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Mathematica Bohemica, Vol. 128 (2003), No. 4, 419–438

Persistent URL: http://dml.cz/dmlcz/134007

128 (2003) MATHEMATICA BOHEMICA No. 4, 419–438

STEREOLOGY OF EXTREMES; SIZE OF SPHEROIDS

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(Received February 10, 2003)

Abstract. The prediction of size extremes in Wicksell's corpuscle problem with oblate spheroids is considered. Three-dimensional particles are represented by their planar sections (profiles) and the problem is to predict their extremal size under the assumption of a constant shape factor. The stability of the domain of attraction of the size extremes is proved under the tail equivalence condition. A simple procedure is proposed of evaluating the normalizing constants from the tail behaviour of appropriate distribution functions and its results are employed for the estimation of the spheroid size. Examples covering families of Gamma, Pareto and Weibull distributions are provided. A short discussion of maximum likelihood estimators of the normalizing constants is also included.

Keywords: sample extremes, domain of attraction, normalizing constants, FGM system of distributions

MSC 2000: 60G70, 62G32, 62P30

1. Introduction

Spheroidal particles of random size and shape distributed in a given volume may serve as a suitable model of many situations occurring in material science, biology, petrography etc. The embedding medium is frequently opaque and the particles cannot be observed directly. Consequently, only their planar sections called the profiles are simply accessible and the estimation of the properties of the original particles from the properties of their profiles is of particular interest.

Let us suppose that the particle arrangement is stationary isotropic, which means that the underlying point process of their centroids is stationary and their orientation is uniformly distributed random variable. We shall consider the *oblate spheroids* in

This research was supported by the project GAČR 201/03/P138 "Statistical and probabilistic analysis of real processes" of Czech Grant Agency, and by the project MSM 113200008 "Mathematical methods in stochastics" of MŠMT ČR.

what follows. They can be completely characterized e.g. by the length of the two major semiaxes, which will be called here the *size* x, and by their *shape factor*

$$t = \frac{x^2}{w^2} - 1,$$

where w is the length of the minor semiaxis. The profiles of spheroids are ellipses which can be again fully characterized by their *size* y (the length of the major semiaxis) and the shape factor

$$z = \frac{y^2}{v^2} - 1,$$

where v is the length of the minor semiaxis.

The classical solution of the particle size reconstruction based on the profile characteristics is *Wicksell's corpuscle problem* [16], [17], a detailed treatment of oblate and prolate spheroids is in [2]. In the past decade, the prediction of extremal size in Wicksell's corpuscle problem has been intensively studied, see [3], [12]–[14]. Basic references to the theory of sample extremes may be [4], [5] or [10].

In this paper, the problem of extremal spheroid size as related to its shape factor is examined. Therefore the results of [13], [14] are generalized and also the problem complementary to [7], [8] (the extremal shape factor as related to the size) is dealt with.

In Section 2, the joint distribution of random size (Y) and shape factor (Z) of profiles and the joint distribution of random size (X) and shape factor (T) of spheroids are introduced; the former is the simple transform of the latter. Then it is shown in Section 3 that the distribution of Y conditioned by T belongs to the same domain of attraction as the distribution of X given T. Under a supplementary tail equivalence condition, also the distribution of Y conditioned by Z and the marginal distribution of Y belong to the same domain of attraction. Normalizing constants are studied in Section 4, and Section 5 contains the explicit form of normalizing constants in bivariate FGM distribution for three different tails of density (each of them representing one of the limit distributions). In Section 6 several examples are given and in Section 7 a statistical application of the theory is briefly outlined. Simulation study of a similar problem concerning the extremal shape factor of spheroids may be found in [1].

2. Distribution of spheroid characteristics

Let the joint density of the spheroid size and shape factor be denoted by $g(x,t)$ and let it be absolutely continuous w.r.t. two-dimensional Lebesgue measure. This density can be transformed to the joint density $h(y,t)$ of the profile size Y and original shape factor T, and also to the density $f(y,z)$ of the profile size Y and profile shape factor Z. It follows e.g. from [2] that $0 \leqslant Y \leqslant X < \omega$ and $0 \leqslant Z \leqslant T < \eta$, where ω and η are the upper endpoints of the marginal distributions of X and T, respectively. The values ω and η need not be finite. Further, let us also denote by $g(x)$ and $g(x|t)$ the marginal and conditional densities of the size, respectively.

Theorem 1. *The joint density of* (Y,T) *is*

$$h(y,t) = \frac{y}{M_t}\int_y^\omega \frac{g(x,t)}{\sqrt{x^2-y^2}}\,\mathrm{d}x, \quad \text{where } M_t = \int_0^\omega x g_t(x)\,\mathrm{d}x. \tag{1}$$

The joint density of (Y,Z) *is*

$$f(y,z) = \frac{y\sqrt{1+z}}{2M}\int_y^\omega\int_z^\eta \frac{g(x,t)\,\mathrm{d}t\,\mathrm{d}x}{\sqrt{t}\sqrt{1+t}\sqrt{t-z}\sqrt{x^2-y^2}}, \tag{2}$$

where M *is half of the mean caliper diameter in the population of particles.*

It is clear that M_t is the conditional mean size of the spheroid given the shape factor $T=t$.

Proof. Let us denote by θ the angle of the sectioning plane and the main plane of the spheroid. Further, let p be the distance of the sectioning plane from the centre of the spheroid and let $\uparrow$ denote the event that the sectioning plane hits the particle. According to [2] we have

$$l(p,\theta|x,t,\uparrow) = [2\mathrm{E}_\theta B]^{-1}\sin\theta,\ \theta\in[0,\pi/2],\ |p| \leqslant x\left(\frac{1+t\sin^2\theta}{1+t}\right)^{1/2} = B,$$

where $l(p,\theta|x,t,\uparrow)$ is the conditional density of the distance p and the angle θ given the spheroid size and shape factor and provided that the spheroid is hit by the sectioning plane. The value B is the half caliper diameter of the particle (half of its breadth equal to the length of the particle projection onto the section plane normal). Under the condition that the particle is hit by the section plane, the section size and shape factor are given by the transformation

$$y = \left(x^2 - \frac{p^2(1+t)}{1+t\sin^2\theta}\right)^{1/2},\ z = t\sin^2\theta. \tag{3}$$

It follows that after the transformation $(p,\theta) \mapsto (y,\theta)$ the conditional density of (y,θ) given the values of (x,t) is

$$h(y,\theta|x,t,\uparrow) = \frac{y}{\sqrt{x^2-y^2}} \frac{\sin\theta}{\mathrm{E}_\theta B} \sqrt{\frac{1+t\sin^2\theta}{1+t}}, \ \theta \in [0,\pi/2], \ y \in [0,x],$$

and the integration w.r.t. θ gives

$$h(y|x,t,\uparrow) = \frac{y}{2\mathrm{E}_\theta B\sqrt{x^2-y^2}} \left(\frac{1}{\sqrt{1+t}} + \sqrt{\frac{1+t}{t}} \arctan\sqrt{t} \right), \ y \in [0,x].$$

Hence

$$h(y|t) = \int_y^\omega h(y|x,t,\uparrow) g(x|t) [\mathrm{E}_\theta B / \mathrm{E}_{g_t}\mathrm{E}_\theta B] \,\mathrm{d}x$$

and since

$$\begin{aligned} \mathrm{E}_{g_t}\mathrm{E}_\theta B &= \int_0^\omega x g(x|t) \int_0^{\pi/2} \left(\frac{1+t\sin^2\theta}{1+t} \right)^{1/2} \sin\theta \,\mathrm{d}\theta \,\mathrm{d}x \\ &= \frac{1}{2} \left(\frac{1}{\sqrt{1+t}} + \sqrt{\frac{1+t}{t}} \arctan\sqrt{t} \right) \int_0^\omega x g(x|t) \,\mathrm{d}x \end{aligned}$$

it follows that $h(y,t)$ is of the form (1).

The distribution $f(y,z)$ may be easily calculated from the transformation (3). It is completely derived in [2]. The reader may note that in [2] the eccentricity $\frac{t}{1+t}$ instead of the shape factor t is used. $\square$

N o t a t i o n r e m a r k. We shall use fixed notation for specific densities, distribution functions and variables. The density $g(x,t)$ and the distribution function $G(x,t)$ will always mean the joint density and the corresponding distribution function of the spheroid characteristics, and similarly $f(y,z)$ and $F(y,z)$ relates to profiles. For the marginal density of the size we shall use simply $g(x)$. The marginal density of the shape factor will be denoted by the subscript—$g_T(t)$. A similar notation is used for distribution functions. Further, $h(y)$ is the marginal density of the profile size and $H(y)$ is its distribution function. As only the distribution of the size given the shape factor will be used, $g_t(x) = g(x|t)$ and $f_z(y) = f(y|z)$ are the conditional densities, $G_t(x)$ and $F_z(y)$ the conditional distribution functions.

3. Limit behaviour of size extremes

Let us recall that there are three possible limit distributions of sample maximum under affine transformation, namely

$$L_{i,\alpha}(v) = \begin{cases} \exp(-v^{-\alpha}), & v \geqslant 0,\ \alpha > 0,\ i = 1,\ \text{“Fréchet”}, \\ \exp(-(-v)^{\alpha}), & v \leqslant 0,\ \alpha > 0,\ i = 2,\ \text{“(reversed) Weibull”}, \\ \exp(-\mathrm{e}^{-v}), & v \in \mathbb{R},\qquad\qquad i = 3,\ \text{“Gumbel”}. \end{cases}$$

We shall write $K \in \mathcal{D}(L)$ if a distribution function K is in the *domain of attraction* (DA) of L.

The following conditions based on the limit behaviour of the density makes it possible to decide into which domain of attraction a considered distribution with a density k belongs:

$$\begin{aligned} &\lim_{s\to\infty} \frac{k(vs)}{k(s)} = v^{-(\alpha+1)}, && \alpha, v > 0,\ \omega = +\infty && \Rightarrow k \in \mathcal{D}(L_{1,\alpha}), \\ &\lim_{s\searrow 0} \frac{k(\omega - vs)}{k(\omega - s)} = v^{\alpha-1}, && \alpha, v > 0,\ \omega < +\infty && \Rightarrow k \in \mathcal{D}(L_{2,\alpha}), \\ &\lim_{s\nearrow \omega} \frac{k(s + vb(s))}{k(s)} = \mathrm{e}^{-v}, && v \in \mathbb{R} && \Rightarrow k \in \mathcal{D}(L_3), \end{aligned} \tag{4}$$

where the auxiliary function b can be chosen such that it is differentiable for $s < \omega$, $\lim_{s\to\omega} b'(s) = 0$, and $\lim_{s\to\infty} b(s)/s = 0$ if $\omega = \infty$, or $\lim_{s\to\omega} b(s)/(\omega - s) = 0$ if $\omega < \infty$, see e.g. [5].

An important question concerns the limit behaviour of $h_t(y)$, $f_z(y)$ and $f(y)$ if one of the conditions (3) is obeyed by the density $g_t(x)$. The meaning of this assumption is clarified by the following theorem.

Theorem 2. *Let us assume that the density $g_t(x)$ fulfils one of the conditions (4) and, moreover, $\alpha > 1$ for L_1.*

(i) *Then the density $h_t(y)$ belongs to the same domain of attraction as the density $g_t(x)$.*

(ii) *If the conditions (4) are fulfilled by the densities $g_t(x)$ uniformly in t then the densities $f_z(y)$ and $f(y)$ also belong to the same domain of attraction as $g_t(x)$.*

The parameters of the limit distributions are in both the cases changed to $\beta = \alpha - 1$ for L_1 and to $\beta = \alpha + 1/2$ for L_2.

P r o o f. The first step is to prove part (i) of the theorem. We shall distinguish the three limiting behaviours of $g_t(x)$.

" F r e c h e t " c a s e. Let us study

$$\lim_{y\to\infty}\frac{h_t(yv)}{h_t(y)}=\lim_{y\to\infty} v\frac{\int_{yv}^{\infty}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-(yv)^2}}}{\int_{y}^{\infty}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-(y)^2}}}.$$

Changing the variable in the numerator, $x\mapsto xv$, one obtains

$$\lim_{y\to\infty}\frac{\int_{y}^{\infty}\frac{g_t(xv)\,\mathrm{d}x}{\sqrt{x^2-(y)^2}}}{\int_{y}^{\infty}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-(y)^2}}}=\lim_{x\to\infty}\frac{g_t(xv)}{g_t(x)}=v^{-(\alpha+1)}.$$

Hence

$$\lim_{y\to\infty}\frac{h_t(yv)}{h_t(y)}=v^{-\alpha}=v^{-([\alpha-1]+1)}\Rightarrow h_t\in\mathcal{D}(L_{1,\alpha-1}).$$

One may try to check also the sufficient and necessary condition for $H_t\in\mathcal{D}(L_{1,\alpha-1})$, namely

$$\lim_{y\to\infty}\frac{1-H_t(yv)}{1-H_t(y)}=v^{-\alpha+1}.$$

Indeed we have

$$\begin{aligned}1-H_t(yv)&=\frac{1}{M_t}\int_{yv}^{\infty}s\int_{s}^{\infty}\frac{g_t(u)\,\mathrm{d}u\,\mathrm{d}s}{\sqrt{u^2-s^2}}=\frac{1}{M_t}\int_{yv}^{\infty}g_t(u)\int_{yv}^{u}\frac{s\,\mathrm{d}s}{\sqrt{u^2-s^2}}\,\mathrm{d}u\\&=\frac{1}{M_t}\int_{yv}^{\infty}g_t(u)\sqrt{u^2-y^2v^2}\,\mathrm{d}u=\frac{v^2}{M_t}\int_{y}^{\infty}\sqrt{u^2-y^2}g_t(uv)\,\mathrm{d}u,\end{aligned}$$

and similarly

$$1-H_t(y)=\frac{1}{M_t}\int_{y}^{\infty}\sqrt{u^2-y^2}g_t(u)\,\mathrm{d}u.$$

Hence

$$\lim_{y\to\infty}\frac{1-H_t(yv)}{1-H_t(y)}=v^2\lim_{y\to\infty}\frac{\int_{y}^{\infty}\sqrt{u^2-y^2}g_t(uv)\,\mathrm{d}u}{\int_{y}^{\infty}\sqrt{u^2-y^2}g_t(u)\,\mathrm{d}u}=v^2\lim_{y\to\infty}\frac{g_t(yv)}{g_t(y)}=v^{-\alpha+1}.$$

" W e i b u l l " c a s e. The following relation should be proved:

$$\lim_{y\to 0+}\frac{h_t(\omega-yv)}{h_t(\omega-y)}=\lim_{y\to 0+}\frac{(\omega-yv)\int_{\omega-yv}^{\omega}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-(\omega-yv)^2}}}{(\omega-y)\int_{\omega-y}^{\omega}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-(\omega-y)^2}}}=v^{\alpha-1/2}.$$

Substituting $x\mapsto\omega-xv$ in the numerator and $x\mapsto\omega-x$ in the denominator the limit becomes

$$v\lim_{y\to 0}\frac{g_t(\omega-xv)}{g_t(\omega-x)}\sqrt{\frac{(\omega-x)^2-(\omega-y)^2}{(\omega-xv)^2-(\omega-yv)^2}}=v\cdot v^{\alpha-1}\cdot\frac{1}{\sqrt{v}}=v^{\alpha-1/2},$$

where $x\leqslant y$. Hence $h\in\mathcal{D}(L_{2,\alpha+1/2})$ holds. The sufficient and necessary condition for the distribution function H_t can be checked as in the previous case.

"G u m b e l" c a s e. The Gumbel limit distribution is the most difficult case. We have to study the behaviour of

$$\lim_{y\to\omega-}\frac{h_t(y+vb(y))}{h_t(y)}=\lim_{y\to\omega-}\frac{(y+vb(y))\int_{y+vb(y)}^{\omega}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-(y+vb(y))^2}}}{y\int_y^{\omega}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-y^2}}}.$$

There is again an appropriate substitution, $x\mapsto x+vb(x)$, in the numerator. Since $b(y)/y\to 0$ as $y\to\omega$ the limit can be rewritten as

$$\lim_{y\to\omega}\frac{g_t(x+vb(x))}{g_t(x)}\sqrt{\frac{x^2-y^2}{x^2-y^2+2v(xb(x)-yb(y))+v^2(b^2(x)-b^2(y))}}(1+vb'(x)),$$

where $y<x<\omega$, and then simplified to

$$\lim_{y\to\omega}\underbrace{\frac{g_t(x+vb(x))}{g_t(x)}}_{\to\mathrm{e}^{-v}}\left[1+2v\underbrace{\frac{xb(x)-yb(y)}{(x-y)(x+y)}}_{\to 0}+v^2\underbrace{\frac{b^2(x)-b^2(y)}{x^2-y^2}}_{\to 0}\right]^{-1/2}\underbrace{(1+vb'(x))}_{\to 1},$$

which completes the proof of part (i).

Part (ii) of the theorem can be proved with help of part (i) and regardless of the limiting behaviour of the density h_t.

The conditional density $h_t(y)$ fulfils the conditions (4) uniformly as the conditions are fulfilled uniformly by the densities $g_t(x)$. Indeed,

$$\frac{h_t(\varphi(y))}{h_t(y)}=\frac{\varphi(y)\int_{\varphi(y)}^{\omega}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-\varphi^2(y)}}}{y\int_y^{\omega}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-y^2}}},$$

where $\varphi(y)$ is the appropriate transformation. Substituting $x\mapsto\varphi(x)$ in the numerator,

$$\frac{h_t(\varphi(y))}{h_t(y)}=\frac{\varphi(y)\int_y^{\omega}\frac{g_t(\varphi(x))\varphi'(x)\,\mathrm{d}x}{\sqrt{\varphi^2(x)-\varphi^2(y)}}}{y\int_y^{\omega}\frac{g_t(x)\,\mathrm{d}x}{\sqrt{x^2-y^2}}}$$

follows. Since the only term depending on t is the conditional density g_t, the uniformity of the density h_t is proved.

Let us now examine the limit

$$\lim_{y\to\xi}\frac{f_z(\varphi(y))}{f_z(y)}=\lim_{y\to\xi}\frac{\int_z^{\eta}\frac{M_t h_t(\varphi(y))g(t)\,\mathrm{d}t}{\sqrt{t(1+t)(t-z)}}}{\int_z^{\eta}\frac{M_t h_t(y)g(t)\,\mathrm{d}t}{\sqrt{t(1+t)(t-z)}}},$$

where ξ and $\varphi(y)$ are appropriate for the limiting case. Denoting by l the corresponding right hand side of the conditions (4), then for any $\varepsilon > 0$ and for y large enough we have

$$\left| \frac{\int_z^\eta \frac{M_t h_t(\varphi(y)) g(t)\,\mathrm{d}t}{\sqrt{t(1+t)(t-z)}}}{\int_z^\eta \frac{M_t h_t(y) g(t)\,\mathrm{d}t}{\sqrt{t(1+t)(t-z)}}} - l \right| = \left| \frac{\int_z^\eta \frac{M_t h_t(y) g(t)}{\sqrt{t(1+t)(t-z)}} \left(\frac{h_t(\varphi(y))}{h_t(y)} - l \right) \mathrm{d}t}{\int_z^\eta \frac{M_t h_t(y) g(t)\,\mathrm{d}t}{\sqrt{t(1+t)(t-z)}}} \right|$$

$$\leqslant \frac{\int_z^\eta \frac{M_t h_t(y) g(t)}{\sqrt{t(1+t)(t-z)}} \left| \frac{h_t(\varphi(y))}{h_t(y)} - l \right| \mathrm{d}t}{\int_z^\eta \frac{M_t h_t(y) g(t)\,\mathrm{d}t}{\sqrt{t(1+t)(t-z)}}} \leqslant \varepsilon \cdot l.$$

It remains to prove the assertion for the marginal density $f(y)$, but this is the same as the marginal density $h(y)$. Hence we need to consider

$$\lim_{y\to\xi} \frac{h(\varphi(y))}{h(y)} = \lim_{y\to\xi} \frac{\int_0^\eta \frac{g(t)}{M_t} \int_{\varphi(y)}^\omega \frac{g_t(x)\varphi(y)\,\mathrm{d}x}{\sqrt{x^2-\varphi^2(y)}}\,\mathrm{d}t}{\int_0^\eta \frac{g(t)}{M_t} \int_y^\omega \frac{g_t(x) y\,\mathrm{d}x}{\sqrt{x^2-y^2}}\,\mathrm{d}t}.$$

Repeating the argument of the preceding part, the proof of Theorem 2 is completed. □

We have mentioned already that the density $h_t(y)$ is not very useful in applications as the true shape factor of the particle is usually unknown. The uniformity condition is, as follows from the proof, quite natural and can be expected in this context. It is obvious that the ellipses observed in the sections are profiles of spheroids the size and shape factor of which are definitely greater. All these greater spheroids may contribute to our observation, hence the assumption of uniformity of the tail behaviour follows. On the other hand, we don't know how far is the assumed uniformity condition from the necessary condition of our theorem.

It is shown in [8] that a bivariate distribution function of the *Farlie-Gumbel-Morgenstern* (FGM) class fulfils the uniformity condition.

Lemma 1. *Consider that for all values of t and large values of x (large independently of t) the joint density $g(x,t)$ of the spheroid size and the shape factor is of the form of FGM class. Assume that the conditional distribution $g_{t_0}(x)$ satisfies the condition $C_{i,\alpha}$ for some i and t_0. Then the condition $C_{i,\alpha}$ is fulfilled by the densities $g_t(x)$ uniformly in t.*

Let us recall that a density function $g(x,t)$ is of the FGM class if it has the form

$$g(x,t) = g(x) g_T(t)[1 + \lambda\{2G(x) - 1\}\{2G_T(t) - 1\}],$$

where g, g_T and G, G_T are the marginal densities and distribution functions respectively, and $|\lambda| < 1$ is a parameter. Note that the components of a random vector (X, T) obeying the FGM distribution are mutually independent for $\lambda = 0$ and positively (negatively) correlated for $\lambda > 0$ ($\lambda < 0$).

4. Normalizing constants

The next step in the application of the proposed approach is to find a way how to calculate the normalizing constants. As was already mentioned above, we are looking for such a suitable affine transformation of the sample extreme $X_{1:n} = \max(X_i | 1 \leqslant i \leqslant n)$ (here conditioned by the size T) that

$$\mathrm{P}\left[\frac{X_{1:n} - b_n}{a_n} < x \,\middle|\, T = t\right] \xrightarrow{w} L_{i,\alpha}(x)$$

for some i and α. The pairs (a_n, b_n) (forming a sequence) are called the *normalizing constants* and their estimation can be based on the tail behaviour of the distribution functions, in particular on the tails $1 - G_t(x)$, $1 - F_z(y)$ and $1 - F(y)$.

Let us recall that the normalizing constants are based on quantiles of the studied distribution. The normalizing constants are not given uniquely, they rather form an "equivalence class" with respect to the limit behaviour. Namely, if (a_n, b_n) are normalizing constants, then the pair (a'_n, b'_n) such that

$$\lim_{n\to\infty} \frac{a'_n}{a_n} = 1, \quad \lim_{n\to\infty} \frac{b'_n - b_n}{a_n} = 0$$

may be also viewed as normalizing constants (but the convergence rates may differ). Therefore when evaluating quantiles in order to obtain normalizing constants, one can make an approximation (or simplification) keeping in mind the limit "uniqueness" mentioned above. The choice of the normalizing constants for three types of the limit behaviour of the distribution functions (each type represents one domain of attraction) is given by the following lemma:

Lemma 2. *Suppose that a distribution function K has an upper end point ω.*

(i) *If $\omega = \infty$ the distribution function K belongs to the Gumbel domain of attraction and if there exist constants $\alpha > 0$, β, $\gamma > 0$, $\delta > 0$ such that*

$$\lim_{v\to\infty} \frac{1 - K(v)}{\alpha v^{\beta} \mathrm{e}^{-\gamma v^{\delta}}} = 1,$$

then the normalizing constants may be chosen as

$$a_n = \left(\frac{\log n}{\gamma}\right)^{1/\delta - 1} \frac{1}{\gamma\delta},\ b_n = \left(\frac{\log n}{\gamma}\right)^{1/\delta} + \frac{\frac{\beta}{\delta}(\log\log n - \log\gamma) + \log\alpha}{(\frac{\log n}{\gamma})^{1-1/\delta}\gamma\delta}.$$

(ii) *If the distribution function K belongs to the Fréchet domain of attraction and if there exist constants $\alpha > 0$, β, $\gamma > 0$ such that*

$$\lim_{v\to\infty} \frac{1 - K(v)}{\alpha(\log v)^{\beta} v^{-\gamma}} = 1$$

then the normalizing constants may be chosen as

$$a_n = \left[n\left(\frac{\log n}{\gamma}\right)^{\beta}\alpha\right]^{1/\gamma},\ b_n = 0.$$

(iii) *If the distribution function K belongs to the Weibull domain of attraction and if there exist constants $\alpha, \gamma > 0$ such that*

$$\lim_{v\to\omega} \frac{1 - K(v)}{\gamma(v/\omega)^{\beta}(\omega - v)^{\alpha}} = 1,$$

then the normalizing constants may be chosen as

$$a_n = (n\gamma)^{-1/\alpha},\ b_n = \omega.$$

For the proof of the lemma see [4] or [11]. The tail behaviours assumed in the lemma cover the most important probability distributions. It can be expected that appropriate normalizing constants can be found by a similar examination of the extremal quantiles even for more complicated models of the limit behaviour of the tails $1 - K(v)$.

5. Tail behaviour in the FGM class

We shall assume that the density $g(x, t)$ of spheroid size and shape factor belongs to the Farlie-Gumbel-Morgenstern class for large values of x and all t in what follows. It means that there is x_0 such that for $\{(x, t)\colon\ x > x_0, t \in [0, \eta]\}$ we have

$$\begin{aligned} g(x,t) &= g(x)g_T(t)[1 + \lambda(2G(x) - 1)(2G_T(t) - 1)] \\ &= g(x)g_T(t)[1 + \lambda\{2(1 - G(x)) - 1\}(1 - 2G_T(t))] \\ &= g(x)g_T(t)[1 - \lambda(1 - 2G_T(t))] + g(x)(1 - G(x))2\lambda g_T(t)(1 - 2G_T(t)). \end{aligned}$$

Therefore one can write (asymptotically for large x and y)

$$
\begin{aligned}
1 - G_t(x) &= [1 - \lambda(1 - 2G_T(t))] \int_x^\omega g(u)\,\mathrm{d}u \\
&\quad + 2\lambda(1 - 2G_T(t)) \int_x^\omega g(u)(1 - G(u))\,\mathrm{d}u, \\
1 - F_z(y) &= \frac{\sqrt{1+z}}{2Mf(z)} \Bigg[\int_z^\eta \frac{g_T(t)[1 - \lambda(1 - 2G_T(t))]}{\sqrt{t(1+t)(t-z)}}\,\mathrm{d}t \\
&\quad \times \int_y^\omega g(x)\sqrt{x^2 - y^2}\,\mathrm{d}x \\
&\quad + \int_z^\eta \frac{2\lambda g_T(t)(1 - 2G_T(t))}{\sqrt{t(1+t)(t-z)}}\,\mathrm{d}t \\
&\quad \times \int_y^\omega g(x)(1 - G(x))\sqrt{x^2 - y^2}\,\mathrm{d}x \Bigg], \\
1 - H(y) &= \int_0^\eta \frac{g_T(t)[1 - \lambda(1 - 2G_T(t))]}{M_t}\,\mathrm{d}t \\
&\quad \times \int_y^\omega g(x)\sqrt{x^2 - y^2}\,\mathrm{d}x \\
&\quad + \int_0^\eta \frac{2\lambda g_T(t)(1 - 2G_T(t))}{M_t}\,\mathrm{d}t \\
&\quad \times \int_y^\omega g(x)(1 - G(x))\sqrt{x^2 - y^2}\,\mathrm{d}x.
\end{aligned}
\tag{5}
$$

It follows from (5) that the main attention must be paid to the integrals containing the marginal density $g(x)$ because the integrals containing the marginal density $g_T(t)$ approach constant values when the tail behaviour of the spheroid size is concerned.

Three typical tails. Three different forms of tails will be now compared, each of them representing one domain of attraction. Our classes cover such important distributions as Gamma, Normal, Weibull, Pareto or Beta. Examples will be given in the next section together with a discussion how to estimate the normalizing constants from the sample.

Following the order of cases covered by Lemma 2, the representative of the Gumbel domain of attraction will be investigated first.

Let the support of the density $g(x)$ be unbounded from the right and let it be possible to approximate $g(x)$ by

$$g(x) \approx a x^b \mathrm{e}^{-cx^d} \Rightarrow 1 - G(x) \approx \frac{a}{cd} x^{b-d+1} \mathrm{e}^{-cx^d}$$

for x large enough, where $a > 0$, b, $c > 0$ and $d > 0$ are some parameters. Here the sign $\approx$ for probability density means that

$$\lim_{x\to\infty} \frac{\int_x^\infty g(u)\,\mathrm{d}u}{\int_x^\infty au^b \mathrm{e}^{-cu^d}\,\mathrm{d}u} = 1.$$

Then

$$\begin{aligned}
(6)\qquad &\int_y^\infty ax^b \mathrm{e}^{-cx^d} \sqrt{x^2 - y^2}\,\mathrm{d}x \\
&= \int_0^\infty a(y+w)^b \mathrm{e}^{-c(y+w)^d} \sqrt{(y+w)^2 - y^2}\,\mathrm{d}w \\
&= ay^{b+1/2} \int_0^\infty \left(1 + \frac{w}{y}\right)^b \sqrt{1 + \frac{w}{2y}} \sqrt{2w} \exp\left\{ - cy^d \left(1 + \frac{w}{y}\right)^d \right\} \mathrm{d}w \\
&= ay^{b+2-3d/2} \int_0^\infty \left(1 + \frac{z}{y^d}\right)^b \sqrt{1 + \frac{z}{2y^d}} \sqrt{2z} \exp\left\{ - cy^d \left(1 + \frac{z}{y^d}\right)^d \right\} \mathrm{d}z \\
&\approx ay^{b+2-3d/2} \mathrm{e}^{-cy^d} \int_0^\infty \sqrt{2z} \mathrm{e}^{-c\,\mathrm{d}z}\,\mathrm{d}z \\
&= \frac{a}{(cd)^{3/2}} y^{b+2-3d/2} \mathrm{e}^{-cy^d} \sqrt{\frac{\pi}{2}}.
\end{aligned}$$

Further, evaluating the integral

$$\begin{aligned}
\int_y^\infty ax^b \mathrm{e}^{-cx^d} \frac{a}{cd} x^{b-d+1} \mathrm{e}^{-cx^d} \sqrt{x^2 - y^2}\,\mathrm{d}x& \\
&= \int_y^\infty \frac{a^2}{cd} x^{2b-d+1} \mathrm{e}^{-2cx^d} \sqrt{x^2 - y^2}\,\mathrm{d}x \\
&\approx \frac{a^2}{[2cd]^{3/2}} y^{2b+3-5d/2} \mathrm{e}^{-2cy^d} \sqrt{\frac{\pi}{2}},
\end{aligned}$$

we can conclude that this value is negligible with respect to the approximation (6) when $y \to \infty$. Using Lemma 2(i) and the limit behaviour of $1 - F_z(y)$ and $1 - H(y)$, the normalizing constants can be easily calculated.

The Fréchet domain of attraction contains densities the tail behaviour of which is of the type

$$g(x) \approx a(\log x)^b x^{-c-1} \Rightarrow 1 - G(x) \approx \frac{a}{c} (\log x)^b x^{-c},$$

where $a > 0$, $b > 0$ and $c > 0$ are some parameters. The tail behaviour of the transformed distribution functions F_z and H must be now appreciated:

$$
\begin{aligned}
(7)\qquad & \int_y^\infty a(\log x)^b x^{-c-1}\sqrt{x^2-y^2}\,\mathrm{d}x \\
&= \int_1^\infty a(\log y+\log z)^b z^{-c-1}y^{-c-1}\sqrt{y^2z^2-y^2}\,y\,\mathrm{d}z \\
&= ay^{-c+1}(\log y)^b\int_1^\infty\Big(1+\frac{\log z}{\log y}\Big)^b z^{-c-1}\sqrt{z^2-1} \\
&\approx ay^{-c+1}(\log y)^b\int_1^\infty z^{-c}\sqrt{1-z^{-2}}\,\mathrm{d}z \\
&= ay^{-c+1}(\log y)^b\int_0^1\frac{1}{2}v^{(c-3)/2}(1-v)^{1/2}\,\mathrm{d}v \\
&= \frac{1}{2}\mathrm{B}\Big(\frac{c-1}{2},\frac{3}{2}\Big)ay^{-c+1}(\log y)^b,
\end{aligned}
$$

where $\mathrm{B}(\cdot,\cdot)$ is the Beta function. A similar limit approximation of the limit behaviour of $\int_y^\infty a^2c^{-1}(\log x)^{2b}y^{-2c-1}\,\mathrm{d}x$ leads to the conclusion that its value is negligible in comparison with (7).

Finally, Weibull domain of attraction is represented by the densities of the tail behaviour

$$g(x)\approx a\Big(\frac{x}{\omega}\Big)^{b-1}(\omega-x)^{c-1}\Rightarrow 1-G(x)\approx\frac{a}{c}(\omega-x)^c$$

for x sufficiently close to ω, where $a > 0$, $b > 0$, and $c > 0$ are some parameters. The tail behaviour of F_z and H is determined by the integrals as follows:

(8)

$$
\begin{aligned}
&\int_y^\omega a\Big(\frac{x}{\omega}\Big)^{b-1}(\omega-x)^{c-1}\sqrt{x^2-y^2}\,\mathrm{d}x \\
&\quad= \int_0^{\omega-y} a\Big(1-\frac{v}{\omega}\Big)^{b-1}v^{c-1}\sqrt{(\omega-v)^2-y^2}\,\mathrm{d}v \\
&\quad= a(\omega-y)^c\int_0^1\Big(1-\frac{\omega-y}{\omega}w\Big)^{b-1}w^{c-1} \\
&\qquad\qquad\times\sqrt{\omega^2-2\omega(\omega-y)w+(\omega-y)^2w^2-y^2}\,\mathrm{d}w \\
&\quad= a(\omega-y)^{c+1/2}\int_0^1\Big(1-\frac{\omega-y}{\omega}w\Big)^{b-1}w^{c-1}\sqrt{(\omega+y)-2\omega w+(\omega-y)w^2}\,\mathrm{d}w \\
&\quad\approx a(\omega-y)^{c+1/2}\sqrt{2\omega}\int_0^1 w^{c-1}(1-w)^{1/2}\,\mathrm{d}w \\
&\quad= \mathrm{B}\Big(c,\frac{3}{2}\Big)a(\omega-y)^{c+1/2}\sqrt{2\omega},
\end{aligned}
$$

whereas the limit approximation of $\int_y^\omega a^2 c^{-1}(x/\omega)^{b-1}(\omega - x)^{2c-1}\sqrt{x^2 - y^2}\,\mathrm{d}x$ is, similarly to the two previous cases, negligible in comparison with (8).

Normalizing constants for FGM tails. Using Lemma 2 for the "three typical tails", the normalizing constants can be now calculated. Let us denote by

(1) (a_n, b_n) the normalizing constants for the tails of the size of the spheroid given its shape factor,

(2) (a_n^s, b_n^s) the normalizing constants for the tails of the size of the spheroid section given the section shape factor, and

(3) (a_n^m, b_n^m) the normalizing constants for the tails of the size of the spheroid section marginally.

Let further the terms independent of size be denoted by

$$\begin{aligned} K_G(t) &= 1 - \lambda[1 - 2G_2(t)], \\ K_F(z) &= \frac{\sqrt{1+z}}{2Mf(z)} \int_z^\eta \frac{g_2(t)[1 - \lambda(1 - 2G_2(t))]}{\sqrt{t(1+t)(t-z)}}\,\mathrm{d}z, \\ K_H &= \int_0^\eta \frac{g_2(t)[1 - \lambda(1 - 2G_2(t))]}{M_t}\,\mathrm{d}t. \end{aligned} \tag{9}$$

Then the previous results can be summarized in the following theorem:

Theorem 3 (Normalizing constants). *Let the joint density function $g(x,t)$ attain the FGM form of density asymptotically for large values of x (independently of t).*

(i) *If $g(x) \approx a x^b \mathrm{e}^{-cx^d}$ as $x \to \infty$ and $G \in \mathcal{D}(L_3)$ then*

$$\begin{aligned} a_n = a_n^s = a_n^m &= \left(\frac{\log n}{c}\right)^{1/d-1} \frac{1}{cd}, \\ b_n &= a_n \left\{ d \log n + \frac{b-d+1}{d}(\log\log n - \log c) + \log\left[\frac{aK_G(t)}{cd}\right]\right\}, \\ b_n^s &= a_n \left\{ d \log n + \frac{b-3d/2+2}{d}(\log\log n - \log c) + \log\left[\sqrt{\frac{\pi}{2}}\frac{aK_F(z)}{(cd)^{3/2}}\right]\right\}, \\ b_n^m &= a_n \left\{ d \log n + \frac{b-3d/2+2}{d}(\log\log n - \log c) + \log\left[\sqrt{\frac{\pi}{2}}\frac{aK_H}{(cd)^{3/2}}\right]\right\} \end{aligned}$$

can be used as the normalizing constants for the Gumbel limit distribution.

(ii) *If* $g(x) \approx a(\log x)^b x^{-c-1}$ *as* $x \to \infty$ *and* $G \in \mathcal{D}(L_{1,\alpha})$ *then*

$$\begin{aligned}
b_n &= b_n^s = b_n^m = 0,\\
a_n &= \left[n \left(\frac{\log n}{c} \right)^b K_G(t) \frac{a}{c} \right]^{1/c},\\
a_n^s &= \left[n \left(\frac{\log n}{c-1} \right)^b K_F(z) \mathrm{B}\left(\frac{c-1}{2}, \frac{3}{2} \right) \frac{a}{2} \right]^{1/(c-1)},\\
a_n^m &= \left[n \left(\frac{\log n}{c-1} \right)^b K_H \mathrm{B}\left(\frac{c-1}{2}, \frac{3}{2} \right) \frac{a}{2} \right]^{1/(c-1)}
\end{aligned}$$

can be used as the normalizing constants for the Fréchet limit distribution.

(iii) *If* $g(x) \approx a(x/\omega)^{b-1}(\omega - x)^{c-1}$ *as* $x \to \omega_-$ *and* $G \in \mathcal{D}(L_{2,\alpha})$ *then*

$$\begin{aligned}
b_n &= b_n^s = b_n^m = \omega,\\
a_n &= \left\{ n K_G(t) \frac{a}{c} \right\}^{-1/c},\\
a_n^s &= \left\{ n K_F(z) \mathrm{B}\left(c, \frac{3}{2} \right) a\sqrt{2\omega} \right\}^{-1/(c+0.5)},\\
a_n^m &= \left\{ n K_H \mathrm{B}\left(c, \frac{3}{2} \right) a\sqrt{2\omega} \right\}^{-1/(c+0.5)}
\end{aligned}$$

can be used as the normalizing constants for the Weibull limit distribution.

It should be noted that because of the tail equivalence in the FGM class, there is one normalizing constant independent of the shape factor distribution in each of these three sets. Consequently, the estimation of the normalizing constants is considerably simplified. On the other hand, some drawbacks of the FGM class have already been mentioned.

6. Examples

Assuming the asymptotic FGM form of the joint density functions and selected parametric forms of the marginal densities g_T (shape factor) and of the tail of g (spheroid size), the normalizing constants can be explicitly calculated.

First, various tails are considered using Theorem 3 (note that the Gamma and Weibull tails belong to the Gumbel DA, the Pareto tail to the Fréchet DA and the Beta tail to the Weibull DA).

Gamma tail. Then for large x we necessarily have

$$g(x) \approx \frac{\mu^{\gamma} x^{\gamma-1}}{\Gamma(\gamma)} \mathrm{e}^{-\mu x},$$

$\mu > 0, \gamma > 0, x > 0$. The normalizing constants can be chosen as

$$\begin{aligned}
a_n &= a_n^s = a_n^m = \frac{1}{\mu},\\
b_n &= a_n\left[\log n + (\gamma - 1)\log\log n + \log\left(\frac{K_G(t)}{\Gamma(\gamma)}\right)\right],\\
b_n^s &= a_n\left[\log n + \left(\gamma - \frac{1}{2}\right)\log\log n - \log\mu + \log\left(\frac{K_F(z)}{\Gamma(\gamma)}\sqrt{\frac{\pi}{2}}\right)\right],\\
b_n^m &= a_n\left[\log n + \left(\gamma - \frac{1}{2}\right)\log\log n - \log\mu + \log\left(\frac{K_H}{\Gamma(\gamma)}\sqrt{\frac{\pi}{2}}\right)\right].
\end{aligned}$$

Weibull tail. In this case

$$g(x) \approx \mu\gamma x^{\gamma-1}\mathrm{e}^{-\mu x^{\gamma}}$$

for x large enough, $\mu > 0$, $\gamma > 0$, $x > 0$ and the suitable normalizing constants are

$$\begin{aligned}
a_n &= a_n^s = a_n^m = \left(\frac{\log n}{\mu}\right)^{1/\gamma - 1},\\
b_n &= a_n[\gamma\log n + \log K_G(t)],\\
b_n^s &= a_n\left[\gamma\log n + \frac{1}{\gamma}(\log\log n - \log\mu) - \frac{1}{2}\log\log n + \log\left(K_F(z)\sqrt{\frac{\pi}{2\gamma}}\right)\right],\\
b_n^m &= a_n\left[\gamma\log n + \frac{1}{\gamma}(\log\log n - \log\mu) - \frac{1}{2}\log\log n + \log\left(K_H\sqrt{\frac{\pi}{2\gamma}}\right)\right].
\end{aligned}$$

Pareto tail. The assumption

$$g(x) \approx \frac{\gamma}{\sigma}\left(\frac{\sigma}{x}\right)^{\gamma+1}$$

for large x, $\gamma > 0, \sigma > 0, x > \sigma$ leads to

$$\begin{aligned}
b_n &= b_n^s = b_n^m = 0,\\
a_n &= [nK_G(t)]^{1/\gamma}\sigma,\\
a_n^s &= \left[nK_F(z)\mathrm{B}\left(\frac{\gamma-1}{2}, \frac{3}{2}\right)\frac{\gamma}{2}\sigma^{\gamma}\right]^{1/(\gamma-1)},\\
a_n^m &= \left[nK_H\mathrm{B}\left(\frac{\gamma-1}{2}, \frac{3}{2}\right)\frac{\gamma}{2}\sigma^{\gamma}\right]^{1/(\gamma-1)}.
\end{aligned}$$

B e t a t a i l. In the last example, let

$$g(x) \approx \frac{x^{b-1}(1-x)^{c-1}}{\mathrm{B}(b,c)}$$

for x close to 1, $0 < x < 1$ and $c > 0$. Then

$$\begin{aligned} b_n &= b_n^s = b_n^m = 1, \\ a_n &= \left[\frac{nK_G(t)}{c\mathrm{B}(b,c)}\right]^{-1/c}, \\ a_n^s &= \left[\sqrt{\frac{\pi}{2}}\frac{\Gamma(c+b)}{\Gamma(c+3/2)\Gamma(b)} nK_F(z)\right]^{-\frac{1}{c+1/2}}, \\ a_n^m &= \left[\sqrt{\frac{\pi}{2}}\frac{\Gamma(c+b)}{\Gamma(c+3/2)\Gamma(b)} nK_H\right]^{-\frac{1}{c+1/2}}. \end{aligned}$$

T h e c o n s t a n t s K_G, K_F a n d K_H. The estimation of

$$K_G(t) = 1 - \lambda[1 - 2G_T(t)]$$

for known parametric distributions of the shape factor is straightforward as the parameter λ can be estimated from the values of a_n^s, b_n^s and a_n^m, b_n^m.

The constants $K_F(z)$ and K_H need a more careful treatment. The assumption of $g_T(t)$ in some parametric form makes it possible to calculate (exactly or numerically) the integral

$$\int_z^\eta \frac{g_T(t)[1-\lambda(1-2G_T(t))]}{\sqrt{t(1+t)(t-z)}}\,\mathrm{d}t.$$

However, in order to evaluate the integral

$$\int_0^\eta \frac{g_T(t)[1-\lambda(1-2G_T(t))]}{M_t}\,\mathrm{d}t$$

and the value of M_t, some parametric form of $g_t(x)$ must be assumed. These obstacles can be overcome by considering M, K_F and K_H (which is independent of t as well as of z—see (9)) as certain constants which could be estimated from the data. Finally,

$$K_F(z) = \frac{\sqrt{1+z}}{2Mf(z)}[(1-\lambda)I_1(z) + 2\lambda I_2(z)],$$

where $I_1(z)$ and $I_2(z)$ can be calculated from the marginal density g_T of the shape factor and the density $f(z)$ may be again estimated from the sections. Moreover,

$$\begin{aligned} \frac{2Mf(z)}{\sqrt{1+z}} &= \int_z^\eta \frac{1}{\sqrt{t(1+t)(t-z)}} \int_0^\omega g(x,t) \int_0^x \frac{y}{\sqrt{x^2-y^2}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_z^\eta \frac{1}{\sqrt{t(1+t)(t-z)}} \int_0^\omega xg(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Consequently, assuming some parametric form of $g(x,t)$, the value of this integral can be calculated and the estimation of M and $f(z)$ from the data is not necessary.

Example-Pareto/uniform distribution. Let us consider a density $g(x,t)$ such that the tail of the marginal density is (for large values of x)

$$g(x) \approx \frac{\gamma}{\sigma}\left(\frac{\sigma}{x}\right)^{\gamma+1},$$

where $\sigma > 0$, $\gamma > 0$, $x > \sigma$, and that the marginal distribution of the shape factor is uniform on $[0,\eta]$: $g_T(t) = 1/\eta$ for $t \in [0,\eta]$. The largest value of the shape factor observed in the section can be used as an asymptotically unbiased estimator of η.

As we shall see in the next section, there are ML estimators for a_n^s (based on the k largest observations with a given shape factor), a_n^m (based on the k largest observations) and γ. Then we can estimate $\widehat{\gamma}$ from all the observations together with $\widehat{a_n^m}$.

Assuming the uniform distribution of the shape factor, it is possible to calculate the integrals $I_1(z)$ and $I_2(z)$ occurring in $K_F(z)$. They lead to elliptic functions and can be evaluated numerically. If two normalizing constants for different shape factor classes (possibly based on different number of observations p and q) are known, λ can be estimated from

$$\frac{a_p^m(z_1)}{a_q^m(z_2)} = \left(\frac{p}{q}\right)^{1/(\gamma-1)}\left[\frac{\sqrt{1+z_1}f(z_2)}{\sqrt{1+z_2}f(z_1)}\cdot\frac{\frac{1-\lambda}{\eta}I_1(z_1)+\frac{2\lambda}{\eta^2}I_2(z_1)}{\frac{1-\lambda}{\eta}I_1(z_2)+\frac{2\lambda}{\eta^2}I_2(z_2)}\right].$$

The estimates $\widehat{\eta}$, $\widehat{\gamma}$, $\widehat{\lambda}$ obtained make it also possible to estimate $\widehat{a_n}(t)$ for the t and n chosen. As we can choose $b_n \equiv 0$ in this case, the distribution of the largest of n spheroids with a shape factor t can be estimated from

$$\widehat{\mathrm{P}}[X_{1:n} < x | T = t] = L_{2,\widehat{\gamma}}(x/\widehat{a_n}(t)).$$

7. Estimation of the normalizing constants

We will briefly discuss the estimation of a_n^s, b_n^s and a_n^m, b_n^m from the observed data. These estimators form the basis for the estimation of the normalizing constants $\widehat{a_n}$, $\widehat{b_n}$. Therefore we base the prediction of the largest spheroid size in a selected shape factor class on the k largest observations of the spheroid section sizes.

Let a random variable with a distribution function K in a domain of attraction of Gumbel or Fréchet limit distributions be observed. Let further $M_1 \geqslant M_2 \geqslant$

$\ldots \geqslant M_k$ be the k largest observations, $\overline{M}_k$ their average and a_n, b_n the set of normalizing constants corresponding to the distribution function K. Then their maximum likelihood estimators are as follows ([4], [6], [9] or [16]):

G u m b e l d o m a i n o f a t t r a c t i o n. The limit distribution has no parameter in this case and the estimators of the normalizing constants are

$$\widehat{a_n} = \overline{M_k} - M_k,\ \widehat{b_n} = \widehat{a_n} \log k + M_k.$$

F r e c h e t d o m a i n o f a t t r a c t i o n. The choice $\widehat{b_n} \equiv 0$ is possible in this case but also the parameter α of the limit distribution must be estimated:

$$\widehat{a_n} = k^{1/\widehat{\alpha}} M_k,\ \widehat{\alpha} = k \left(\sum_{i=1}^{k} (\log M_i - \log M_k) \right)^{-1}.$$

There remain still many unsolved problems in the proposed procedure. One of the most obvious is the choice of the number k of the largest observations brought into calculations and its relation to the number of observations n. As the random vector (X, T) follows a distribution absolutely continuous w.r.t. Lebesgue measure, it is not possible to observe the conditional distribution directly. We must use small intervals of the shape factor (shape factor classes) instead and estimate the distribution of $(X|T = t)$ by the distribution of $(X|T \in [t - \delta, t + \delta])$. There is a problem which shape factor classes should be used. The ones with the most observations? Or those where large values of size are observed? And there is of course another question, namely, how well does the FGM family fit the data? Such statistical problems are, however, beyond the scope of the present paper.

A c k n o w l e d g m e n t. The author would like to thank to the referee for his valuable comments and hints.

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