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POSITIVE SOLUTIONS OF INEQUALITY WITH
 p -LAPLACIAN IN EXTERIOR DOMAINS

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Abstract. In the paper the differential inequality

$$\Delta_p u + B(x, u) \leq 0,$$

where $\Delta_p u = \operatorname{div}(\|\nabla u\|^{p-2} \nabla u)$, $p > 1$, $B(x, u) \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ is studied. Sufficient conditions on the function $B(x, u)$ are established, which guarantee nonexistence of an eventually positive solution. The generalized Riccati transformation is the main tool.

Keywords: p -Laplacian, oscillation criteria

MSC 2000: 35B05

1. INTRODUCTION

In the paper we study positive solutions of the partial differential inequality

$$(1) \quad \Delta_p u + B(x, u) \leq 0,$$

where $\Delta_p u = \operatorname{div}(\|\nabla u\|^{p-2} \nabla u)$ is the p -Laplace operator, $p > 1$, $B(x, u): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $\|\cdot\|$ is the usual Euclidean norm in \mathbb{R}^n . Inequality (1) covers several equations and inequalities studied in literature. If $p = 2$ then (1) reduces to the semilinear Schrödinger inequality

$$(2) \quad \Delta u + B(x, u) \leq 0,$$

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studied in [6], [7]. Another important special case of (1) is the half-linear differential equation

$$(3) \quad \Delta_p u + c(x)|u|^{p-1} \operatorname{sgn} u = 0,$$

studied in [2], [3]. For important applications of equations with p -Laplacian see [1].

The aim of this paper is to introduce sufficient conditions for nonexistence of a solution which would be eventually positive (i.e., positive outside of some ball in \mathbb{R}^n). Remark that in a similar way one can study also negative solutions of the inequality

$$\Delta_p u + B(x, u) \geq 0,$$

and a combination of these results produces criteria for nonexistence of a solution of the inequality

$$(4) \quad u[\Delta_p u + B(x, u)] \leq 0$$

which would have no zero outside of some ball in \mathbb{R}^n , the so called weak oscillation criteria. A simple version of this procedure is used in Corollary 6. A more elaborated version of this procedure can be found in [6].

The following notation is used throughout the paper: $\langle \cdot, \cdot \rangle$ denotes the scalar product, $q = \frac{p}{p-1}$ is the conjugate number to the number p ,

$$\begin{aligned} \Omega(a, b) &= \{x \in \mathbb{R}^n : a \leq \|x\| \leq b\}, \\ \Omega_a &= \Omega(a, \infty) = \{x \in \mathbb{R}^n : a \leq \|x\|\}, \\ S_a &= \partial\Omega_a = \{x \in \mathbb{R}^n : \|x\| = a\}, \end{aligned}$$

and $\omega_1 = \int_{S_1} 1 \, d\sigma$ is the measure of the n -dimensional unit sphere in \mathbb{R}^n .

2. RICCATI TRANSFORMATION

The main tool used for the study of positive solutions is the generalized Riccati transformation. The special case of this transformation has been used in [6], where inequality (2) is studied. A simple version of this transformation, convenient for the half-linear equation, has been introduced in [2].

Our approach combines both these methods. We use the transformation

$$(5) \quad \bar{w}(x) = -\alpha(\|x\|) \frac{\|\nabla u(x)\|^{p-2} \nabla u(x)}{\varphi(u(x))}$$

$\alpha \in C^1([a_0, \infty), \mathbb{R}^+)$, $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ which maps a positive C^1 function $u(x)$ into an n -vector function $\vec{w}(x)$.

Lemma 1. *Let u be a positive solution of (1) on Ω_{a_0} . The n -vector function $\vec{w}(x)$ is well-defined by (5) and satisfies the Riccati-type inequality*

$$(6) \quad \operatorname{div} \vec{w}(x) \geq \frac{\alpha(\|x\|)B(x, u(x))}{\varphi(u(x))} + \frac{\alpha'(\|x\|)}{\alpha(\|x\|)} \langle \vec{\nu}(x), \vec{w}(x) \rangle + \alpha^{1-q}(\|x\|)\varphi^{q-2}(u(x))\varphi'(u(x))\|\vec{w}(x)\|^q$$

on Ω_{a_0} , where $\vec{\nu}(x) = \frac{x}{\|x\|}$ is the outward unit normal vector to the sphere $S_{\|x\|}$.

Proof. Let $u(x) \geq 0$ be a solution of (1) on Ω_{a_0} and let $\vec{w}(x)$ be defined by (5). From (5) it follows that

$$\operatorname{div} \vec{w} = \frac{\alpha}{\varphi(u)} \Delta_p u - \|\nabla u\|^{p-2} \left\langle \nabla u, \nabla \left(\frac{\alpha}{\varphi(u)} \right) \right\rangle$$

and in view of (1)

$$\operatorname{div} \vec{w} \geq \frac{\alpha B(x, u)}{\varphi(u)} - \frac{\alpha' \|\nabla u\|^{p-2}}{\varphi(u)} \langle \nabla u, \vec{\nu} \rangle + \frac{\alpha \varphi'(u)}{\varphi^2(u)} \|\nabla u\|^p$$

holds (the dependence on $x \in \Omega_{a_0}$ is suppressed in the notation). In view of (5) this inequality is equivalent to (6). \square

3. NONEXISTENCE OF POSITIVE SOLUTION

The main result of the paper is the following

Theorem 1. *Let $a_0 \geq 0$. Suppose that there exist functions*

$$\alpha \in C^1([a_0, \infty), \mathbb{R}^+), \quad \varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+), \quad c \in C(\mathbb{R}^n, \mathbb{R}),$$

and numbers $k, l, k > 0, l > 1$, such that

- (i) $B(x, u) \geq c(x)\varphi(u)$ for $x \in \mathbb{R}^n, u > 0$,
- (ii) $\varphi'(u)\varphi^{q-2}(u) \geq k$ for $u > 0$,
- (iii) $\lim_{r \rightarrow \infty} \int_{\Omega(a_0, r)} \left[\alpha(\|x\|)c(x) - \frac{1}{p} \left(\frac{l}{kq} \right)^{p-1} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|) \right] dx = +\infty$,
- (iv) $\lim_{r \rightarrow \infty} \int_{a_0}^r \alpha^{\frac{1}{1-p}}(r) r^{\frac{1-n}{p-1}} dr = +\infty$.

Then (1) has no positive solution on Ω_a for arbitrary $a > 0$.

P r o o f. Suppose, by contradiction, that u is a solution of (1) positive on Ω_a for some $a > a_0$. Lemma 1 and the assumptions (i), (ii) imply

$$\begin{aligned} \operatorname{div} \vec{w} &\geq \alpha c + \frac{\alpha'}{\alpha} \langle \vec{v}, \vec{w} \rangle + \alpha^{1-q} k \|\vec{w}\|^q \\ &= \alpha c + \alpha^{1-q} \frac{kq}{l} \left[\frac{\|\vec{w}\|^q}{q} + \left\langle \vec{w} \frac{l\alpha^{q-2}\alpha'}{kq} \vec{v} \right\rangle \right] + \alpha^{1-q} \frac{k}{l^*} \|\vec{w}\|^q, \end{aligned}$$

where $l^* = \frac{l}{l-1}$ is the conjugate number to the number l . The Young inequality implies

$$\frac{\|\vec{w}\|^q}{q} + \left\langle \vec{w} \frac{l\alpha^{q-2}\alpha'}{kq} \vec{v} \right\rangle + \frac{1}{p} \left(\frac{l\alpha^{q-2}|\alpha'|}{kq} \right)^p \geq 0.$$

Combining both these inequalities we obtain

$$\begin{aligned} \operatorname{div} \vec{w} &\geq \alpha c - \alpha^{1-q} \frac{kq}{lp} \left(\frac{l\alpha^{q-2}|\alpha'|}{kq} \right)^p + \alpha^{1-q} \frac{k}{l^*} \|\vec{w}\|^q \\ &= \alpha c - \frac{1}{p} \left(\frac{l}{kq} \right)^{p-1} |\alpha'|^p \alpha^{1-p} + \alpha^{1-q} \frac{k}{l^*} \|\vec{w}\|^q. \end{aligned}$$

Integration of the last inequality over $\Omega(a, r)$ and the Gauss-Ostrogradski divergence theorem gives

$$\begin{aligned} \int_{S_r} \langle \vec{w}, \vec{v} \rangle d\sigma - \int_{S_a} \langle \vec{w}, \vec{v} \rangle d\sigma &\geq \frac{k}{l^*} \int_{\Omega(a,r)} \alpha^{1-q} \|\vec{w}\|^q dx \\ &\quad + \int_{\Omega(a,r)} \left[\alpha c - \frac{1}{p} \left(\frac{l}{kq} \right)^{p-1} |\alpha'|^p \alpha^{1-p} \right] dx. \end{aligned}$$

By assumption (iii) there exists $r_0, r_0 > a$, such that

$$\int_{\Omega(a,r)} \left[\alpha c - \frac{1}{p} \left(\frac{l}{kq} \right)^{p-1} |\alpha'|^p \alpha^{1-p} \right] dx + \int_{S_a} \langle \vec{w}, \vec{v} \rangle d\sigma \geq 0 \quad \text{for } r > r_0.$$

Hence

$$(7) \quad \int_{S_r} \langle \vec{w}, \vec{v} \rangle d\sigma \geq \frac{k}{l^*} g(r)$$

holds for $r > r_0$, where

$$g(r) = \int_{\Omega(a,r)} \alpha^{1-q} (\|x\|) \|\vec{w}(x)\|^q dx.$$

The Hölder inequality gives

$$(8) \quad \int_{S_r} \langle \vec{w}, \vec{v} \rangle d\sigma \leq \left(\int_{S_r} \|\vec{w}\|^q d\sigma \right)^{\frac{1}{q}} \left(\int_{S_r} 1 d\sigma \right)^{\frac{1}{p}} = \alpha^{\frac{1}{p}}(r) (g'(r))^{\frac{1}{q}} \omega_1^{\frac{1}{p}} r^{\frac{n-1}{p}}.$$

From (7) and (8) we obtain

$$(g'(r))^{\frac{1}{q}} \alpha^{\frac{1}{p}}(r) \omega_1^{\frac{1}{p}} r^{\frac{n-1}{p}} \geq \frac{k}{l^*} g(r) \quad \text{for } r \geq r_0$$

and equivalently

$$\frac{g'(r)}{g^q(r)} \omega_1^{\frac{q}{p}} \geq \left(\frac{k}{l^*}\right)^q \alpha^{-\frac{q}{p}}(r) r^{(1-n)\frac{q}{p}} = \left(\frac{k}{l^*}\right)^q \alpha^{\frac{1}{1-p}}(r) r^{\frac{1-n}{p-1}} \quad \text{for } r \geq r_0.$$

Integration of this inequality over the interval (r_0, ∞) gives a convergent integral on the left-hand side and a divergent integral on the right-hand side of this inequality, by virtue of the assumption (iv). This contradiction completes the proof. \square

Remark 1. For $\varphi(u) = u^{p-1}$ we have $\varphi'(u)\varphi^{q-2}(u) = p-1$ and the assumption (ii) holds with $k = p-1$. Conversely, $\varphi(u) \geq \left(\frac{k}{p-1}\right)^{p-1} u^{p-1}$ is necessary for (ii) to be satisfied. Remark also that neither sign restrictions, nor radial symmetry, are supposed for the function $c(x)$ in (i).

Corollary 2 (Leighton type criterion). *Let $p \geq n$. Suppose that there exists a continuous function $c(x)$ such that*

$$(9) \quad B(x, u) \geq c(x)u^{p-1} \quad \text{for } u > 0$$

and

$$(10) \quad \lim_{r \rightarrow \infty} \int_{\Omega(1,r)} c(x) \, dx = +\infty.$$

Then (1) has no positive solution on Ω_a for arbitrary $a > 0$.

Proof. Follows from Theorem 1 for $\alpha(r) \equiv 1$ and $\varphi(u) = u^{p-1}$. \square

Remark 2. Remark that (10) is known to be a sufficient condition for oscillation of (3) provided $p \geq n$, see [2]. It is also known that the condition $p \geq n$ in this criterion cannot be omitted.

Corollary 3. *Suppose that (9) holds and there exists $m > 1$ such that*

$$(11) \quad \lim_{r \rightarrow \infty} \int_{\Omega(1,r)} \left[\|x\|^{p-n} c(x) - m \left| \frac{p-n}{p} \right|^p \frac{1}{\|x\|^n} \right] dx = +\infty.$$

Then (1) has no positive solution on Ω_a for arbitrary $a > 0$.

Proof. Follows from Theorem 1 for $\alpha(r) = r^{p-n}$ and $\varphi(u) = u^{p-1}$, $m = l^{p-1}$. \square

Remark 3. If the limit $\lim_{r \rightarrow \infty} \frac{1}{\ln r} \int_{\Omega(1,r)} \|x\|^{p-n} c(x) dx$ exists, or if this limit equals $+\infty$, then (11) is equivalent to the condition

$$\lim_{r \rightarrow \infty} \frac{1}{\ln r} \int_{\Omega(1,r)} \|x\|^{p-n} c(x) dx > \omega_1 \left| \frac{p-n}{p} \right|^p.$$

This condition is very close to the criterion for oscillation of the half-linear equation [5, Corollary 2.1], which contains “lim sup” instead of “lim” and one additional condition

$$\liminf_{r \rightarrow \infty} \left[r^{p-1} \left(C_0 - \int_{\Omega(1,r)} \|x\|^{1-n} c(x) dx \right) \right] > -\infty,$$

where

$$C_0 = \lim_{r \rightarrow \infty} \frac{p-1}{r^{p-1}} \int_1^r t^{p-2} \int_{\Omega(1,t)} \|x\|^{1-n} c(x) dx dt.$$

Among other, the constant $\left| \frac{p-n}{p} \right|^p$ is here shown to be optimal.

Corollary 4. Let $p \geq n$, $p > 2$, (9) and

$$\lim_{r \rightarrow \infty} \int_{\Omega(\cdot,r)} \ln(\|x\|) c(x) dx = +\infty.$$

Then (1) has no positive solution on Ω_a for arbitrary $a > 0$.

Proof. Let $a > e$, $p \geq n$, $p > 2$, $\alpha(r) = \ln r$. Since

$$\lim_{r \rightarrow \infty} \frac{\alpha^{\frac{1}{1-p}}(r) r^{\frac{1-n}{p-1}}}{\frac{1}{r \ln r}} = \lim_{r \rightarrow \infty} r^{\frac{p-n}{p-1}} \ln^{\frac{p-2}{p-1}} r \geq 1,$$

the condition (iv) of Theorem 1 holds. Further,

$$\begin{aligned} \int_{\Omega(e,r)} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|) dx &= \omega_1 \int_e^r \xi^{n-1-p} \ln^{1-p} \xi d\xi \\ &\leq \omega_1 \int_e^r \xi^{-1} \ln^{1-p} \xi d\xi = \omega_1 \frac{1}{p-2} [1 - \ln^{2-p} r]. \end{aligned}$$

Hence $\lim_{r \rightarrow \infty} \int_{\Omega(e,r)} |\alpha'(\|x\|)|^p \alpha^{1-p}(\|x\|) dx$ exists and (12) is equivalent to the condition (iii) of Theorem 1. Now Theorem 1 implies the conclusion. \square

The choice $\alpha(r) = \ln^\beta r$ leads to

Corollary 5. Let $p \geq n$, let (9) hold and suppose that there exists β , $\beta \in (0, p-1)$ such that

$$\lim_{r \rightarrow \infty} \int_{\Omega(\cdot,r)} \ln^\beta(\|x\|) c(x) dx = +\infty.$$

Then (1) has no positive solution on Ω_a for arbitrary $a > 0$.

Proof. The proof is a complete analogue of the proof of Corollary 4. \square

Following terminology in [6], a function $f: \Omega \rightarrow \mathbb{R}$ is called *weakly oscillatory* if and only if $f(x)$ has a zero in $\Omega \cap \Omega_a$ for every $a > 0$. The inequality (4) is called *weakly oscillatory* in Ω whenever every solution u of the inequality is oscillatory in Ω .

Corollary 6. *Let $B(x, u): \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a continuous function which is odd with respect to the variable u , i.e. let $B(x, -u) = -B(x, u)$. Let the assumptions of Theorem 1 be satisfied. Then inequality (4) is weakly oscillatory in \mathbb{R}^n .*

Proof. Suppose that there exists $a > 0$ such that inequality (4) has a solution u without zeros on Ω_a . If u is a positive function, then Theorem 1 yields a contradiction. Further, if u is a negative solution on Ω_a , then $v(x) := -u(x)$ is a positive solution of (4) on Ω_a and the same argument as in the first part of this proof leads to a contradiction. \square

3.1. Perturbed half-linear differential inequality. Let us consider a perturbed half-linear differential inequality

$$(13) \quad \Delta_p u + c(x)|u|^{p-1} \operatorname{sgn} u + \sum_{i=1}^m q_i(x)\psi_i(u) \leq 0,$$

where $c(x)$, $q_i(x)$ are continuous functions, $\psi_i(u)$ are continuously differentiable, positive and nondecreasing for $u > 0$. Define

$$q(x) = \min\{c(x), q_1(x), q_2(x), \dots, q_m(x)\}$$

and

$$\varphi(u) = u^{p-1} + \sum_{i=1}^m \psi_i(u).$$

Then

$$c(x)|u|^{p-1} \operatorname{sgn} u + \sum_{i=1}^m q_i(x)\psi_i(u) \geq q(x)\varphi(u), \quad \varphi'(u)\varphi^{q-2}(u) \geq p - 1$$

and hence Theorem 1 can be applied. Remark that since q_i may change sign, a standard argument based on the Sturmian majorant and a comparison with half-linear differential equation (3) cannot be applied (as has been explained for $p = 2$ already in [6]).

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