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SEQUENTIAL CONVERGENCES ON GENERALIZED BOOLEAN ALGEBRAS

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Abstract. In this paper we investigate convergence structures on a generalized Boolean algebra and their relations to convergence structures on abelian lattice ordered groups.

Keywords: generalized Boolean algebra, abelian lattice ordered group, sequential convergence, elementary Carathéodory functions

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The system $\operatorname{Conv} B$ of all sequential convergences on a Boolean algebra B which are compatible with the structure of B was investigated in [5], [7], [9].

Some concrete types of sequential convergences on a Boolean algebra were dealt with by Löwig [10], Novák and Novotný [10] and Papangelou [12].

Let A be a generalized Boolean algebra. We define the system $\operatorname{Conv} A$ of sequential convergences on A in such a way that in the case when A is a Boolean algebra the new definition coincides with that given in [5].

For a lattice ordered group G the system $\operatorname{Conv} G$ of sequential convergences on G was studied in several papers; cf., e.g., [2], [3], [7].

Both $\operatorname{Conv} A$ and $\operatorname{Conv} G$ are partially ordered by the set-theoretical inclusion.

In this paper we prove that for each generalized Boolean algebra A there exists an abelian lattice ordered group G such that the partially ordered set $\operatorname{Conv} A$ is isomorphic to a convex subset of the partially ordered set $\operatorname{Conv} G$.

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From this we conclude that each interval of the partially ordered set $\operatorname{Conv} A$ is a complete lattice satisfying the infinite distributive law

$$(*) \qquad \left(\bigvee_{i\in I} \alpha_i\right) \wedge \beta = \bigvee_{i\in I} (\alpha_i \wedge \beta).$$

This generalizes a result from [9].

For an analogous relation between sequential convergences on MV-algebras and sequential convergences on lattice ordered groups cf. [8].

We apply the results and methods of [5], [6], [7].

1. Preliminaries

Through the paper A denotes a generalized Boolean algebra with the least element 0. Let $\mathbb{N}$ be the set of all positive integers. Then the direct power $A^{\mathbb{N}}$ is also a generalized Boolean algebra; its elements will be denoted by $(x_n)_{n\in\mathbb{N}}$ or, shortly, by (x_n). They are called sequences in A. If $a \in A$ and $x_n = a$ for each $n \in \mathbb{N}$, then we put $(x_n) = \operatorname{const} a$.

For $x, y \in A$ with $x \leqslant y$ we denote by $y \ominus x$ the relative complement of the element x in the interval $[0, y]$ of A.

If $\alpha \leqslant A^{\mathbb{N}} \times A$, then the relation $((x_n), x) \in \alpha$ will be expressed by writing

$$x_n \to_\alpha x.$$

1.1. Definition. A subset α of $A^{\mathbb{N}} \times A$ is said to be a convergence on A if the following conditions are satisfied:

(i) If $x_n \to_\alpha x$ and (y_n) is a subsequence of (x_n), then $y_n \to_\alpha x$.

(ii) If $(x_n) \in A^{\mathbb{N}}$, $x \in A$ and if for each subsequence (y_n) of (x_n) there exists a subsequence (z_n) of (y_n) such that $z_n \to_\alpha x$, then $x_n \to_\alpha x$.

(iii) If $a \in A$ and $(x_n) = \operatorname{const} a$, then $x_n \to_\alpha a$.

(iv) If $x_n \to_\alpha x$ and $x_n \to_\alpha y$, then $x = y$.

(v) If $x_n \to_\alpha x$ and $y_n \to_\alpha y$, then $x_n \vee y_n \to_\alpha x \vee y$, $x_n \wedge y_n \to_\alpha x \wedge y$.

(vi) If $x_n \leqslant y_n \leqslant z_n$ is valid for each $n \in \mathbb{N}$ and if $x_n \to_\alpha x$, $z_n \to_\alpha x$, then $y_n \to_\alpha x$.

(vii) For $x \in A$ and $(x_n) \in A^{\mathbb{N}}$ the relation $x_n \to_\alpha x$ holds if and only if the relations

$$x \ominus (x \wedge x_n) \to_\alpha 0, \quad (x \vee x_n) \ominus x \to_\alpha 0$$

are valid.

We denote by Conv A the system of all convergences on A; this system is partially ordered by the set-theoretical inclusion.

By an elementary calculation we can verify

1.2. Lemma. *Let A be a Boolean algebra and let $u, v \in A$, $u \leqslant v$. Then*

$$v \ominus u = v \wedge u',$$

where u' is the complement of u in A.

1.3. Lemma. *Let A be a Boolean algebra and $\alpha \subseteq A^{\mathbb{N}} \times A$. Suppose that the conditions* (iii), (v), (vi) *from* 1.1 *are satisfied and that, moreover, the implication*

$$\text{(c)} \qquad t_n \to_\alpha t \Rightarrow t'_n \to_\alpha t'$$

holds. Then the condition (vii) *from* 1.1 *is also valid.*

P r o o f. Assume that $x_n \to_\alpha x$. Then in view of (iii) and (v) we obtain

$$u_n \to_\alpha x,$$

where $u_n = x \wedge x_n$. Thus according to (c),

$$u'_n \to_\alpha x'.$$

Applying (iii) and (v) we get

$$x_n \wedge u'_n \to_\alpha x \wedge x'.$$

Since $x \wedge u'_n = x \ominus u_n$ (cf. 1.2), we have

$$x \ominus (x \wedge x_n) \to_\alpha 0.$$

By a similar argument we obtain

$$(u \vee x_n) \ominus x \to_\alpha 0.$$

Conversely, suppose that the conditions

$$x \ominus (x \wedge x_n) \to_\alpha 0, \quad (x \vee x_n) \ominus x \to_\alpha 0$$

are satisfied. Thus under the notation as above we have $x \ominus u_n \to_\alpha 0$. In view of 1.2,

$$x \wedge u_n' \to_\alpha 0.$$

Hence by (c) we get $x' \vee u_n \to_\alpha 1$, where 1 is the greatest element of A. According to (iii) and (v),

$$x \wedge (x' \vee u_n) \to_\alpha x \wedge 1,$$

thus $u_n \to_\alpha x$. Similarly we can verify that $v_n \to_\alpha x$, where $v_n = x \vee x_n$. Then we conclude from (vi) that $x_n \to_\alpha x$. □

1.4. Lemma. *Let A be a Boolean algebra, $\alpha \in \operatorname{Conv} A$, $x_n \to_\alpha x$. Then $x_n' \to_\alpha x'$.*

Proof. Let u_n and v_n be as in the proof of 1.3. Thus

$$u_n \to_\alpha x, \quad v_n \to_\alpha x$$

and $u_n \leqslant x_n \leqslant v_n$ for each $n \in \mathbb{N}$. Hence $u_n' \geqslant x_n' \geqslant v_n'$ for each $n \in \mathbb{N}$. In view of (vi) it suffices to verify that the relations

$$u_n' \to_\alpha x', \quad v_n' \to_\alpha x'$$

hold. Let us prove the first of these relations.

In view of (vii) we have to show that

$$x' \ominus (x' \wedge u_n') \to_\alpha 0 \quad \text{and} \quad (x' \vee u_n') \ominus x' \to_\alpha 0.$$

Since $u_n' \geqslant x'$, we have

$$x' \ominus (x' \wedge u_n') = x' \ominus x' = 0,$$

whence $x' \ominus (x' \wedge u_n') \to_\alpha 0$. Further,

$$(x' \vee u_n') \ominus x' = u_n' \ominus x'.$$

Thus according to 1.2,

$$(x' \vee u_n') \ominus x' = u_n' \wedge x = x \ominus u_n.$$

Since $u_n \to_\alpha x$, we conclude from (vii) that $x \ominus u_n \to_\alpha 0$, thus

$$(x' \vee u_n') \ominus x' \to_\alpha 0.$$

Therefore $u_n' \to_\alpha x'$. Similarly we obtain $v_n' \to_\alpha x'$. Thus $x_n' \to_\alpha x'$. □

Let us recall that Definition 1.1 in [5] differs from the above Definition 1.1 only in the points that

(α) it is assumed that the structure under consideration is a Boolean algebra, and

(β) instead of the condition (vii) it is assumed that the condition (c) is satisfied.

Hence in view of 1.3 and 1.4 we have

1.5. Proposition. *If A is a Boolean algebra, then the definition of* $\operatorname{Conv} A$ *given in* 1.1 *coincides with that considered in* 1.1 *of* [5].

2. The system $\operatorname{Conv}_0 A$

For each $\alpha \subseteq A^{\mathbb{N}} \times A$ we put

$$\alpha_0 = \{(x_n) \in A^{\mathbb{N}} \colon ((x_n), 0) \in \alpha\}.$$

Further we denote

$$\operatorname{Conv}_0 A = \{\alpha_0 \colon \alpha \in \operatorname{Conv} A\}.$$

The system $\operatorname{Conv}_0 A$ is partially ordered by the set-theoretical inclusion.

2.1. Lemma. *Let $\alpha, \beta \in \operatorname{Conv} A$, $\alpha_0 = \beta_0$. Then $\alpha = \beta$.*

Proof. Assume that $(x_n) \in A^{\mathbb{N}}$, $x \in A$, $x_n \to_\alpha x$. Hence in view of (vii),

$$x \ominus (x \wedge x_n) \to_\alpha 0, \quad (x \vee x_n) \ominus x \to_\alpha 0.$$

Thus we have also

$$x \ominus (x \wedge x_n) \to_\beta 0, \quad (x \vee x_n) \ominus x \to_\beta 0.$$

Applying (vii) again we get $x_n \to_\beta x$. Hence $\alpha \leqslant \beta$. In the same way we obtain $\beta \leqslant \alpha$. Therefore $\alpha = \beta$. □

The following lemma generalizes Lemma 1.5 of [5] (some steps in the proof are the same as in the proof of the lemma mentioned).

2.2. Lemma. *Let T_1 be a nonempty subset of $A^{\mathbb{N}}$. There exists $\alpha \in \operatorname{Conv} A$ with $\alpha_0 = T_1$ if and only if the following conditions are satisfied:*

(i_1) *If $(x_n) \in T_1$, then each subsequence of (x_n) belongs to T_1.*

(ii_1) *If $(x_n) \in A^{\mathbb{N}}$ and if each subsequence (y_n) of (x_n) has a subsequence which belongs to T_1, then $(x_n) \in T_1$,*

(iii_1) *For $a \in A$ we have $\operatorname{const} a \in T_1$ if and only if $a = 0$.*

(iv_1) *If (x_n) and (y_n) belong to T_1, then $(x_n \vee y_n) \in T_1$.*

(v_1) *If (x_n) belongs to T_1, $(y_n) \in A^{\mathbb{N}}$ and $y_n \leqslant x_n$ for each $n \in \mathbb{N}$, then $(y_n) \in T_1$.*

P r o o f. Assume that there is $\alpha \in \operatorname{Conv} A$ such that $T_1 = \alpha_0$. Then from 1.1 we immediately obtain that the conditions (i$_1$)–(v$_1$) hold.

Conversely, assume that T_1 is a subset of $A^{\mathbb{N}}$ such that the conditions (i$_1$)–(v$_1$) are satisfied. For $(x_n) \in A^{\mathbb{N}}$ and $x \in A$ we put

$$x_n \to_\alpha x$$

if

$$(*_1) \qquad (x \ominus (x \wedge x_n)) \in T_1 \quad \text{and} \quad ((x \vee x_n) \ominus x) \in T_1.$$

Consider the conditions (i)–(v) from 1.1.

(i)–(iii): These conditions easily follow from (i$_1$)–(iii$_1$).

(v): Assume that $x_n \to_\alpha x$ and $y_n \to_\alpha y$. Denote

$$\begin{aligned} x_n \vee y_n &= z_n, & x \vee y &= z,\\ z \wedge z_n &= u_n, & z \vee z_n &= v_n,\\ x \wedge x_n &= u_n^1, & x \vee x_n &= v_n^1,\\ y \wedge y_n &= u_n^2, & y \vee y_n &= v_n^2. \end{aligned}$$

Let n be a fixed element of $\mathbb{N}$. Consider the lattice $[0, v_n] = L$; for $t \in L$ let t' be the complement of t in the lattice L. In view of 1.2 we have

$$z \ominus u_n = z \wedge u_n',$$

whence

$$\begin{aligned} z \ominus u_n = z \wedge (z \wedge z_n)' = z \wedge (z' \vee z_n') = z \wedge z_n' = (x \vee y) \wedge (x_n \vee y_n)'\\ = (x \vee y) \wedge (x_n' \wedge y_n') = (x \wedge x_n' \wedge y_n') \vee (y \wedge x_n' \wedge y_n'). \end{aligned}$$

Applying 1.2 again we obtain

$$x \ominus u_n^1 = x \wedge x_n', \quad y \ominus u_n^2 = y \wedge y_n'.$$

Thus

$$(1) \qquad z \ominus u_n \leqslant (x \ominus u_n^1) \vee (y \ominus u_n^2).$$

In view of the assumption we have

$$(x \ominus u_n^1) \in T_1, \quad (y \ominus u_n^2) \in T_1$$

and then, according to (iv$_1$), (v$_1$) and (1) we get

$$(z \ominus u_n) \in T_1. \tag{2}$$

By an analogous method we prove

$$(v_n \ominus z) \in T_1. \tag{3}$$

Hence, in view of (2) and (3), the definition of α yields $z_n \to_\alpha z$. We have verified that $x_n \vee y_n \to_\alpha x \vee y$. Similarly we can verify that the relation $x_n \wedge y_n \to_\alpha x \wedge y$ is valid.

(vi): Suppose that $x_n \leqslant y_n \leqslant z_n$ for each $n \in \mathbb{N}$ and that $x_n \to_\alpha x$, $z_n \to_\alpha x$. Then

$$x \ominus (x \wedge z_n) \geqslant x \ominus (x \wedge y_n),$$
$$(x \vee z_n) \ominus x \geqslant (x \vee y_n) \ominus x$$

for each $n \in \mathbb{N}$, and

$$(x \ominus (x \wedge x_n)) \in T_1, \quad ((x \vee z_n) \ominus x) \in T_1.$$

Thus in view of (v$_1$),

$$(x \ominus (x \wedge y_n)) \in T_1, \quad ((x \vee y_n) \ominus x) \in T_1.$$

Hence $y_n \to_\alpha x$.

(iv): Assume that $x_n \to_\alpha x$ and $x_n \to_\alpha y$. By way of contradiction, suppose that $x \neq y$. Then in view of (v),

$$x_n = x_n \wedge x_n \to_\alpha x \wedge y.$$

We have either $x \wedge y \neq x$ or $x \wedge y \neq y$. Thus without loss of generality we can suppose that $x < y$.

Put $t_n = (x_n \vee x) \wedge y$. Then $x \leqslant t_n \leqslant y$. Applying (iii) and (v) we obtain

$$t_n \to_\alpha x, \quad t_n \to_\alpha y. \tag{4}$$

Let us consider the lattice $[0, y] = L$ and for $p \in L$ let p' be the complement of p in L. In view of (4),

$$(t_n \ominus x) \in T_1, \quad (y \ominus t_n) \in T_1,$$

hence according to 1.2,

$$(t_n \wedge x') \in T_1, \quad (y \wedge t'_n) \in T_1.$$

The second relation yields $(t'_n) \in T_1$. Thus from (iv$_1$) we conclude

$$((t_n \wedge x') \vee t'_n) \in T_1.$$

Hence $(x' \vee t'_n) \in T_1$. Clearly $x' \vee t'_n = x'$, whence $\operatorname{const} x' \in T_1$. Then in view of (iii$_1$) we get $x' = 0$ and thus $x = y$; we arrived at a contradiction.

(vii): For proving the validity of this condition it suffices to verify that

$$T_1 = \alpha_0.$$

Let $(x_n) \in \alpha_0$, hence $x_n \rightarrow_\alpha 0$. Then the condition $(*_1)$ is satisfied for $x = 0$. The second relation in $(*_1)$ yields $(x_n) \in T_1$.

Conversely, suppose that (x_n) belongs to T_1. We have

$$0 \ominus (0 \wedge x_n) = 0, \quad (0 \vee x_n) \ominus 0 = x_n,$$

hence in view of $(*_1)$, $x_n \rightarrow_\alpha 0$. □

For each $\alpha \in \operatorname{Conv} A$ we put $f_1(\alpha) = \alpha_0$.

2.3. Proposition. *f_1 is an isomorphism of the partially ordered set* $\operatorname{Conv} A$ *onto the partially ordered set* $\operatorname{Conv}_0 A$.

P r o o f. According to the definition of $\operatorname{Conv}_0 A$, f_1 is a mapping of $\operatorname{Conv} A$ onto the set $\operatorname{Conv}_0 A$. Moreover, it is obvious that if $\alpha, \beta \in A$ and $\alpha \leqslant \beta$, then $f_1(\alpha) \leqslant f_1(\beta)$.

Let $T_1 \in \operatorname{Conv}_0 A$. We apply Lemma 2.2. By means of the condition $(*_1)$ we assign to T_1 an element α of $\operatorname{Conv} A$; we denote

$$f_2(T_1) = \alpha.$$

In view of $(*_1)$, whenever $T_1, T_2 \in \operatorname{Conv}_0 A$ and $T_1 \leqslant T_2$, then $f_2(T_1) \leqslant f_2(T_2)$. Next, from that part of the proof of 2.2 which concerns the condition (vii) we conclude that

$$f_2(T) = \alpha \Rightarrow f_1(\alpha) = T,$$

whence $f_2 = f_1^{-1}$. Thus f_1 is an isomorphism of $\operatorname{Conv} A$ onto $\operatorname{Conv}_0 A$. □

3. Auxiliary results

Let A be as above and let A_1 be a nonempty subset of $A^{\mathbb{N}}$. We denote by

δA_1—the set of all subsequences of sequences belonging to A_1;

A_1^*—the set of all $(x_n) \in A^{\mathbb{N}}$ such that for each subsequence (y_n) of (x_n) there is a subsequence (z_n) of (y_n) which belongs to A_1;

$[A_1]$—the ideal of the generalized Boolean algebra $A^{\mathbb{N}}$ generated by the set A_1.

3.1. Definition. Let A_1 be as above. A_1 is called regular in $A^{\mathbb{N}}$ if there exists $\alpha_0 \in \operatorname{Conv}_0 A$ such that $A_1 \subseteq \alpha_0$.

By the same method as in Section 2 of [5] we obtain the following results 3.2 and 3.3.

3.2. Proposition. *Let $\emptyset \neq A_1 \subseteq A^{\mathbb{N}}$. Then the following conditions are equivalent:*

(i) *A_1 is regular in $A^{\mathbb{N}}$.*

(ii) *If $(y_n^1), (y_n^2), \ldots, (y_n^m)$ are elements of δA_1 and b is an element of A such that $b \leqslant y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$ is valid for each $n \in \mathbb{N}$, then $b = 0$.*

3.3. Lemma. *Let A_1 be a regular subset of $A^{\mathbb{N}}$. Then*

(i) *$[\delta A_1]^* \in \operatorname{Conv}_0 A$.*

(ii) *If $\alpha_0 \in \operatorname{Conv}_0 A$ and $A_1 \subseteq \alpha_0$, then $[\delta A_1]^* \subseteq \alpha_0$.*

If A_1 is regular in A, then in view of 3.3 we say that $[\delta A_1]^*$ is the element of $\operatorname{Conv}_0 A$ which is generated by the set A_1.

Now let G be an abelian lattice ordered group. For the definition of $\operatorname{Conv} G$, cf., e.g., [6]. Thus $\operatorname{Conv} G$ is a nonempty subset α of $G^{\mathbb{N}} \times G$ satisfying conditions analogous to (i)–(vi) in 1.1 with the distinction that in (v) also the validity of the relation $x_n + y_n \to_\alpha x + y$ is assumed. Similarly as in the case of a generalized Boolean algebra we define $\operatorname{Conv}_0 G$. Both the systems $\operatorname{Conv} G$ and $\operatorname{Conv}_0 G$ are partially ordered by the set-theoretical inclusion and, under this partial order, they are isomorphic.

A nonempty subset M of $(G^+)^{\mathbb{N}}$ is called regular in $(G^+)^{\mathbb{N}}$ if there exists $\alpha_0 \in \operatorname{Conv}_0 G$ with $M \subseteq \alpha_0$.

Let $\emptyset \neq M \subseteq (G^+)^{\mathbb{N}}$. The sets δM, M^* and $[M]$ are defined analogously as above (instead of the lattice A_1 we consider now the lattice G^+). Further, let $\langle M \rangle$ be the subsemigroup of the semigroup $(G^+)^{\mathbb{N}}$ generated by the set M.

3.4. Proposition. (Cf. [3]). *Let $\emptyset \neq M \subseteq (G^+)^{\mathbb{N}}$. Then the following conditions are equivalent:*

(a) *M is regular in $(G^+)^{\mathbb{N}}$.*

(b) *If $g \in G$, $\operatorname{const} g \in [\langle \delta M \rangle]$, then $g = 0$.*

3.5. Lemma. *Let $\emptyset \neq M \subseteq (G^+)^{\mathbb{N}}$. Then the following conditions are equivalent:*

(i) *M is regular in $(G^+)^{\mathbb{N}}$.*

(ii) *If $(h_n^1), (h_n^2), \ldots, (h_n^k)$ are subsequences of some sequences belonging to M and if $h_n = h_n^1 \vee h_n^2 \vee \ldots \vee h_n^k$ $(n = 1, 2, \ldots)$, then $\bigwedge_{n \in \mathbb{N}} h_n = 0$.*

P r o o f. The method is the same as in the proof of Lemma 2.5 in [6] with the distinction that the set $\{(g_n)\}$ considered in the lemma mentioned is replaced by the set M (we have to apply Proposition 3.4 above and Lemma 2.4 from [6]). □

An element $x \in G^+$ is called singular if the interval $[0, x]$ of G is a Boolean algebra. Let $S(G)$ be the set of all singular elements of G. The following assertion is easy to verify.

3.6. Lemma. *$S(G)$ is a convex sublattice of the lattice $(G^+, \leqslant)$.*

3.7. Corollary. *$S(G)$ is a generalized Boolean algebra.*

Let us denote $S(G) = A$.

3.8. Lemma. *Let $\emptyset \neq A_1 \subseteq A^{\mathbb{N}}$. Then the following conditions are equivalent:*

(i) *A_1 is regular in $A^{\mathbb{N}}$.*

(ii) *A_1 is regular in $(G^+)^{\mathbb{N}}$.*

P r o o f. This is implied by 3.2 and 3.5. □

Let $\alpha_1 \in \operatorname{Conv}_0 A$. Then α_1 is regular in $A^{\mathbb{N}}$. Hence in view of 3.8, α_1 is regular in $(G^+)^{\mathbb{N}}$. Then according to [2] there exists $T(\alpha_1) \in \operatorname{Conv}_0 G$ such that

(i) $\alpha_1 \subseteq T(\alpha_1)$,

(ii) if $\beta \in \operatorname{Conv}_0 A$ and $\alpha_1 \subseteq \beta$, then $T(\alpha_1) \subseteq \beta$.

(Namely, $T(\alpha_1) = [\langle \delta \alpha_1 \rangle]^*$).

3.9. Lemma. (Cf. [7], Lemma 3.3). *Let $(x_n) \in (G^+)^{\mathbb{N}}$. Under the above assumptions and notation, the following conditions are equivalent:*

(i) $(x_n) \in T(\alpha_1)$.

(ii) *There are $m \in \mathbb{N}$ and $(z_n) \in (\alpha_1)$ such that $x_n \leqslant m z_n$ for each $n \in \mathbb{N}$.*

3.10. Lemma. *Let $x, y \in A$, $m \in \mathbb{N}$, $x \leqslant my$. Then $x \leqslant y$.*

Proof. Denote $v = x \vee y$. Then in view of 3.6, $v \in A$, hence the interval $[0, v]$ of G is a Boolean algebra. By way of contradiction, assume that $x \nleqslant y$. Then there is $x_1 \in [0, v]$ such that $0 < x_1 \leqslant x$ and $x_1 \wedge y = 0$. Hence $x_1 \wedge my = 0$, which is a contradiction. □

For a related result (under a stronger assumption) cf. [7], Lemma 3.5.

Applying 3.9 and 3.10 and using the same method as in the proof of 3.6 in [7] we get

3.11. Lemma. *The mapping T is an isomorphism of the partially ordered set $\operatorname{Conv}_0 A$ into the partially ordered set $\operatorname{Conv}_0 G$.*

The system $\operatorname{Conv}_0 A$ has the least element, let us denote it by α^0. A sequence (x_n) in A belongs to α^0 if and only if there is $m \in \mathbb{N}$ such that $x_{m+n} = 0$ for each $n \in \mathbb{N}$. It is obvious that $T(\alpha^0)$ is the least element of $\operatorname{Conv}_0 G$.

3.12. Lemma. *Let $x \in G^+$, $a \in A$, $m \in \mathbb{N}$ and $x \leqslant ma$. Put $a_1 = x \wedge a$. Then $x \leqslant ma_1$.*

Proof. Since the interval $[0, a]$ of G is a Boolean algebra, there exists $a_2 \in [0, a]$ such that $a_1 \wedge a_2 = 0$ and $a_1 \vee a_2 = a$. Denote $x \wedge a_2 = a_3$. If $a_3 > 0$, then $a_1 \vee a_3 \leqslant x$. Moreover, $a_1 \wedge a_3 = 0$, whence $a_1 \vee a_3 = a_1 + a_3 > a_1$, which is a contradiction. Thus $a_3 = 0$ and hence $x \wedge a_2 = 0$. This yields that $x \wedge ma_2 = 0$. Therefore

$$x = x \wedge ma = x \wedge m(a_1 \vee a_2) = x \wedge (ma_1 \vee ma_2) = x \wedge ma_1.$$

□

Now let $\alpha_1 \in \operatorname{Conv}_0 A$ and $\beta \in \operatorname{Conv}_0 G$. Assume that $\beta \leqslant T(\alpha_1)$. Let $(x_n) \in \beta$. Thus $(x_n) \in T(\alpha_1)$. Hence the condition (ii) from 3.9 is valid. For each $n \in \mathbb{N}$ we put

$$z_n^1 = x_n \wedge z_n. \tag{1}$$

Then we have $(z_n^1) \in \beta$. Let us denote by Z_1 the system of all sequences (z_n^1) which can be constructed in this way. Hence $Z_1 \subseteq \beta$ and thus Z_1 is regular in $(G^+)^{\mathbb{N}}$. Moreover, $Z_1 \subseteq A^{\mathbb{N}}$ and consequently, in view of 3.8, Z_1 is regular in $A^{\mathbb{N}}$. Thus there exists $\alpha_2 \in \operatorname{Conv}_0 A$ such that α_2 is generated by Z_1. The relation $Z_1 \subseteq \beta$ implies $T(\alpha_2) \leqslant \beta$.

If (x_n) is as above, then in view of (1) and 3.12 we get

$$x_n \leqslant m z_n^1 \quad \text{for each } n \in \mathbb{N}.$$

From this and from 3.9 we infer that $\beta \leqslant T(\alpha_2)$. Summarizing, $\beta = T(\alpha_2)$. Hence we have

3.13. Lemma. *$T(\mathrm{Conv}_0 A)$ is a convex subset of the partially ordered set $\mathrm{Conv}_0 G$.*

4. Elementary Carathéodory functions

The system $E(B)$ of elementary Carathéodory functions corresponding to a Boolean algebra B was used by Gofman [1] and the author [4], [8].

The definition of $E(B)$ can be applied without any modification for the case when instead of a Boolean algebra B we have a generalized Boolean algebra A. For the sake of completeness, we recall the definition. For any $u, v \in A$ we put

$$v \ominus_1 u = v \ominus (v \wedge u).$$

Let A be a generalized Boolean algebra. If $x, y \in A$ and $x \leqslant y$, then the symbol $y \ominus x$ has the same meaning as above.

We denote by $E(A)$ the set consisting of all forms

$$f = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n, \tag{1}$$

where $a_i \neq 0$ are reals and $b_i \in A$, $b_i > 0$, $b_{i(1)} \wedge b_{i(2)} = 0$ for any distinct $i(1), i(2) \in \{1, 2, \ldots, n\}$, and of the "empty form". If g is another such form,

$$g = a_1^0 b_1^0 + a_2^0 b_2^0 + \ldots + a_m^0 b_m^0,$$

then f and g are considered as equal if

$$\bigvee_{i=1}^{n} b_i = \bigvee_{j=1}^{m} b_j^0, \tag{i}$$

$$a_i = a_j^0 \quad \text{whenever} \quad b_i \wedge b_j^0 \neq 0. \tag{ii}$$

The operation $+$ in $E(A)$ is defined by

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + a_j^0)(b_i \wedge b_j^0) + \sum_{i=1}^{n} a_i \Bigl(b_i \ominus_1 \bigvee_{j=1}^{m} b_j^0 \Bigr) + \sum_{j=1}^{m} a_j^0 \Bigl(b_j^0 \ominus_1 \bigvee_{i=1}^{n} b_i \Bigr),$$

where in the summation only those terms are taken into account in which $a_i + a_j^0 \neq 0$ and the elements

$$b_i \wedge b_j^0, \quad b_i \ominus_1 \bigvee_{j=1}^{m} b_j^0, \quad b_j^0 \ominus_1 \bigvee_{i=1}^{n} b_i$$

are non-zero. The multiplication by a real $a \neq 0$ is defined by

$$af = (aa_1)b_1 + \ldots + (aa_n)b_n;$$

$0f$ is the empty form. The form f is positive if $a_i > 0$ for $i = 1, 2, \ldots, n$. Then $E(A)$ is a vector lattice; the empty form is the zero element of $E(A)$.

If we disregard the multiplication by reals, then $E(A)$ is an abelian lattice ordered group.

Let $G(A)$ be the subset of $E(A)$ consisting of the empty form f_0 and of all forms (1) such that all a_i are integers, $a_i \neq 0$. Then $G(A)$ is an ℓ-subgroup of the lattice ordered group $E(A)$.

If we identify the element f_0 with the zero element of A and if, moreover, for each $0 \neq b \in A$ we identify the form $f = 1b$ with the element b, then A turns out to be a subset of $G(A)$.

The following assertion is easy to verify.

4.1. Lemma. *A is the set of all singular elements of $G(A)$.*

4.2. Theorem. *Let A be a generalized Boolean algebra and let $G = G(A)$. Then the mapping T defined in* Section 3 *is an isomorphism of the partially ordered set* $\mathrm{Conv}_0\, A$ *into the partially ordered set* $\mathrm{Conv}_0\, G$ *such that* $T(\mathrm{Conv}_0\, A)$ *is a convex subset of* $\mathrm{Conv}_0\, G$ *containing the least element of* $\mathrm{Conv}_0\, G$.

P r o o f. This is a consequence of 4.1 and of the results of Section 3 (cf. 3.12 and 3.13). □

In view of 2.3 and of the fact that $\mathrm{Conv}\, G$ is isomorphic to $\mathrm{Conv}_0\, G$ for each lattice ordered group we also have

4.3. Corollary. *Let A be a generalized Boolean algebra. There exists an abelian lattice ordered group G such that the partially ordered set* $\mathrm{Conv}\, A$ *is isomorphic to a convex subset of the partially ordered set* $\mathrm{Conv}\, G$.

Further, from 2.2 and 3.3 we immediately obtain

4.4. Corollary. *Let A be a generalized Boolean algebra. Then each interval of the partially ordered set* $\mathrm{Conv}\, A$ *is a complete lattice satisfying identically the relation* $(*)$.

References

[1] *C. Gofman*: Remarks on lattice ordered groups and vector lattices, I, Carathéodory functions. Trans. Amer. Math. Soc. *88* (1958), 107–120.
[2] *M. Harminc*: Sequential convergences on abelian lattice-ordered groups. Convergence structures, 1984, Math. Research, Band. vol. 24, Akademie Verlag, Berlin, 1985, pp. 153–158.
[3] *M. Harminc*: The cardinality of the system of all sequential convergences on an abelian lattice ordered group. Czechoslovak Math. J. *37* (1987), 533–546.
[4] *J. Jakubík*: Cardinal properties of lattice ordered groups. Fundamenta Math. *74* (1972), 85–98.
[5] *J. Jakubík*: Sequential convergences in Boolean algebras. Czechoslovak Math. J. *38* (1988), 520–530.
[6] *J. Jakubík*: Lattice ordered groups having a largest convergence. Czechoslovak Math. J. *39* (1989), 717–729.
[7] *J. Jakubík*: Convergences and higher degrees of distributivity of lattice ordered groups and of Boolean algebras. Czechoslovak Math. J. *40* (1990), 453–458.
[8] *J. Jakubík*: Sequential convergences on MV-algebras. Czechoslovak Math. J. *45* (1995), 709–726.
[9] *J. Jakubík*: Disjoint sequences in Boolean algebras. Math. Bohem. *123* (1998), 411–418.
[10] *H. Löwig*: Intrinsic topology and completion of Boolean rings. Ann. Math. *43* (1941), 1138–1196.
[11] *J. Novák, M. Novotný*: On the convergence in σ-algebras of point-sets. Czechoslovak Math. J. *3* (1953), 291–296.
[12] *F. Papangelou*: Order convergence and topological completion of commutative lattice-groups. Math. Ann. *155* (1964), 81–107.

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