INFINITE PATHS IN LOCALLY FINITE GRAPHS AND IN THEIR SPANNING TREES

Bohdan Zelinka, Liberec

(Received August 16, 2001)

To the memory of Jiří Sedláček

Abstract. The paper concerns infinite paths (in particular, the maximum number of pairwise vertex-disjoint ones) in locally finite graphs and in spanning trees of such graphs.

Keywords: locally finite graph, one-way infinite path, two-way infinite path, spanning tree, Hamiltonian path

MSC 2000: 05C38, 05C05, 05C45

A graph G is called locally finite if every vertex of G has a finite degree. Obviously every finite graph is also locally finite. We will treat locally finite graphs which themselves are infinite.

Let G be a connected infinite locally finite graph. It is well-known that the vertex set of G is countable and G contains at least one infinite path.

There are two types of infinite paths. A one-way infinite path is an infinite connected graph which has one vertex of degree one (initial vertex) and in which all other vertices are of degree two. A two-way infinite path is an infinite connected graph which is regular of degree two. A general symbol for a one-way (or two-way) infinite path will be W_1 (or W_2 respectively). A finite path having length n (i.e. having n edges and $n+1$ vertices) will be denoted by P_n.

We will use also the symbol of the block graph of a given graph G. Let G be a graph, let $A(G)$ be the set of all cutvertices (articulations) of G, let $B(G)$ be the set of all blocks of G. The block graph $BG(G)$ of G is the bipartite graph with vertex sets $A(G)$, $B(G)$ such that $a \in A(G)$ is adjacent to $b \in B(G)$ in $BG(G)$ if and only if a is an articulation of G belonging to the block b.

Let G be a connected infinite locally finite graph. We will study the numerical invariant $IW(G)$ which denotes the maximum number of pairwise vertex-disjoint one-way infinite paths in G. Evidently $IW(G) \geqslant 1$ and it may be even infinite (countable).

Proposition 1. *Let G be an infinite locally finite connected graph. Then $IW(G) = 1$ if and only if G contains no two-way infinite path.*

Proof. A two-way infinite path is the union of two edge-disjoint one-way infinite paths and thus evidently it is also the union of two vertex-disjoint ones with one edge added. □

As usual, a circuit in a graph G is a subgraph of G which is finite, connected and regular of degree 2.

We recall the definition of a block of a graph which will be used here similarly as in the case of finite graphs. Let $\circ$ be a binary relation on the set $E(G)$ of edges of G such that $e_1 \circ e_2$ if and only if either $e_1 = e_2$, or there exists a circuit in G which contains both e_1 and e_2. The relation $\circ$ is an equivalence relation on $E(G)$. A subgraph B of G whose edge is one class $\circ$ and whose vertex set is the set of all end vertices of these edges is a block of G.

Now we shall study a special type of infinite graphs, namely the graph consisting of infinitely many blocks, each of which is finite. We will call them finite-block graphs, shortly FB-graphs.

Theorem 1. *Let G be an infinite locally finite FB-graph. The graph G contains no two-way infinite path if and only if its block graph $BG(G)$ contains no two-way infinite path.*

Proof. Suppose that G contains a two-way infinite path W_2. Then there exists a two-way infinite sequence $\ldots, B_{-2} \ldots, B_{-2}, B_{-1}, B_0, B_1, B_2, \ldots$ of G such that the intersection of W_2 with B_n for each integer n is a finite path D_n and each path D_n is immediately followed by D_{n+1} in W_2. Now we denote each block B_n by b_n and the articulation between b_n and b_{n+1} by a_n; we have a two-way infinite path in $BG(G)$ with the vertices

$$\ldots, a_{-2}, b_{-1}, a_{-1}, b_0, a_0, b_1, a_1, b_2, \ldots$$

On the other hand, let W_2' be a two-way infinite path in $BG(G)$ with the vertices

$$\ldots, b'_{-2}, a'_{-2}, b'_{-1}, a'_{-1}, b'_0, a'_0, b'_1, a'_1, \ldots$$

Each block b'_n is a connected graph, therefore there exists a finite path D'_k in it connecting a'_{n-1} with a'_n. The union of the paths D'_n is a two-way infinite path W_2 in G. □

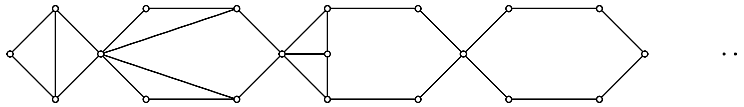

Figure 1

An example of an FB-graph without a two-way infinite path is in Fig. 1.

Proposition 2. *Let G be a connected infinite locally finite graph. Then each edge of G belongs to a one-way infinite path in G.*

Proof. Let e be an edge of G, let W_1 be a one-way infinite path in G. As G is connected, there exists a finite path D in G containing e and a vertex of W_1. The union of D and W_1 is the required path. □

Theorem 2. *Let G be a connected infinite locally finite graph. Let B be a block of G. Then either all edges of B belong to two-way infinite paths in G, or none does.*

Proof. Consider the relation $\circ$ and let e be an edge of B. Let B contain an edge f belonging to a two-way infinite path W_2 in G. Then $e \circ f$. If $e = f$, the assertion is true. Otherwise there exists a circuit D in B which contains both e and f. Let D_0 be a finite path in D which is a subpath of D, contains e and is edge-disjoint with W_2. If e belonged to W_2, then the assertion would be true. Let u, v be the end vertices of D_0. If we omit the subpath of W_2 connecting u and v and replace it by D_0, we obtain a two-way infinite path in G containing e. □

Remark. Let again G be a connected infinite locally finite graph. The subgraph of G formed by all edges which belong to two-way infinite paths is connected. On the other hand, all other edges may be deleted without changing the structure of two-way infinite paths.

Now we turn our attention to spanning trees.

Theorem 3. *Let G be a connected infinite locally finite graph. Then there exists a spanning tree T of G such that $IW(T) = IW(G)$.*

Proof. Let $IW(G) = p$. Let $D_1, \ldots, D_p$ be pairwise vertex-disjoint one-way infinite paths in G. The tree T will be constructed in several steps. In the first step we have the forest F_0 whose connected components are $D_1, \ldots, D_p$ and isolated vertices. In the second step a tree T_0 is obtained from F_0 in such a way that for any path D_k with $k \geqslant 2$ a finite path connecting a vertex of D_k with a vertex of D_1 is chosen and added to the forest. If some circuits occur, edges are deleted where it is necessary. At the end of this step a tree T_0 is obtained. Further trees $T_1, T_2, \ldots$

are formed in such a way that a connected component of F_0 distinct from the trees T_i already constructed is chosen and one of its vertices is connected by a finite path with a vertex of T_i with maximal i from those which have been already constructed. The last tree from such trees is then required to fullfil $|V(T_i)| = |V(G)|$. □

There may exist a spanning tree T of G such that $IW(T) < IW(G)$. In particular, this occurs with graphs which have a one-way Hamiltonian or a two-way Hamiltonian path. A two-way Hamiltonian path is an analogue of a Hamiltonian circuit; it is regular of degree 2 (and obviously infinite). A one-way infinite Hamiltonian path is an analogue of a finite Hamiltonian path; it has one vertex of degree 1 and all others of degree 2.

E x a m p l e 1. For any positive integer k the direct product $P_k \times W_1$ has a one-way infinite Hamiltonian path, while $IW(P_k \times W_1) = k + 1$.

E x a m p l e 2. The direct product $W_1 \times W_1$ has a one-way infinite Hamiltonian path, while $IW(W_1 \times W_1)$ is infinite.

Both paths mentioned are seen in Fig. 2 and in Fig. 3.

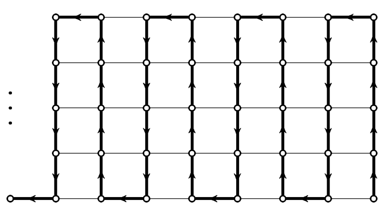

Figure 2

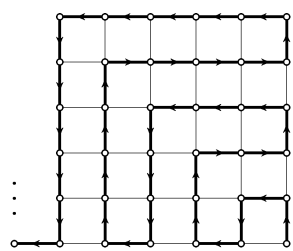

Figure 3

Using the ideas of proof of Theorem 1, the following two theorems may be proved.

Theorem 4. *Let G be a connected infinite locally finite FB-graph. The graph G contains a two-way infinite Hamiltonian path H_2 if and only if its block graph $BG(G)$ contains a two-way infinite path H_2' with a sequence of vertices*

$$\ldots, a_{-2}, b_{-1}, a_{-1}, b_0, a_0, b_1, a_1, \ldots$$

such that in each block b_n a finite Hamiltonian path connecting a_{n-1} and a_n exists (for each integer n).

Theorem 5. *Let G be a connected infinite locally finite FB graph. The graph G contains a one-way infinite Hamiltonian path H_1 if and only if its block graph*

$BG(G)$ contains a one-way infinite path H_1' with the sequence of vertices

$$b_0, a_0, b_1, a_1, b_2, a_2, b_3, a_3, \ldots$$

such that in each block b_n for a positive integer n there exists a finite Hamiltonian path connecting a_{n-1} and a_n and in the block b_0 there exists a finite Hamiltonian path ending in a_0.

At the end we shall prove a formula for $IW(T)$, where T is a tree.

Theorem 6. *Let T be a finite or infinite locally finite tree. For each positive integer k let d denote the number of vertices of T of degree k. Suppose that $\sum\limits_{k=1}^{\infty}(k-2)d_k$ is finite. Then*

$$IW(T) = 2 + \sum_{k=1}^{\infty}(k-2)d_k.$$

P r o o f. We shall do the proof by induction with respect to $IW(T)$. First let $IW(T) = 0$. Then T is a locally finite tree without infinite paths and therefore it is finite. Denote $D(T) = (k-2)d_k$. We have $D(T) = \sum\limits_{k=1}^{\infty} kd_k - 2\sum\limits_{k=1}^{\infty} d_k$; both the sums on the right-hand side are finite. The sum $\sum\limits_{k=1}^{\infty} kd_k$ is the sum of degrees of all vertices of T. Let n be the number of vertices of T; then the number of edges is $n-1$. Hence $\sum\limits_{k=1}^{\infty} kd_k = 2n-2$. The sum $\sum\limits_{k=1}^{\infty} d_k = n$ and $D(T) = -2$, hence $2 + D(T) = 0 = IW(T)$. Now suppose that the assertion is true for $IW(T) = p \geqslant 0$ and let T be a locally finite tree with $D(T)$ finite and with $IW(T) = p+1$. Let W be a one-way infinite path in T. For $k = 2$ we have $(k-2)d_k = 0$ and thus $D(T) = -d_1 + \sum\limits_{k=3}^{\infty}(k-2)d_k$. As this number is finite, both d_1 and $\sum\limits_{k=1}^{\infty}(k-2)d_k$ must be finite and thus d_k is finite for all $k \neq 2$. In particular, in W there are only finitely many vertices having degrees different from 2 in T. There exists a one-way infinite subpath W' of W, all of whose vertices have degree 2 in T. Let u be the initial vertex of W'. Let T' be the tree obtained from T by deleting all vertices and edges of W' except u. Then $IW(T') = IW(T) - 1 = p$. If d_k' denotes the number of vertices of degree k in T', then $d_1' = d_1' + 1$ and $d_k' = d_k$ for $k \geqslant 3$. By the induction hypothesis we have $2 + D(T') = 2 - d_1' + \sum\limits_{k=3}^{\infty}(k-2)d_k' = IW(T') = IW(T) - 1$ and thus $2 + D(T) = 2 - d_1 - 1 + \sum\limits_{k=3}^{\infty}(k-2)d_k = 1 + D(T') = IW(T') + 1 = IW(T)$. $\square$

References

[1] *D. Kömig*: Theorie der endlichen und unendlichen Graphen. Teubner, Leipzig, 1936.
[2] *O. Ore*: Theory of Graphs. AMS, Providence, 1963.

Author's address: *Bohdan Zelinka*, Department of Applied Mathematics, Technical University of Liberec, Voroněžská 13, 460 01 Liberec, Czech Republic, e-mail: `bohdan.zelinka@vslib.cz`.