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STATISTICAL CONVERGENCE OF SUBSEQUENCES OF A GIVEN SEQUENCE

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Abstract. This paper is closely related to the paper of Harry I. Miller: Measure theoretical subsequence characterization of statistical convergence, Trans. Amer. Math. Soc. *347* (1995), 1811–1819 and contains a general investigation of statistical convergence of subsequences of an arbitrary sequence from the point of view of Lebesgue measure, Hausdorff dimensions and Baire's categories.

Keywords: asymptotic density, statistical convergence, Lebesgue measure, Hausdorff dimension, Baire category

MSC 2000: 40A05, 18B05, 11K55

Introduction

The concept of statistical convergence was introduced in papers [9] of H. Fast and [22] of I. J. Schoenberg, generalized and developed in many later papers (e.g. [2], [3], [4], [5], [6], [7], [10], [11], [12], [13], [15], [16], [21]).

The statistical convergence can be viewed as a regular method of summability of sequences. This evokes the question about its relation to other methods of summability. This question is considered in [3], where it is shown that the statistical convergence is equivalent in the space l_∞ of all bounded sequences with the strong Cesàro method of summability. The results from [3] are extended in [4].

The concept of the statistical convergence is based on the notion of the asymptotic density of sets $A \subseteq \mathbb{N} = \{1, 2, \dots, n, \dots\}$.

In [10] an axiomatic approach is given for introducing the concept of density of sets $A \subseteq \mathbb{N}$. This makes it possible to extend the concept of statistical convergence

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(e.g. by using non-negative regular matrices). Applications of such matrices can be found also in [6].

In [13] a class $\mathcal{K}$ of matrices similar to the Cauchy matrix is defined and it is proved there that the statistical convergence of an arbitrary sequence $x \in l_\infty$ is equivalent to the summability of x by each matrix of the class $\mathcal{K}$.

In [10] the μ-statistical convergence and convergence in μ-density is introduced, where μ is a two-valued finitely additive measure defined on a field of subsets of $\mathbb{N}$. These convergences are extensions of the usual statistical convergence.

In connection with the concept of the usual limit point of a sequence, A. Fridy in [12] introduced the notion of a statistical limit point of a sequence. A number $L \in \mathbb{R}$ called a statistical limit point of a sequence $x = (x_n)_1^\infty$ if there is a set $\{n_1 < n_2 < \ldots < n_k < \ldots\} \subseteq \mathbb{N}$, the asymptotic density of which is not zero (i.e. it is greater than zero or does not exist), such that $\lim_{k\to\infty} x_{n_k} = L$. The notion of statistical limit points is extended in [7] to the notion of T-statistical limit points, T being a non-negative regular matrix. In [15] topological properties of the set Λ_x of all statistical limit points of x are investigated and the relation of Λ_x to distribution functions of x is established. The set Λ_x is equal to the set of discontinuity points of a distribution function of x.

The concept of convergence of subsequences in the usual convergence and the above mentioned notion of statistical limit points suggests the study of statistical convergence of subsequences of a given sequence. This study was started in [16] by H. I. Miller. The purpose of our paper is to investigate the structure of the set $C_{\mathrm{stat}}(y)$ from various points of view, where $C_{\mathrm{stat}}(y) = \{t \in (0,1]\colon y(t)$ converges statistically$\}$, $y(t) = y_{k_1}, y_{k_2}, \ldots, y_{k_n}, \ldots$ if $t = \sum_{n=1}^{\infty} 2^{-k_n} \in (0,1]$. The set $C_{\mathrm{stat}}(y)$ and the related sets depend on $y = (y_n)_1^\infty$, but we will see that these sets have some common properties for all $y = (y_n)_1^\infty$. In the first section of the paper we will describe the fundamental metric, in the second section the topological properties of the set $C_{\mathrm{stat}}(y)$ and the related sets for an arbitrary $y = (y_n)_1^\infty$.

Definitions and notation

We recall the concept of the asymptotic density of sets $B \subseteq \mathbb{N} = \{1, 2, \ldots, n, \ldots\}$. If $B \subseteq \mathbb{N}$ we put $B(n) = |B \cap \{1, 2, \ldots, n\}|$, where $|M|$ denotes the cardinality of M. The numbers $\underline{d}(B) = \liminf_{n\to\infty} \frac{B(n)}{n}$, $\overline{d}(B) = \limsup_{n\to\infty} \frac{B(n)}{n}$ are called the lower and upper density of B. If $\underline{d}(B) = \overline{d}(B) = \lim_{n\to\infty} \frac{B(n)}{n}$ then $d(B) = \lim_{n\to\infty} \frac{B(n)}{n}$ is called the asymptotic density of B (cf. [14], p. xix; [17], p. 70–72).

We recall the concept of statistical convergence (cf. [9], [22]).

Definition A. A sequence $x = (x_n)_1^\infty$ of real numbers is said to converge statistically to $\xi \in \mathbb{R}$ provided that for every $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in \mathbb{N}\colon\ |x_n - \xi| \geqslant \varepsilon\}$.

If $x = (x_n)_1^\infty$ converges statistically to ξ then we write $\operatorname{lim-stat} x_n = \xi$ or $\operatorname{lim-stat} x = \xi$.

The statistical convergence is a natural generalization of the usual convergence. If $\lim\limits_{n\to\infty} x_n = \xi$ (in the usual sense), then $\operatorname{lim-stat} x_n = \xi$. The converse in general does not hold.

We will often use the following characterization of statistical convergence (cf. [21], Lemma 1.1).

Theorem A. *A sequence $x = (x_n)_1^\infty$ converges statistically to $\xi \in \mathbb{R}$ if and only if there is a set $M = \{m_1 < m_2 < \ldots\} \subseteq \mathbb{N}$ with $d(M) = 1$ such that $\lim\limits_{k\to\infty} x_{m_k} = \xi$.*

We recall the above mentioned correspondence between the numbers of $(0,1]$ and the subsequences of a given sequence $y = (y_n)_1^\infty$ (cf. [16]). If $t \in (0,1]$, then t has a unique non-terminating dyadic expansion

$$t = \sum_{k=1}^{\infty} c_k(t) 2^{-k}, \tag{1}$$

$c_k(t) = 0$ or 1 $(k = 1, 2, \ldots)$ and $c_k(t) = 1$ for infinitely many k's.

If we put $\{k,\ c_k(t) = 1\} = \{k_1 < k_2 < \ldots\}$, then (1) has the form $t = \sum\limits_{k=1}^{\infty} 2^{-k_n}$. Put $y(t) = y_{k_1}, y_{k_2}, \ldots, y_{k_n}, \ldots$

So we get a one-to-one correspondence between the numbers of $(0,1]$ and the subsequences of y. This correspondence enables us "to measure" the magnitude of a class $\mathcal{S}$ of subsequences of y by a corresponding set $A \subseteq (0,1]$ of all t's from $(0,1]$ that correspond to subsequences from $\mathcal{S}$.

We will suppose in the whole paper that $y = (y_n)_1^\infty$ is a fixed sequence of real numbers. We will deal with the following subsets of $(0,1]$:

$$\begin{aligned}
C(y) &= \{t \in (0,1]\colon\ y(t) \text{ is convergent}\},\\
D(y) &= \{t \in (0,1]\colon\ y(t) \text{ is divergent}\},\\
C_{\mathrm{stat}}(y) &= \{t \in (0,1]\colon\ y(t) \text{ converges statistically}\},\\
D_{\mathrm{stat}}(y) &= \{t \in (0,1]\colon\ y(t) \text{ does not converge statistically}\}.
\end{aligned}$$

Further, if $y = (y_n)_1^\infty$ converges statistically, then

$$\begin{aligned}
C^*_{\mathrm{stat}}(y) &= \{t \in (0,1]\colon\ \operatorname{lim-stat} y(t) = \operatorname{lim-stat} y\}\\
D^*_{\mathrm{stat}}(y) &= (0,1] \setminus C^*_{\mathrm{stat}}(y).
\end{aligned}$$

From the previous definitions we immediately get:

$$C^*_{\text{stat}}(y) \subseteq C_{\text{stat}}(y), \tag{2}$$

$$C(y) \subseteq C_{\text{stat}}(y), \tag{2$'$}$$

$$D_{\text{stat}}(y) \subseteq D^*_{\text{stat}}(y). \tag{2$''$}$$

In what follows $\lambda(M)$ $(M \subseteq \mathbb{R})$ denotes the Lebesgue measure of M and $\dim M$ the Hausdorff dimension of M (cf. [20]).

1. Metric results

In [16] (Theorem 3 in [16]) the following result is proved which in our terminology can be formulated as follows.

Theorem B. *A sequence $y = (y_n)_1^\infty$ converges statistically to $\xi \in \mathbb{R}$ if and only if $\lambda(C^*_{\text{stat}}(y)) = 1$.*

Hence, if $y = (y_n)_1^\infty$ converges statistically then $C^*_{\text{stat}}(y)$ is a measurable set and has full measure. This fact evokes the question whether the set $C_{\text{stat}}(y)$ is Lebesgue measurable for an arbitrary $y = (y_n)_1^\infty$. We will give the affirmative answer to this question.

In connection with the question mentioned we recall the following classical fact concerning the sets $C(y)$ and $D(y)$:

If a sequence $y = (y_n)_1^\infty$ converges, then $C(y) = (0, 1]$ and if $y = (y_n)_1^\infty$ diverges, then $\lambda(D(y)) = 1$ (and so $\lambda(C(y)) = 0$, cf. [1], [8] p. 404).

In the first place we will deal with the measurability of the set $C_{\text{stat}}(y)$ in the case that $y = (y_n)_1^\infty$ is a bounded sequence.

The interval $(0, 1]$ is considered as a metric space with the Euclidean metric.

Theorem 1.1. *Let $y = (y_n)_1^\infty$ be a bounded sequence of real numbers. Then the set $C_{\text{stat}}(y)$ is an $F_{\sigma\delta}$ set in $(0, 1]$.*

Corollary 1.1. *Under the assumption of Theorem 1.1 the set $D_{\text{stat}}(y)$ is a $G_{\delta\sigma}$ set in $(0, 1]$.*

Proof of Theorem 1.1. We will define functions $g_{m,n}$ as follows:

If $t = \sum\limits_{k=1}^{\infty} c_k(t) 2^{-k}$ is the non-terminating dyadic expansion of t, then we put $p(n,t) = \sum\limits_{i=1}^{n} c_i(t)$ $(n = 1, 2, \ldots)$. Then $p(n,t) > 0$ for all sufficiently large n's.

Suppose that $p(h,t) > 0$ for $h = \min\{m,n\}$. Then we put

$$g_{m,n}(t) = \frac{1}{p(m,t)}\frac{1}{p(n,t)} \sum_{i\leqslant m, j\leqslant n} c_i(t)c_j(t)|y_i - y_j|.$$

If $p(h,t) = 0$, then we put $g_{m,n}(t) = 1$.

In [24] a test for statistical convergence of bounded sequences is established which can be formulated in our terminology as follows:

$$t \in C_{\mathrm{stat}}(y) \text{ if and only if } \lim_{m,n\to\infty} g_{m,n}(t) = 0.$$

From this we get the following expression for $C_{\mathrm{stat}}(y)$:

$$C_{\mathrm{stat}}(y) = \bigcap_{k=1}^{\infty}\bigcup_{m_0=1}^{\infty}\bigcup_{n_0=1}^{\infty}\bigcap_{m=m_0}^{\infty}\bigcap_{n=n_0}^{\infty} B(k,m,n) \tag{3}$$

where

$$B(k,m,n) = \{t \in (0,1]\colon\ g_{m,n}(t) < \tfrac{1}{k}\}.$$

Put $Y = (0,1] \setminus \mathbb{Q}$ where $\mathbb{Q}$ is the set of all rationals. The set Y is considered as a metric space with the Euclidean metric. Then (3) implies

$$C_{\mathrm{stat}}(y) \cap Y = \bigcap_{k=1}^{\infty}\bigcup_{m_0=1}^{\infty}\bigcup_{n_0=1}^{\infty}\bigcap_{m=m_0}^{\infty}\bigcap_{n=n_0}^{\infty} D(k,m,n) \tag{4}$$

where $D(k,m,n) = Y \cap B(k,m,n)$.

Observe that for fixed m, n the function $g_{m,n}$ is constant on every interval

$$\Big(\frac{l}{2^p}, \frac{l+1}{2^p}\Big] \quad (0 \leqslant l \leqslant 2^p - 1) \tag{5}$$

where $p = \max\{m,n\}$. From this fact it can be easily deduced that $D(k,m,n)$ is a closed set in Y (for fixed k, m, n). But then we get from (4) that $C_{\mathrm{stat}}(y) \cap Y$ is an $F_{\sigma\delta}$ set in Y and so an $F_{\sigma\delta}$ set in $(0,1]$ as well.

From the equality

$$C_{\mathrm{stat}}(y) = (C_{\mathrm{stat}}(y) \cap Y) \cup (C_{\mathrm{stat}}(y) \cap \mathbb{Q})$$

the theorem follows immediately. $\square$

Let us analyse the statement that a number t belongs to $C_{\text{stat}}(y)$. If $t \in C_{\text{stat}}(y)$ then there exists $\text{lim-stat}\, y(t) = \xi(t) \in \mathbb{R}$. If $t = \sum_{j=1}^{\infty} 2^{-l_j}$, $l_1 < l_2 < \ldots$, then by Theorem A there exists a set $J \subseteq \mathbb{N}$ with $d(J) = 1$ such that $\lim\limits_{j\to\infty,\ j\in J} y_{l_j} = \xi(t)$. The converse is also true.

Note that $\lim\limits_{n\to\infty,\ n\in A} z_n = L$ has the following meaning: For every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for each $n > n_0$, $n \in A$ we have $|z_n - L| < \varepsilon$ (we say shortly that $(z_n)_1^\infty$ converges to L along the set A). In a similar manner the convergences $\lim\limits_{n\to\infty,\ n\in A} z_n = +\infty$ and $\lim\limits_{n\to\infty,\ n\in A} z_n = -\infty$ can be interpreted.

It seems to be useful for further purposes to introduce the statistical limits $+\infty$, $-\infty$.

Definition 1.1. The number $+\infty$ ($-\infty$) is called a statistical limit of the sequence $x = (x_n)_1^\infty$ ($\text{lim-stat}\, x_n = \text{lim-stat}\, x = +\infty$ ($-\infty$)) provided that for each $K \in \mathbb{R}$ we have

$$d(\{n \in \mathbb{N}\colon\ x_n \geqslant K\}) = 1 \ (d(\{n \in \mathbb{N}\colon\ x_n \leqslant K\}) = 1).$$

Remark. Observe that $\text{lim-stat}\, x_n = +\infty$ if and only if $d(\{n \in \mathbb{N}\colon\ x_n \leqslant K\}) = 0$ for every $K \in \mathbb{R}$. A similar statement holds for $-\infty$.

The existence of the infinite statistical limit $+\infty$ can be characterized in a similar way as the existence of a finite limit (cf. [21], Lemma 1.1, our Theorem A).

Lemma 1.1. *The statement*

$$\text{lim-stat}\, x_n = +\infty \tag{6}$$

holds if and only if there exists a set $M \subseteq \mathbb{N}$ *with* $d(M) = 1$ *such that* $\lim\limits_{n\to\infty, n\in M} x_n = +\infty$.

Proof. 1. The proof of $\Leftarrow$ is easy and can be omitted.

2. Suppose that (6) holds. Put $M_m = \{n \in \mathbb{N}\colon\ x_n > m\}$ $(m = 1, 2, \ldots)$. Then

$$M_1 \supseteq M_2 \supseteq \ldots \supseteq M_m \supseteq M_{m+1} \supseteq \ldots, \tag{7}$$

$$d(M_m) = 1 \quad (m = 1, 2, \ldots). \tag{8}$$

Choose $v_1 \in M_1$. According to (8) there is a $v_2 > v_1$, $v_2 \in M_2$, such that for every $n \geqslant v_2$

$$\frac{M_2(n)}{n} > \frac{1}{2}.$$

Again by (8) there is a $v_3 > v_2$, $v_3 \in M_3$, such that for every $n \geqslant v_3$

$$\frac{M_3(n)}{n} > \frac{2}{3}, \text{a.s.o.}$$

So we get (by induction) a sequence $v_1 < v_2 < \ldots < v_j < \ldots$, $v_j \in M_j$ $(j = 1, 2, \ldots)$, such that for every $n \geqslant v_j$

$$\frac{M_j(n)}{n} > \frac{j-1}{j} \quad (j = 2, 3, \ldots).$$

We construct the set M as follows: We insert into M the interval $[1, v_1) \cap \mathbb{N}$; further, a number $n \in [v_j, v_{j+1}) \cap \mathbb{N}$ $(j \geqslant 1)$ will belong to the set M if and only if it belongs to M_j.

We prove that $d(M) = 1$. Let $n \geqslant v_1$. Then n belongs to an interval $[v_j, v_{j+1})$ for some j. Thus by (7), (8) we get

$$\frac{M(n)}{n} \geqslant \frac{M_j(n)}{n} \geqslant \frac{j-1}{j}. \tag{9}$$

From (9), $d(M) = 1$ follows at once.

Let $m \in \mathbb{N}$ be an arbitrary positive integer. If $n \in \mathbb{N}$, $n \geqslant v_m$, then n belongs to an interval $[v_j, v_{j+1})$ for some $j \geqslant m$, hence it belongs to M_j and also to M_m (see (7)). But then $x_n \geqslant m$ by the definition of M_m. Thus

$$\lim_{n \to \infty,\ n \in M} x_n = \infty.$$

$\square$

R e m a r k. Observe that if $\text{lim-stat}\, x_n = \xi$, $\xi \in \mathbb{R} \cup \{+\infty, -\infty\}$ then evidently we can choose the set M mentioned in Theorem A and Lemma 1.1 in such a way that the elements x_k $(k \in M)$ belong to an arbitrary chosen neighbourhood of ξ.

In what follows we introduce the following notation:

$$\begin{aligned} C_{\text{stat}}^{\infty}(y) &= \{t \in (0,1]\colon \text{lim-stat}\, y(t) = +\infty\}, \\ C_{\text{stat}}^{-\infty}(y) &= \{t \in (0,1]\colon \text{lim-stat}\, y(t) = -\infty\}, \\ K_{\text{stat}}(y) &= C_{\text{stat}}(y) \cup C_{\text{stat}}^{\infty}(y) \cup C_{\text{stat}}^{-\infty}(y). \end{aligned}$$

Observe that the "summands" from the right-hand side are pair-wise disjoint.

Let $x = (x_n)_1^\infty$ be a sequence of real numbers. For $m \in \mathbb{N}$ we define a new sequence $x^{(m)} = (x_j^{(m)})_{j=1}^\infty$ as follows:

$$x_j^{(m)} = \begin{cases} x_j \text{ if } x_j \in (-m, m), \\ m \text{ if } x_j \geqslant m, \\ -m \text{ if } x_j \leqslant -m. \end{cases}$$

R e m a r k. It is easy to see that $|x_j - t| \geqslant |x_j^{(m)} - t|$ for every $t \in [-m, m]$.

The next lemma is of fundamental importance in our further considerations.

Lemma 1.2. *We have*

$$K_{\mathrm{stat}}(y) = \bigcap_{m=1}^{\infty} C_{\mathrm{stat}}(y^{(m)}). \tag{10}$$

P r o o f. 1. Let $t = \sum\limits_{j=1}^{\infty} 2^{-l_j} \in (0,1]$. Suppose that t belongs to $K_{\mathrm{stat}}(y)$. We can suppose that $\text{lim-stat}\, y(t) = \xi \geqslant 0$ (if not, we take the sequence $-y = (-y_n)_1^\infty$). Let $m \in \mathbb{N}$. We have two possibilities:

a) $\xi \leqslant m$. Then $\{j \in \mathbb{N}\colon\ |y_{l_j}^{(m)} - \xi| < \varepsilon\} \supseteq \{j \in \mathbb{N}\colon\ |y_{l_j} - \xi| < \varepsilon\}$ for every $\varepsilon > 0$ (see the remark above). This immediately yields that $\text{lim-stat}\, y^{(m)}(t) = \xi$.

b) $\xi > m$. Then there exists a set $J \subset \mathbb{N}$, $d(J) = 1$, such that $y_{l_j} \in (m, \infty)$ for every $j \in J$. Thus for every $j \in J$ we have $y_{l_j}^{(m)} = m$ and $\text{lim-stat}\, y^{(m)}(t) = m$.

2. Let $t \in \bigcap\limits_{m=1}^{\infty} C_{\mathrm{stat}}(y^{(m)})$. We show that t belongs to $K_{\mathrm{stat}}(y)$. Define $\xi^{(m)}(t) = \text{lim-stat}\, y^{(m)}(t)$. We have two possibilities:

a) There exists an m_0 such that $\xi^{(m_0)}(t) \in (-m_0, m_0)$. Then there exists a set $J \subseteq \mathbb{N}$ with $d(J) = 1$ such that $\lim\limits_{j\to\infty,\ j\in J} y_{l_j}^{(m_0)} = \xi^{(m_0)}(t)$ and $y_{l_j}^{(m_0)} \in (-m_0, m_0)$ $(j \in J)$. Thus $y_{l_j} \in (-m_0, m_0)$ $(j \in J)$ and $\text{lim-stat}\, y(t) = \xi^{(m_0)}(t)$.

b) Such m_0 does not exist. It means that for every $m \in \mathbb{N}$ we have $\xi^{(m)}(t) = m$ or $\xi^{(m)}(t) = -m$, so we can suppose that $\xi^{(m)}(t) = m$ for infinitely many m's. We show that in this case $\text{lim-stat}\, y(t) = +\infty$.

Let K be an arbitrary positive real number. Choose $m_0 \in \mathbb{N}$ such that $m_0 > K$ and $\text{lim-stat}\, y^{m_0}(t) = m_0$. Then there is a set $J \subseteq \mathbb{N}$ with $d(J) = 1$ such that $\lim\limits_{j\to\infty,\ j\in J} y_{l_j}^{(m_0)} = m_0$ and $y_{l_j}^{(m_0)} \in (K, \infty)$ $(j \in J)$. So $y_{l_j} \in (K, \infty)$ for each $j \in J$ and $\text{lim-stat}\, y(t) = +\infty$ follows immediately. $\square$

Lemma 1.3. *Each of the sets $C_{\mathrm{stat}}^{\infty}(y)$, $C_{\mathrm{stat}}^{-\infty}(y)$ is a $G_{\delta\sigma\delta}$ set in $(0,1]$.*

P r o o f. Let $n \in \mathbb{N}$. The interval $(0,1]$ is the union of the following intervals $\mathrm{i}_n^{(j)}$ (of the n-th order):

$$\mathrm{i}_n^{(j)} = \Big(\frac{j}{2^n}, \frac{j+1}{2^n}\Big] \ (0 \leqslant j \leqslant 2^n - 1).$$

To every interval $\mathrm{i}_n^{(j)}$ $(0 \leqslant j \leqslant 2^n - 1)$ a sequence $c_1^{(j)}, \dots, c_n^{(j)}$ of 0's and 1's corresponds in such a manner that the dyadic expansion of every $t \in \mathrm{i}_n^{(j)}$, $t = \sum\limits_{k=1}^{\infty} c_k(t)2^{-k}$, satisfies the conditions $c_k(t) = c_k^{(j)}$ $(k = 1, 2, \dots, n)$. We say shortly that $\mathrm{i}_n^{(j)}$ belongs to the sequence $c_1^{(j)}, \dots, c_n^{(j)}$.

Let m, k, K be positive integers. Denote by $A(m,k,K)$ the union of all intervals $\mathrm{i}_m^{(j)}$ of the m-th order that belong to such sequences $c_1^{(j)}, \dots, c_n^{(j)}$ of 0' and 1's that

$$\sum_{h=1}^{m} c_h^{(j)} \chi_{J_K}(h) > \frac{k-1}{k} \sum_{h=1}^{m} c_h^{(j)}, \tag{11}$$

where $J_K = \{l \in \{1, 2, \dots, m\} \colon\ y_l > K\}$ and χ_M denotes the characteristic function of the set M.

We show that

$$C_{\mathrm{stat}}^{\infty}(y) = \bigcap_{K=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m_0=1}^{\infty} \bigcap_{m=m_0}^{\infty} A(m,k,K). \tag{12}$$

Let $t = \sum\limits_{k=1}^{\infty} c_k(t)2^{-k} = \sum\limits_{l=1}^{\infty} 2^{-k_l} \in C_{\mathrm{stat}}^{\infty}(y)$. By Lemma 1.1 there is a set $L \subseteq \mathbb{N}$ with $d(L) = 1$, such that

$$\lim_{l\to\infty,\ l\in L} y_{k_l} = +\infty \text{ and } \forall_{l\in L} \colon\ y_{l_k} > K.$$

Indeed, if $k \in \mathbb{N}$, the previous facts imply that for every $m \geqslant m_0$ (m_0 being suitably chosen) the number of all l's with $k_l \leqslant m$, $y_{k_l} > K$ is greater than $\frac{k-1}{k} p(m,t)$, $p(m,t) = \sum\limits_{i=1}^{m} c_i(t) = |\{l \in \mathbb{N} \colon\ k_l \leqslant M\}|$ (see (11)).

From this we see that t belongs to the right hand side of (12).

Let t belong to the right-hand side of (18). We will show that t belongs to $C_{\mathrm{stat}}^{\infty}(y)$.

Let $k, K \in \mathbb{N}$. Choose an $m_0' \in \mathbb{N}$ such that t belongs to $\bigcap\limits_{m=m_0'}^{\infty} A(m,k,K)$. Then by the definition of the set $A(m,k,K)$ we have (for $m \geqslant m_0'$)

$$q(m,t) > \frac{k-1}{k} p(m,t),$$

where $q(m,t)$ denotes the number of all i's, $i \leqslant m$ for which $c_i(t) > 0$ and $y_i > K$.

Since the values of the sums $p(m,t) = \sum_{i=1}^{m} c_i(t)$ $(m \geqslant m_0')$ coincide with $\left(\sum_{i=1}^{m_0'} c_i(t), +\infty\right) \cap \mathbb{N}$, it is easy to deduce from the previous facts that $d(J_K(t)) = 1$, where $J_K(t) = \{l \in \mathbb{N}\colon\ c_l(t)y_l > K\}$. But this by definition of $C_{\text{stat}}^{\infty}(y)$ means that t belongs to the left-hand side of (12).

Now, take into account that each of the sets $A(m,k,K)$ is a G_δ set in $(0,1]$. Then the theorem follows from (12) immediately. $\square$

Now we are able to formulate our result about measurability of $C_{\text{stat}}(y)$ and the related sets.

Theorem 1.2. *Let $y = (y_n)_1^{\infty}$ be an arbitrary sequence of real numbers. Then each of the sets $C_{\text{stat}}(y)$, $C_{\text{stat}}^{\infty}(y)$, $C_{\text{stat}}^{-\infty}(y)$, $K_{\text{stat}}(y)$ is (L)-measurable, the set $K_{\text{stat}}(y)$ is an $F_{\sigma\delta}$, each of the sets $C_{\text{stat}}^{\infty}(y)$, $C_{\text{stat}}^{-\infty}(y)$ is a $G_{\delta\sigma\delta}$ set and the set $C_{\text{stat}}(y)$ is an $F_{\sigma\delta\sigma}$ in $(0,1]$.*

Proof. The part of Theorem 1.2 concerning the set $K_{\text{stat}}(y)$ follows from Theorem 1.1 and Lemma 1.2 (see (10)), since each of the sequences $y^{(m)}$ $(m = 1,2,\ldots)$ is bounded.

The part concerning the sets $C_{\text{stat}}^{\infty}(y)$, $C_{\text{stat}}^{-\infty}(y)$ follows from Lemma 1.3. Further,

$$C_{\text{stat}}(y) = K_{\text{stat}}(y) \setminus (C_{\text{stat}}^{\infty}(y) \cup C_{\text{stat}}^{-\infty}(y)).$$

By Lemma 1.2 and Lemma 1.3 the set on the right-hand side is a difference of an $F_{\sigma\delta}$ set and a $G_{\delta\sigma\delta}$ set (see Lemma 1.3). Therefore it is an $F_{\sigma\delta\sigma}$ set in $(0,1]$. $\square$

Using Theorem 1.1 and Theorem 1.2 we can give a certain general information about the Lebesgue measure of the sets $C_{\text{stat}}(y)$, $C_{\text{stat}}^{\infty}(y)$, $C_{\text{stat}}^{-\infty}(y)$, $K_{\text{stat}}(y)$:

Theorem 1.3. *Let $y = (y_n)_1^{\infty}$ be an arbitrary sequence of real numbers. Then the Lebesgue measure of each of the sets $C_{\text{stat}}(y)$, $C_{\text{stat}}^{\infty}(y)$, $C_{\text{stat}}^{-\infty}(y)$, $K_{\text{stat}}(y)$ is either 0 or 1.*

Proof. It is obvious that if $t = \sum_{k=1}^{\infty} c_k(t)2^{-k} \in (0,1]$ belongs to some of the mentioned sets, then $t' = \sum_{k=1}^{\infty} c_k(t')2^{-k}$, $c_k(t) \neq c_k(t')$ only for a finite number of k's, belongs to the same set as well. From this the homogeneity of each of the sets follows (cf. [19], Lemma 1.1), thus each of these sets has measure 0 or 1 (cf. [25]). $\square$

We give yet some further metric results about the above sets.

Theorem 1.4. *Let $y = (y_n)_1^\infty$ be a sequence of real numbers.*

(i) *If y converges, then $C(y) = C_{\mathrm{stat}}(y) = C^*_{\mathrm{stat}}(y) = (0, 1]$, $D(y) = D_{\mathrm{stat}}(y) = D^*_{\mathrm{stat}}(y) = \emptyset$,*

(ii) *If y is a bounded divergent sequence and y converges statistically, then $\lambda(C^*_{\mathrm{stat}}(y)) = \lambda(C_{\mathrm{stat}}(y)) = 1$, $\lambda(D^*_{\mathrm{stat}}(y)) = \lambda(D_{\mathrm{stat}}(y)) = 0$, $\lambda(D(y)) = 1$, $\lambda(C(y)) = 0$.*

P r o o f. (i) This part can be obtained by some well-known facts from analysis and by inclusions (2), (2′), (2″).

(ii) From Theorem 3 of [16] (our Theorem B) we get $\lambda(C^*_{\mathrm{stat}}(y)) = 1$. Further, it is well-known that $\lambda(D(y)) = 1$ (cf. [1] and [8], p. 404). The rest of (ii) follows from (2), (2′), (2″). □

We now present some applications of the Hausdorff dimension to the investigation of metric properties of our sets.

The next result shows that the statistical convergence of a sequence y guarantees that the magnitude of the set $C(y)$ is maximal from the point of view of the Hausdorff dimension.

Theorem 1.5. *If a sequence $y = (y_n)_1^\infty$ converges statistically then* $\dim C(y) = 1$.

Corollary 1.2. *If a sequence $y = (y_n)_1^\infty$ is a divergent sequence which converges statistically then $\lambda(C(y)) = 0$ and* $\dim C(y) = 1$.

P r o o f o f T h e o r e m 1.5. If $y = (y_n)_1^\infty$ converges then the assertion is trivial.

Suppose that $y = (y_n)_1^\infty$ is divergent and simultaneously there exists $\xi = \text{lim-stat}\, y_n \in \mathbb{R}$. By Theorem A there exists a set $M = \{m_1 < m_2 < \ldots\} \subseteq \mathbb{N}$ with $d(M) = 1$ such that

$$\lim_{k\to\infty} y_{m_k} = \xi.$$

Obviously all subsequences of the sequence $(y_{m_k})_{k=1}^\infty$ converge to ξ.

We now use a result from [20] (Theorem 2,7 in [20]) which when applied to dyadic expansions can be formulated as follows:

Let $A \subseteq \mathbb{N}$ be a fixed set. For each $k \in A$ let ε_k^0 be a fixed number (0 or 1). Denote by $Z(A; (\varepsilon_k^0), k \in A)$ the set of all $t = \sum\limits_{j=1}^{\infty} c_j(t) 2^{-j} \in (0, 1]$ for which $c_j(t) = \varepsilon_k^0$ if $j \in A$ and $c_j(t) = 0$ or 1 if $j \notin A$. Then

$$\dim Z(A; (\varepsilon_k^0), k \in A) = \liminf_{n\to\infty} \frac{\log \prod\limits_{j \leqslant n, j \in \mathbb{N}\setminus A} 2}{n \log 2} = \underline{d}(A).$$

Choose $A = \mathbb{N} \setminus M$, $\varepsilon_k^0 = 0$ for every $k \in \mathbb{N} \setminus M$. Then obviously

$$Z(\mathbb{N} \setminus M; (\varepsilon_k^0), k \in \mathbb{N} \setminus M) \subseteq C(y) \tag{13}$$

and so by the result of [20] $\dim Z(\mathbb{N} \setminus M; (\varepsilon_k^0), k \in \mathbb{N} \setminus M) = \underline{d}(M) = d(M) = 1$. Theorem 1.5 follows from (13). □

In connection with Theorem 1.5 a question arises what can be stated about the set $C(y)$ if $y = (y_n)_1^\infty$ has an infinite statistical limit. In this case $\lambda(C(y)) = 0$ since y is a divergent sequence (cf. [1]; [8], p. 404). But we prove that a stronger statement holds for $C(y)$:

Theorem 1.6. *Suppose that $y = (y_n)_1^\infty$ has an infinite statistical limit. Then $\dim C(y) = 0$.*

P r o o f. We restrict ourselves to the case lim-stat $y_n = +\infty$. Then by Lemma 1.1 there exists a set $M = \{m_1 < m_2 < \ldots\} \subseteq \mathbb{N}$ with $d(M) = 1$ such that $\lim\limits_{k\to\infty} y_{m_k} = +\infty$.

If the set $\mathbb{N} \setminus M$ is finite, then $C(y) = \emptyset$ and the assertion is clear.

Suppose that $\mathbb{N} \setminus M$ is infinite. If a subsequence $y_{j_1}, y_{j_2}, \ldots$ of y converges the set $\{j_1, j_2, \ldots, j_m, \ldots\}$ has only a finite number of common elements with the set M. From this observation we immediately get the identity

$$C(y) = \bigcup_B D_B, \tag{13'}$$

where B runs over all finite subsets of M and D_B denotes the set of all $t = \sum\limits_{k=1}^{\infty} c_k(t) = 2^{-k} \in (0,1]$, where $c_k(t) = 0$ if $k \in M \setminus B$, $c_k(t) = 1$ if $k \in B$ and $c_k(t) = 0$ or 1 if $k \in \mathbb{N} \setminus M$.

The Hausdorff dimension of the set D_B (by fixed B) can be calculated using the methods applied in the proof of Theorem 1.5. For A (in $Z(A; (\varepsilon_k^0), k \in A)$) we take the set M and put $\varepsilon_k^0 = 1$ if $k \in B$ and $\varepsilon_k^0 = 0$ if $k \in M \setminus B$. Then we get $\dim D_B = \underline{d}(\mathbb{N} \setminus M) = d(\mathbb{N} \setminus M) = 0$.

Since the class of all finite subsets of M is countable, we get from (13′) (using Lemma 4 of [18]):

$$\dim(C(y)) \leqslant \sup_B \dim D_B = 0,$$

hence $\dim C(y) = 0$. □

Using the idea of the proof of Theorem 3 from [16] we can establish the following result (Theorem 1.7).

Lemma 1.4. *Suppose that* lim-stat $y_n = \xi \in \mathbb{R}$. *Put* $P(t) = \{k \in \mathbb{N}\colon\ c_k(t) = 1\}$ *for* $t = \sum\limits_{k=1}^{\infty} c_k(t)2^{-k} \in (0,1]$. *If* $\underline{d}(P(t)) > 0$, *then* lim-stat $y(t) = \xi$.

P r o o f. Let $P(t) = \{n_1 < n_2 < \ldots\}$. By assumption

$$\liminf_{k\to\infty} \frac{k}{n_k} > 0.$$

Therefore there exist $c > 0$ and $k_0 \in \mathbb{N}$ such that for every $k > k_0$ we have $\frac{k}{n_k} > c > 0$, thus

$$\frac{n_k}{k} < \frac{1}{c} \quad (k > k_0). \tag{14}$$

Using a simple estimation we get from (14)

$$\frac{1}{k}|\{i \leqslant k\colon\ |y_{n_i} - \xi| \geqslant \varepsilon\}| \leqslant \frac{n_k}{k}\frac{|\{i \leqslant n_k\colon\ |y_i - \xi| \geqslant \varepsilon\}|}{n} \leqslant \frac{1}{c}\frac{|\{i \leqslant n_k\colon\ |y_i - \xi| \geqslant \varepsilon\}|}{n_k}.$$

If $k \to \infty$ the assertion follows immediately. $\square$

The following result can be considered to be a completion of Theorem 1.4:

Theorem 1.7. *Suppose that* $y = (y_n)_1^\infty$ *converges statistically. Then*

$$\dim D^*_{\text{stat}}(y) = 0.$$

Corollary 1.3. (a) *Under the assumption of Theorem* 1.7 *we have* $\dim D_{\text{stat}}(y) = 0$.

(b) *If* $y = (y_n)_1^\infty$ *converges statistically then* $\dim C(y) = 1$ *and* $\dim D(y) = 0$

(see $(2'')$ and Theorem 1.5).

P r o o f o f T h e o r e m 1.7. By Lemma 1.4, if $\underline{d}(P(t)) > 0$, then the number t belongs to $C^*_{\text{stat}}(y)$. Hence if t does not belong to $C^*_{\text{stat}}(y)$ (i.e. if t belongs to $D^*_{\text{stat}}(y)$) then $\underline{d}(P(t)) = 0$.

Take into account that $P(t)(m) = \sum\limits_{i=1}^{m} c_i(t) = p(m,t)$. So Lemma 1.4. asserts in fact that if $t \in D^*_{\text{stat}}(y)$, then

$$\liminf_{m\to\infty} \frac{p(m,t)}{m} = 0,$$

hence

$$D^*_{\mathrm{stat}}(y) \subseteq \left\{t \colon \liminf_{m\to\infty} \frac{p(m,t)}{m} = 0\right\} = H_0.$$

But it is a well-known fact that $\dim H_0 = 0$ (cf. [17], p. 194). So from the previous inclusion the theorem follows at once. $\square$

2. Topological results

In this part we give some applications of the concept of Baire's categories of sets to the study of the structure of $C_{\mathrm{stat}}(y)$ and the related sets.

In what follows we suppose that the set of limit points of the sequence $y = (y_n)_1^\infty$ can also contain points $+\infty$ and $-\infty$. In the investigation of the set $C_{\mathrm{stat}}(y)$ it seems to be convenient to distinguish two cases concerning the structure of the set of all limit points of y.

Theorem 2.1. *Let $y = (y_n)_1^\infty$ be an arbitrary sequence of real numbers. If y has only one limit point ξ then*

(i) *If ξ is finite then $C_{\mathrm{stat}}(y) = C^*_{\mathrm{stat}}(y) = (0,1]$.*
(ii) *If $\xi = +\infty$ then $C^\infty_{\mathrm{stat}}(y) = (0,1]$.*
(iii) *If $\xi = -\infty$ then $C^{-\infty}_{\mathrm{stat}}(y) = (0,1]$.*
(iv) *$K_{\mathrm{stat}}(y) = (0,1]$.*

Theorem 2.2. *Let $y = (y_n)_1^\infty$ be an arbitrary set of real numbers which has at least two limit points. Then*

(i) *If one of them is finite, then $C_{\mathrm{stat}}(y)$ is a dense set of the first Baire category in $(0,1]$.*
(ii) *If one of them is $+\infty$, then $C^\infty_{\mathrm{stat}}(y)$ is a dense set of the first Baire category in $(0,1]$.*
(iii) *If one of them is $-\infty$, then $C^{-\infty}_{\mathrm{stat}}(y)$ is a dense set of the first Baire category in $(0,1]$.*
(iv) *$K_{\mathrm{stat}}(y)$ is a dense set of the first Baire category in $(0,1]$.*
(v) *If y converges statistically then $C^*_{\mathrm{stat}}(y)$ is a dense set of the first Baire category in $(0,1]$.*

Proof of Theorem 2.2 is based on the following lemma:

Lemma 2.1. *Suppose that a sequence $y = (y_n)_1^\infty$ has two distinct finite limit points. Then $C_{\mathrm{stat}}(y)$ is a set of the first Baire category in $(0,1]$.*

Proof. Let $t = \sum\limits_{k=1}^{\infty} c_k(t)2^{-k} \in (0,1]$. For $v \in \mathbb{R}$ we put

$$h_n^{(v)}(t) = 1, \quad \text{if } p(n,t) = \sum_{k=1}^{n} c_k(t) = 0,$$
$$h_n^{(v)}(t) = \frac{1}{p(n,t)} \sum_{k=1}^{n} c_k(t) \mathrm{e}^{\mathrm{i}vc_k(t)y_k}, \quad \text{if } p(n,t) > 0.$$

By the Schoenberg criterion for statistical convergence (cf. [22]) a sequence $x = (x_n)_1^{\infty}$ converges statistically to $\xi \in \mathbb{R}$ if and only if

$$\forall v \in \mathbb{R} \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}vx_k} = \mathrm{e}^{\mathrm{i}v\xi}.$$

Using this criterion we see that $y(t)$ converges statistically to $\xi = \xi(t) \in \mathbb{R}$ if and only if

$$\forall v \in \mathbb{R} \ \lim_{n\to\infty} \frac{1}{p(n,t)} \sum_{k=1}^{n} c_k(t) \mathrm{e}^{\mathrm{i}c_k(t)vx_k} = \mathrm{e}^{\mathrm{i}v\xi(t)}. \tag{15}$$

Hence we get

$$C_{\mathrm{stat}}(y) \subseteq H^{(v)} \tag{16}$$

where $H^{(v)} = \{t \in (0,1]\colon$ there exists $\lim\limits_{n\to\infty} h_n^{(v)}(t) = h^{(v)}(t) \in C\}$, C denotes the set of all complex numbers, $h^{(v)}(t) = \mathrm{e}^{\mathrm{i}v\xi(t)}$.

Suppose that $\xi_1 \neq \xi_2$ are two distinct finite limit points of $y = (y_n)_1^{\infty}$. We can assume that $\xi_1 \not\equiv \xi_2 (\mathrm{mod}\, 2\pi)$ and put $v = 1$ in (16). In the opposite case we should choose an irrational number v such that $v\xi_1 \not\equiv v\xi_2 (\mathrm{mod}\, 2\pi)$ and replace $h = h^{(1)}$, $h_n = h_n^{(1)}$ and $H = H^{(1)}$ by $h^{(v)}$, $h_n^{(v)}$ and $H^{(v)}$, respectively.

For $v = 1$ we obtain from (16)

$$C_{\mathrm{stat}}(y) \subseteq H. \tag{16'}$$

Since $\xi_1 \neq \xi_2$, there are two disjoint sets K, $L \subseteq \mathbb{N}$, $K = \{k_1 < k_2 < \ldots\}$, $L = \{l_1 < l_2 < \ldots\}$ such that

$$\lim_{n\to\infty} y_{k_n} = \xi_1, \ \lim_{n\to\infty} y_{l_n} = \xi_2. \tag{17}$$

We can obviously assume that

$$\lim_{n\to\infty} (k_{n+1} - k_n) = \lim_{n\to\infty} (l_{n+1} - l_n) = +\infty. \tag{17'}$$

With respect to (16′) it suffices to prove that H is the set of the first category in $(0,1]$.

We sketch the plan of this proof. Put $Y=(0,1]\setminus\mathbb{Q}$, $\mathbb{Q}$ the set of all rational numbers, Y being considered as a metric space with the Euclidean metric.

If m is a fixed positive integer, then Y can be expressed as a union of sets (see (5)):

$$Y_m^{(j)}=\Big(\frac{j}{2^m},\frac{j+1}{2^m}\Big]\cap\mathbb{Q}'\ \ (0\leqslant j\leqslant 2^m-1),$$

$\mathbb{Q}'=\mathbb{R}\setminus\mathbb{Q}$. The sets $Y_m^{(j)}$ $(j=0,1,\ldots,2^m-1)$ are called the intervals in Y of the m-th order.

Functions h, h_n are considered as partial functions restricted to Y. We show that the oscillation of h at every point of $H\cap Y$ is $\geqslant\delta=|\mathrm{e}^{\mathrm{i}\xi_1}-\mathrm{e}^{\mathrm{i}\xi_2}|>0$. From this the discontinuity of h on $H\cap Y$ follows. But the functions h_n $(n=1,2,\ldots)$ are continuous on Y (h_n is constant on each $Y_n^{(j)}$, $j=0,1,\ldots,2^n-1$). Further, $\lim\limits_{n\to\infty}h_n=h$ on $H\cap Y$. Therefore h is a function in the first Baire class on $H\cap Y$ and so the set of its discontinuity points (i.e. the set $H\cap Y$) is a set of the first Baire category in $H\cap Y$ (cf. [23], p. 185). Thus $H\cap\mathbb{Q}'$ is a set of the first category in Y and so in $(0,1]$ as well. Then from $H=(H\cap\mathbb{Q})\cup(H\cap\mathbb{Q}')$ we see that H is a set of the first category in $(0,1]$.

Hence it remains to prove that the function h has at each point $t_0\in H\cap Y$ the oscillation $\geqslant\delta=|\mathrm{e}^{\mathrm{i}\xi_1}-\mathrm{e}^{\mathrm{i}\xi_2}|>0$.

Let $t_0=\sum\limits_{k=1}^{\infty}c_k(t_0)2^{-k}\in H\cap Y$. For each $m\in\mathbb{N}$ there is a $j=j(t_0)$ such that $t_0\in i_m^{j(t_0)}$ $(m=1,2,\ldots)$. It suffices to prove that in $i_m^{j(t_0)}$ there are two points t_1, $t_2\in H\cap Y$ such that $|h(t_1)-h(t_2)|=\delta$.

Define t_1, $t_2\in(0,1]$ in the following manner:

$$\begin{aligned}
&c_k(t_1)=c_k(t_0)\quad \text{if } k\leqslant m,\\
&c_k(t_1)=0\quad \text{if } k>m, k\neq k_s\ (s=1,2,\ldots),\\
&c_k(t_1)=1\quad \text{if } k>m, k=k_s\ (s=1,2,\ldots).
\end{aligned}$$

Similarly

$$\begin{aligned}
&c_k(t_2)=c_k(t_0)\quad \text{if } k\leqslant m,\\
&c_k(t_2)=0\quad \text{if } k>m, k\neq l_s\ (s=1,2,\ldots),\\
&c_k(t_2)=1\quad \text{if } k>m, k=l_s\ (s=1,2,\ldots).
\end{aligned}$$

From the definitions of $c_k(t_1)$, $c_k(t_2)$ $(k=1,2,\ldots)$ it follows that

$$t_1=\sum_{k=1}^{\infty}c_k(t_1),\ t_2=\sum_{k=1}^{\infty}c_k(t_2)$$

are irrational numbers in $(0,1]$ and $\lim y(t_1)=\xi_1$, $\lim y(t_2)=\xi_2$ (see (17)); further, $t_1,\ t_2 \in Y_m^{j(t_0)}$. So $h(t_1)=\mathrm{e}^{\mathrm{i}\xi_1}$, $h(t_2)=\mathrm{e}^{\mathrm{i}\xi_2}$ (see (15)) and $|h(t_1)-h(t_2)|=\delta>0$ follows at once. $\square$

Recall the meaning of $y^{(m)}=(y_j^{(m)})_{j=1}^{\infty}$ connected with $y=(y_n)_1^{\infty}$ (see Lemma 1.2).

P r o o f o f T h e o r e m 2.2. The density of the above sets follows from the fact that if $L=\text{lim-stat}\, y(t)\in\mathbb{R}\cup\{+\infty,-\infty\}$, then $L=\text{lim-stat}\, y(t')$, where the dyadic expansions of t and t' differ only in a finite number of digits.

Since $y=(y_n)_1^{\infty}$ has two limit points ξ, η there is an $m_0\in\mathbb{N}$ such that $\xi^{(m_0)}\neq\eta^{(m_0)}$, where $\xi^{(m_0)}$ and $\eta^{(m_0)}$ are the corresponding (finite) limit points of the sequence $y^{(m_0)}$. Therefore $C_{\text{stat}}(y^{(m_0)})$ is a set of the first category by Lemma 2.1.

However, then each of the sets is a set of the first category in $(0,1]$ by virtue of (10). $\square$

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