begin{aligned} \liminf_{t\to\infty} (8\pi^2 t)^{\frac{n}{2}} \|\mathrm{e}^{-tA}a\|_2^2 &= \liminf_{t\to\infty} \int \mathrm{e}^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}})|^2 \,\mathrm{d}\eta \\ &\geqslant \int_0^\infty \mathrm{e}^{-r^2} \Big(\liminf_{t\to\infty} \int_{S^{n-1}} |\hat{a}(r(2t)^{-\frac{1}{2}},\omega)|^2 \,\mathrm{d}\omega \Big) r^{n-1} \,\mathrm{d}r > 0. \end{aligned}$$

This proves the second estimate of (1.12). The first estimate follows from $\|\mathrm{e}^{-tA}a\|_2 \leqslant \|a\|_2$ and

$$\begin{aligned} \|\mathrm{e}^{-tA}a\|_2^2 &= (8\pi^2 t)^{-\frac{n}{2}} \int \mathrm{e}^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}})|^2 \,\mathrm{d}\eta \\ &\leqslant Ct^{-\frac{n}{2}} \Big\| \int_{S^{n-1}} |\hat{a}(\cdot,\omega)|^2 \,\mathrm{d}\omega \Big\|_\infty \int_0^\infty \mathrm{e}^{-r^2} r^{n-1} \,\mathrm{d}r. \end{aligned}$$

The proof is complete. □

R e m a r k s. (i) Condition (1.11) implies that $\hat{a}$ is discontinuous at $\xi = 0$. Indeed, since $\nabla \cdot a = 0$, we have $\xi \cdot \hat{a}(\xi) = 0$; so if $\hat{a}$ is continuous at $\xi = 0$, we get $\omega \cdot \hat{a}(0) = 0$ for all unit vectors ω, and $\hat{a}(0) = 0$. (For this reason, $a \in \boldsymbol{L}^1$ implies (1.3)).

(ii) The assumption of Lemma is not vacuous. Indeed, suppose $\hat{a}$ is written in the form

$$\hat{a}(\xi) = f(|\xi|)g(\xi/|\xi|),$$

in terms of functions $f(r)$ and $g(\omega)$ such that

$$g \in \boldsymbol{L}^2(S^{n-1}), \quad g \not\equiv 0, \quad \omega \cdot g(\omega) \equiv 0 \quad (\omega \in S^{n-1})$$

and

$$f \in BC([0,\infty)), \quad \int_0^\infty |f(r)|^2 r^{n-1}\,\mathrm{d}r < \infty, \quad f(0) \neq 0.$$

Then, $\hat{a}$ satisfies condition (1.11).

(iii) In this connection, we note that under condition (1.2) we have

$$\|\mathrm{e}^{-tA}a\|_2 \geqslant ct^{-\frac{n+2}{4}} \quad \text{for large } t > 0 \tag{1.10$'$}$$

if and only if $(b_{k\ell}) \neq 0$. Indeed, using (1.2) and (1.3), we have (see Section 4)

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|\mathrm{e}^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2 = 0, \quad k = 1,\ldots,n. \tag{1.4$'$}$$

Suppose $(b_{k\ell}) \neq 0$. Then $(\sum\limits_k \|\partial_\ell E_t b_{k\ell}\|_2^2)^{1/2} = Ct^{-\frac{n+2}{4}}$ with $C > 0$; so we get

$$\|\mathrm{e}^{-tA}a\|_2 \geqslant \big(\sum_k \|\partial_\ell E_t b_{k\ell}\|_2^2\big)^{1/2} - \big(\sum_k \|\mathrm{e}^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2^2\big)^{1/2} \geqslant ct^{-\frac{n+2}{4}}$$

for large $t > 0$. Conversely, if we assume (1.10$'$), then (1.4$'$) implies

$$\big(\sum_k \|\partial_\ell E_t b_{k\ell}\|_2^2\big)^{1/2} \geqslant \|\mathrm{e}^{-tA}a\|_2 - \big(\sum_k \|\mathrm{e}^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2^2\big)^{1/2} \geqslant ct^{-\frac{n+2}{4}}$$

for large $t > 0$. Hence $\sum\limits_k \|\partial_\ell E_t b_{k\ell}\|_2^2 > 0$ for large $t > 0$, which implies $(b_{k\ell}) \neq 0$.

The L^2 decay problem for weak solutions of the Navier-Stokes equations was successfully studied for the first time by [5] and the result was then systematically developed by [3, 6, 10]. Estimates (1.6) and (1.9) are studied in [6]–[9] in case $n = 2, 3$, and some sufficient conditions are obtained. Our Theorems A and B provide *necessary and sufficient conditions* for those estimates to hold. We further note that our lower bound estimates (1.6) and (1.9) hold in all space dimensions $n \geqslant 2$, although the function $\|u(t)\|_2$ is known only to be lower semicontinuous when $n \geqslant 3$. As will be seen in the proof below, this is due to (1.4) and the fact that the functions $\partial_\ell E_t(x)$ and $F_{\ell,jk}(x,t)$ are written in the form $t^{-\frac{n+1}{2}} K(xt^{-\frac{1}{2}})$ in terms of some bounded, integrable and uniformly continuous functions K.

We finally consider an example of two-dimensional flows u with $(b_{k\ell}) = 0$, $(c_{k\ell}) = (c\delta_{k\ell})$, which was first treated by [7].

Theorem C. *When $n = 2$, there is a smooth weak solution u such that $(b_{k\ell}) = 0$, $(c_{k\ell}) = (c\delta_{k\ell})$, and, with some constant $\gamma > 0$,*

$$\|u(t)\|_q \leqslant C_q \mathrm{e}^{-\gamma t} \quad \text{and} \quad |u(x,t)| \leqslant C_m \mathrm{e}^{-\gamma t}(1+|x|)^{-m} \tag{1.13}$$

for all $1 \leqslant q \leqslant \infty$ and all integers $m \geqslant 0$.

The above example was studied by [7, 8, 9], in which is given the exponential decay of $\|u(t)\|_q$ for $2 \leqslant q \leqslant \infty$. Our estimates (1.13) include the case $1 \leqslant q < 2$ as well as the decay estimates in the spatial direction. Theorem C is proved in [2].

In what follows we prove Theorems A and B, and conclude the paper with the proof of (1.4) which was given also in [2].

2. Proof of Theorem A

We begin with the following

Proposition 2.1. *Let $(b_{k\ell})$ and $(c_{k\ell})$ be real $n \times n$ matrices and let $(c_{k\ell})$ be symmetric. Then*

$$b_{k\ell}\partial_\ell E_t(x)\delta_{jk} + c_{k\ell}F_{\ell,jk}(x,t) = 0, \quad j = 1,\ldots,n, \tag{2.1}$$

for all $x \in \mathbb{R}^n$ and for some $t > 0$, if and only if

$$(b_{k\ell}) = 0 \quad \text{and} \quad (c_{k\ell}) = (c\delta_{k\ell}) \quad \text{for some } c \in \mathbb{R}. \tag{2.2}$$

Furthermore, (2.2) implies that (2.1) holds for all x and for all $t > 0$.

P r o o f. Assumption (2.1) implies, via the Fourier transformation,

$$\begin{aligned} b_{k\ell}\xi_\ell \mathrm{e}^{-t|\xi|^2}\delta_{jk} &= -c_{k\ell}\xi_\ell\Big(\mathrm{e}^{-t|\xi|^2}\delta_{jk} - \xi_j\xi_k\int_t^\infty \mathrm{e}^{-s|\xi|^2}\,\mathrm{d}s\Big) \\ &= -(c_{j\ell} - |\xi|^{-2}c_{k\ell}\xi_j\xi_k)\xi_\ell \mathrm{e}^{-t|\xi|^2} \end{aligned}$$

for some $t > 0$, and we get $|\xi|^2(b_{j\ell} + c_{j\ell})\xi_\ell = \xi_j c_{k\ell}\xi_k\xi_\ell$. Taking $\xi_j = 0$ for any fixed j, $\xi_\ell = 1$ for any fixed $\ell \neq j$, and $\xi_k = 0$ for all k such that $k \neq j$ and $k \neq \ell$, we easily obtain $b_{j\ell} + c_{j\ell} = 0$ whenever $j \neq \ell$, and so

$$|\xi|^2(b_{jj} + c_{jj})\xi_j = \xi_j c_{k\ell}\xi_k\xi_\ell, \quad j = 1, \ldots, n.$$

We let $\xi_j = 1$ and $\xi_k = 0$ for $k \neq j$, to get $b_{jj} + c_{jj} = c_{jj}$; so $b_{jj} = 0$. This implies

$$|\xi|^2 c_{jj}\xi_j = \xi_j c_{k\ell}\xi_k\xi_\ell, \quad j = 1, \ldots, n. \tag{2.3}$$

Hence, $c_{11} = \ldots = c_{nn} = c_{k\ell}\xi_k\xi_\ell|\xi|^{-2}$. We then set $j = 1$, $\xi_1 = \xi_2 = 1$ and $\xi_k = 0$ for $k \geqslant 3$ in (2.3), to get $2c_{11} = c_{11} + c_{22} + c_{12} + c_{21} = 2(c_{11} + c_{12})$ since $c_{k\ell} = c_{\ell k}$ by assumption. Therefore, $c_{12} = 0$. We thus obtain $c_{j\ell} = 0 = -b_{j\ell}$ whenever $j \neq \ell$; so $(b_{k\ell}) = 0$ and $(c_{k\ell}) = (c\delta_{k\ell})$. That (2.2) implies (2.1) for *all* $t > 0$ is easily seen from

$$F_{k,jk} = \partial_j E_t + \int_t^\infty \partial_j \Delta E_s \,\mathrm{d}s = \partial_j E_t + \int_t^\infty \partial_j \partial_s E_s \,\mathrm{d}s = \partial_j E_t - \partial_j E_t = 0,$$

where $\partial_s = \partial/\partial s$. The proof of Proposition 2.1 is complete. □

To establish Theorem A, it suffices in view of (1.4) to prove the following

Proposition 2.2. *Let a satisfy* (1.2) *and define*

$$b_{k\ell} = \int y_\ell a_k(y)\,\mathrm{d}y, \quad c_{k\ell} = \int_0^\infty \int (u_\ell u_k)(y, s)\,\mathrm{d}y\,\mathrm{d}s.$$

Then we have

$$\text{either} \quad (b_{k\ell}) \neq 0 \quad \text{or} \quad (c_{k\ell}) \neq (c\delta_{k\ell}), \tag{2.4}$$

if and only if a corresponding weak solution u satisfies

$$\|u(t)\|_2 \geqslant c' t^{-\frac{n+2}{4}} \quad \text{for large } t > 0 \tag{2.5}$$

with a constant $c' > 0$ indenpendent of t.

Proof. In what follows we write

$$\boldsymbol{b}_\ell = (b_{1\ell}, \ldots, b_{n\ell}), \quad \boldsymbol{F}_{\ell,k} = (F_{\ell,1k}, \ldots, F_{\ell,nk}).$$

Assume first (2.4). By Proposition 2.1, we have $\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 = Ct^{-\frac{n+2}{4}}$ for all $t > 0$ with some $C > 0$, and so (1.4) implies

$$\begin{aligned}\|u(t)\|_2 &\geqslant \|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 - \|u(t) + \partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\ &= Ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) \geqslant c' t^{-\frac{n+2}{4}}\end{aligned}$$

for large $t > 0$. Assume next (2.5). By (1.4) we have

$$\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \geqslant \|u(t)\|_2 - \|u(t) + \partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \geqslant c' t^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}),$$

and so

$$\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 > 0 \quad \text{for large } t > 0.$$

We thus obtain (2.4) by Proposition 2.1. This proves Proposition 2.2. □

3. Proof of Theorem B

Suppose that $n \geqslant 3$. We have

$$c_{k\ell} = \int_0^\infty \int (u_\ell u_k)(y,s)\, \mathrm{d}y\, \mathrm{d}s < \infty;$$

so the argument given in [2, Sect. 5] applies to our present situation, implying

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \|u(t) - \mathrm{e}^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 = 0. \tag{3.1}$$

Suppose (1.9) holds. Since $\|\boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 = C t^{-\frac{n+2}{4}}$, it follows from (3.1) that

$$\begin{aligned}\|\mathrm{e}^{-tA} a\|_2 &\geqslant \|u(t)\|_2 - \|-u(t) + \mathrm{e}^{-tA} a - \boldsymbol{F}_{\ell,k} c_{k\ell} + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\ &\geqslant \|u(t)\|_2 - \|u(t) - \mathrm{e}^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 - \|\boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\ &\geqslant c t^{-\frac{n}{4}} - C t^{-\frac{n+2}{4}} \geqslant c' t^{-\frac{n}{4}}\end{aligned}$$

for large $t > 0$. This proves (1.10). Conversely, if (1.10) holds, then (3.1) implies

$$\begin{aligned}\|u(t)\|_2 &\geqslant \|\mathrm{e}^{-tA} a\|_2 - \|\boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 - \|u(t) - \mathrm{e}^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\ &\geqslant c t^{-\frac{n}{4}} - C t^{-\frac{n+2}{4}} \geqslant c' t^{-\frac{n}{4}}\end{aligned}$$

for large $t > 0$. This proves (1.9) in case $n \geqslant 3$.

When $n = 2$, we introduce

$$c_{k\ell}(t) = \int_0^{t/2} \int (u_\ell u_k)(y,s)\, \mathrm{d}y\, \mathrm{d}s$$

instead of $c_{k\ell}$. The argument of [2, Sect. 5] is then modified to yield

$$\|u(t) - \mathrm{e}^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell}(t)\|_2 \leqslant C t^{-1} \log(1+t). \tag{3.1$'$}$$

See also Section 4 below. Since

$$\|\boldsymbol{F}_{\ell,k}c_{k\ell}(t)\|_2 \leqslant Ct^{-1}\int_0^{t/2}\|u(s)\|_2^2\,\mathrm{d}s \leqslant Ct^{-1}\log(1+t),$$

this implies $\|u(t)-\mathrm{e}^{-tA}a\|_2 \leqslant Ct^{-1}\log(1+t)$. Now we can prove the result in the same way as in the case $n \geqslant 3$. Indeed, (1.10) implies

$$\|u(t)\|_2 \geqslant \|\mathrm{e}^{-tA}a\|_2 - \|u(t)-\mathrm{e}^{-tA}a\|_2 \geqslant ct^{-\frac{1}{2}} - Ct^{-1}\log(1+t) \geqslant c't^{-\frac{1}{2}}$$

for large $t>0$, while (1.9) yields

$$\|\mathrm{e}^{-tA}a\|_2 \geqslant \|u(t)\|_2 - \|u(t)\mathrm{e}^{-tA}\|_2 \geqslant ct^{-\frac{1}{2}} - Ct^{-1}\log(1+t) \geqslant c't^{-\frac{1}{2}}$$

for large $t>0$. The proof of Theorem B is complete.

4. Proof of (1.4)

Here we present the proof of (1.4) given in [2]. The same method can be applied to the proof of (3.1) and (3.1′) with no essential change. Let a satisfy (1.2) and so (1.3). We first prove

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\left\|\mathrm{e}^{-tA}a + (\partial_k E_t)(\cdot)\int y_k a(y)\,\mathrm{d}y\right\|_2 = 0. \tag{4.1}$$

Direct calculation gives

$$\begin{aligned}\mathrm{e}^{-tA}a &= \int [E_t(x-y)-E_t(x)]a(y)\,\mathrm{d}y = -\iint_0^1 (\partial_k E_t)(x-y\theta)y_k a(y)\,\mathrm{d}\theta\,\mathrm{d}y\\ &= -(\partial_k E_t)(x)\int y_k a(y)\,\mathrm{d}y - \iint_0^1 [(\partial_k E_t)(x-y\theta)-(\partial_k E_t)(x)]y_k a(y)\,\mathrm{d}\theta\,\mathrm{d}y,\end{aligned}$$

so

$$\mathrm{e}^{-tA}a + (\partial_k E_t)(x)\int y_k a(y)\,\mathrm{d}y = -\iint_0^1 [(\partial_k E_t)(x-y\theta)-(\partial_k E_t)(x)]y_k a(y)\,\mathrm{d}\theta\,\mathrm{d}y.$$

We can write $(\partial_k E_t)(x) = t^{-\frac{n+1}{2}}(\partial_k E_1)(xt^{-\frac{1}{2}})$, to obtain

$$\left\|\mathrm{e}^{-tA}a + (\partial_k E_t)(\cdot)\int y_k a(y)\,\mathrm{d}y\right\|_2 \leqslant Ct^{-\frac{n+2}{4}}\iint_0^1 \varphi_t(y,\theta)|y||a(y)|\,\mathrm{d}\theta\,\mathrm{d}y.$$

Here $\varphi_t(y,\theta) = \|(\nabla E_1)(\cdot - y\theta t^{-\frac{1}{2}}) - (\nabla E_1)(\cdot)\|_2$ is bounded and $\lim_{t\to\infty} \varphi_t(y,\theta) = 0$ for any fixed (y,θ). Since $|y||a(y)|$ is integrable by (1.2), the dominated convergence theorem yields

$$\lim_{t\to\infty} \int\int_0^1 \varphi_t(y,\theta)|y||a(y)|\,\mathrm{d}\theta\,\mathrm{d}y = 0.$$

This proves (4.1). Now let u satisfy (1.1). We next show that the function

$$w(t) = u(t) - \mathrm{e}^{-tA}a = -\int_0^t\int \boldsymbol{F}_{\ell,k}(x-y,t-s)(u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

satisfies

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \left\| w(t) + \boldsymbol{F}_{\ell,k}(\cdot,t)\int_0^\infty\int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \right\|_2 = 0. \tag{4.2}$$

Indeed, we have

$$\begin{aligned}
w(t) &+ \boldsymbol{F}_{\ell,k}(x,t)\int_0^\infty\int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&= \boldsymbol{F}_{\ell,k}(x,t)\int_{t/2}^\infty\int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \int_0^{t/2}\int [\boldsymbol{F}_{\ell,k}(x-y,t-s) - \boldsymbol{F}_{\ell,k}(x,t-s)](u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \int_0^{t/2}\int [\boldsymbol{F}_{\ell,k}(x,t-s) - \boldsymbol{F}_{\ell,k}(x,t)](u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \int_{t/2}^t\int \boldsymbol{F}_{\ell,k}(x-y,t-s)(u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

It is easy to see that

$$t^{\frac{n+2}{4}}\|I_1\|_2 \leqslant C\int_{t/2}^\infty (1+s)^{-1-\frac{n}{2}}\,\mathrm{d}s \to 0 \quad \text{as } t\to\infty. \tag{4.3}$$

We write I_3 in the form

$$I_3 = \int_0^{t/2}\int\int_0^1 s(\partial_t \boldsymbol{F}_{\ell,k})(x,t-s\theta)(u_\ell u_k)(y,s)\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}s$$

to get

$$\begin{aligned}
\|I_3\|_2 &\leqslant C\int_0^{t/2}\int\int_0^1 s(t-s\theta)^{-1-\frac{n+2}{4}}|u(y,s)|^2\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant Ct^{-1-\frac{n+2}{4}}\int_0^{t/2} s\|u(s)\|_2^2\,\mathrm{d}s
\end{aligned}$$

and so

$$t^{\frac{n+2}{4}}\|I_3\|_2 \leqslant Ct^{-1}\int_0^t (1+s)^{-\frac{n}{2}}\,\mathrm{d}s \to 0 \quad \text{as } t\to\infty. \tag{4.4}$$

To estimate I_2, note that we can write $\boldsymbol{F}_{\ell,k}(x,t) = t^{-\frac{n+1}{2}}K(xt^{-\frac{1}{2}})$, to get

$$\begin{aligned}\|I_2\|_2 &\leqslant Ct^{-\frac{n+2}{4}}\int_0^{t/2}\!\!\int \|K(\cdot - y(t-s)^{-\frac{1}{2}}) - K(\cdot)\|_2|u(y,s)|^2\,\mathrm{d}y\,\mathrm{d}s\\ &\equiv Ct^{-\frac{n+2}{4}}\int_0^{t/2}\!\!\int \varphi_t(y,s)|u(y,s)|^2\,\mathrm{d}y\,\mathrm{d}s \equiv Ct^{-\frac{n+2}{4}}\int_0^{t/2}\psi_t(s)\,\mathrm{d}s.\end{aligned}$$

Since $\psi_t(s)\leqslant C\|u(s)\|_2^2$, the dominated convergence theorem implies

$$\lim_{t\to\infty}\int_0^M \psi_t(s)\,\mathrm{d}s = 0 \quad \text{for any fixed } M>0.$$

Given $\varepsilon>0$, choose $M>0$ so that $\int_M^\infty \|u(s)\|_2^2\,\mathrm{d}s<\varepsilon$. Then for $t>2M$,

$$\int_0^{t/2}\psi_t(s)\,\mathrm{d}s \leqslant \int_0^M \psi_t(s)\,\mathrm{d}s + C\int_M^\infty \|u(s)\|_2^2\,\mathrm{d}s \leqslant \int_0^M \psi_t(s)\,\mathrm{d}s + C\varepsilon.$$

This implies that

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|I_2\|_2 = 0. \tag{4.5}$$

It remains to prove

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|I_4\|_2 = 0. \tag{4.6}$$

To do so, we follow the arguments of [3, 5]. The function

$$v(t) = -\int_\tau^t\!\!\int \boldsymbol{F}_{\ell,k}(x-y,t-s)(u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s = u(t) - \mathrm{e}^{-(t-\tau)A}u(\tau)$$

defined for $t\geqslant\tau>0$ satisfies

$$\partial_t v + Av = -P(u\cdot\nabla u) \quad (t>\tau), \quad v(\tau)=0.$$

(We may assume v is smooth, replacing u by the approximate solutions u_N given in [3]). Since $(P(u\cdot\nabla v),v) = (u\cdot\nabla v, v) = 0$, the standard energy integral method gives

$$\partial_t\|v\|_2^2 + 2\|A^{1/2}v\|_2^2 = -2(u\cdot\nabla u, v) = 2(u\cdot\nabla v, u) = 2(u\cdot\nabla v, u_0)$$

and

$$2|(u\cdot\nabla v, u_0)| \leqslant 2\|u\|_2\|A^{1/2}v\|_2\|u_0\|_\infty \leqslant C\|u\|_2\|A^{1/2}v\|_2(t-\tau)^{-\frac{n}{4}}\tau^{-\frac{n+2}{4}}$$
$$\leqslant C\|A^{1/2}v\|_2(t-\tau)^{-\frac{n+1}{2}}\tau^{-\frac{n+2}{4}} \leqslant \|A^{1/2}v\|_2^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}},$$

where $u_0(t) = \mathrm{e}^{-(t-\tau)A}u(\tau)$. We thus obtain

$$\partial_t\|v\|_2^2 + \|A^{1/2}v\|_2^2 \leqslant C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

Let $\{E_\lambda\}_{\lambda\geqslant 0}$ be the spectral measure associated to A. Since $\|A^{1/2}v\|_2^2 \geqslant \varrho(\|v\|_2^2 - \|E_\varrho v\|_2^2)$ for any $\varrho > 0$, the above estimate yields

$$\partial_t\|v\|_2^2 + \varrho\|v\|_2^2 \leqslant \varrho\|E_\varrho v\|_2^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

But, $\|E_\varrho v\|_2^2 \leqslant C\varrho^{\frac{n+2}{2}}\Big(\int_\tau^t \|u\|_2^2\,\mathrm{d}s\Big)^2$ as shown in [3, 5]; so

$$\partial_t\|v\|_2^2 + \varrho\|v\|_2^2 \leqslant C\varrho^{\frac{n+4}{2}}\Big(\int_\tau^t \|u\|_2^2\,\mathrm{d}s\Big)^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

Here we set $\varrho = m/(t-\tau)$, $m > 0$, and multiply both sides by $(t-\tau)^m$, to obtain

$$\partial_t((t-\tau)^m\|v\|_2^2) \leqslant C_m(t-\tau)^{m-\frac{n}{2}-2}\Big(\int_\tau^t \|u\|_2^2\,\mathrm{d}s\Big)^2 + C(t-\tau)^{m-n-1}\tau^{-1-\frac{n}{2}}.$$

Now fix m so that $m > n/2+2$ and $m > n+1$, and integrate the above inequality, to get

$$\|v(t)\|_2^2 \leqslant C(t-\tau)^{-2-\frac{n}{2}}\int_\tau^t\Big(\int_\tau^s \|u\|_2^2\,\mathrm{d}\sigma\Big)^2\mathrm{d}s + C(t-\tau)^{-n}\tau^{-1-\frac{n}{2}}.$$

Inserting $\tau = t/2$ yields $v(t) = I_4$, so

$$t^{n+\frac{n}{2}}\|I_4\|_2^2 \leqslant Ct^{n-1}\Big(\int_{t/2}^\infty \|u\|_2^2\,\mathrm{d}s\Big)^2 + Ct^{-1} \leqslant Ct^{-1} \to 0$$

as $t\to\infty$. This proves (4.6).

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