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On the Existence of One-Signed Periodic Solutions of Some Differential Equations of Second Order

Jan LIGĘZA

Institut of Mathematics, Silesian University,
Bankowa 14, 40 007 Katowice, Poland
e-mail: ligeza@ux2.math.us.edu.pl

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Abstract

We study the existence of one-signed periodic solutions of the equations

$$x''(t) - a^2(t)x(t) + \mu f(t, x(t), x'(t)) = 0,$$
$$x''(t) + a^2(t)x(t) = \mu f(t, x(t), x'(t)),$$

where $\mu > 0$, $a : (-\infty, +\infty) \to (0, \infty)$ is continuous and 1-periodic, f is a continuous and 1-periodic in the first variable and may take values of different signs. The Krasnosielski fixed point theorem on cone is used.

Key words: Positive solutions; boundary value problems; cone; fixed point theorem.

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1 Introduction

Nonnegative solutions to varius boundary value problems for ordinary differential equations have been considered by several authors (see for instance in

[1]–[8]). This paper deals with existence of positive (negative) periodic solutions of the nonlinear differential equations of the form

$$x''(t) - a^2(t)x(t) + \mu f(t, x(t), x'(t)) = 0, \tag{1.1}$$

$$x''(t) + a^2(t)x(t) = \mu f(t, x(t), x'(t)), \tag{1.2}$$

where $a : (-\infty, +\infty) \to (0, \infty)$ is continuous, 1-periodic, $\mu > 0$, f is a continuous, 1-periodic function in t and may take values of different signs. Existence in this paper will be established using Krasnosielski fixed point theorem in a cone, which we state here for the convenience of the reader.

Theorem 1.1 (K. Deimling [4], D. Guo, V. Laksmikannthan [5]). *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are bounded and open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$ and let $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be continuous and completely continuous. In addition suppose either $\|Au\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \ge \|u\|$ for $K \cap \partial\Omega_2$ or $\|Au\| \ge \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \le \|u\|$ for $u \in K \cap \partial\Omega_2$ hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

2 Preliminary results

First, we shall give some notation. We define $P_1^m(\mathbb{R})$ $(m \in \mathbb{N})$ to be the subspace of BC$(\mathbb{R})$ (bounded, continuous real functions on $\mathbb{R}$) consisting of all 1-periodic mapping x such that $x^{(m)}$ is an 1-periodic and continuous function on $\mathbb{R}$. For $x \in P_1^1(\mathbb{R})$ we define

$$\|x\|_1 = \sup_{t\in[0,1]} [|x(t)| + |x'(t)|].$$

Note $P_1^1(\mathbb{R}, \|\cdot\|_1)$ is a Banach space.

Let us consider the boundary value problems

$$-\left(x''(t) - a^2(t)x(t)\right) = 0, \quad x(0) = x(1),\ x'(0) = x'(1); \tag{2.1}$$

$$x''(t) + a^2(t)x(t) = 0, \quad x(0) = x(1),\ x'(0) = x'(1), \tag{2.2}$$

In this paper we assume conditions under which the only solution of the problem (2.1) or (2.2) is the trivial one. In the proofs of theorems we will make use the Green functions G_1 and G_2 of the boundary value problems (2.1) and (2.2).

Remark 2.1 If $a \in C[0, 1]$ and $a(t) > 0$ for all $t \in [0, 1]$, then the problem (2.1) has only the trivial solution and $G_1(t, s) > 0$ for all $t, s \in [0, 1]$ (see [7]).

If $a \in C[0, 1]$, $a(t) > 0$ for $t \in [0, 1]$ and $\sup_{t\in[0,1]} a(t) < \pi$, then the problem (2.2) has only the trivial solution and $G_2(t, s) > 0$ for all $t, s \in [0, 1]$ (see [7]).

Remark 2.2 If $a(t) \equiv k > 0$ for $t \in [0, 1]$, then

$$G_1(t, s) = \frac{1}{2k(e^k - 1)} \begin{cases} e^{k(1-s+t)} + e^{k(s-t)}, & 0 \le t \le s \le 1 \\ e^{k(t-s)} + e^{k(1+s-t)}, & 0 \le s \le t \le 1. \end{cases}$$

Remark 2.3 If $a(t) \equiv k > 0$ for $t \in [0,1]$ and $k \neq 2l\pi$ for all $l \in \mathbb{N}$, then

$$G_2(t,s) = \frac{1}{2k \sin k/2} \cos k[1/2 - |s-t|].$$

Before giving the lemmas we shall introduce some notation. We denote

$$\overline{M_i} = \sup_{t,s\in[0,1]} G_i(t,s), \qquad \overline{m_i} = \inf_{t,s\in[0,1]} G_i(t,s),$$
$$\overline{M_i}' = \sup_{t,s\in[0,1]} \left|\frac{\partial G_i}{\partial t}(t,s)\right|, \qquad \overline{m_i}' = \inf_{t,s\in[0,1]} \left|\frac{\partial G_i}{\partial t}(t,s)\right|$$

for $i = 1,2$.

The properties of the functions G_i $(i = 1,2)$ needed later on are described by the following lemmas.

Lemma 2.4 *Suppose that*

(2.3) $\quad f : \mathbb{R}^3 \to \mathbb{R}$ *is continuous,* $a \in C[0,1]$ *and* $a(t) > 0$ *for* $t \in [0,1]$.

Then $x \in C^2[0,1]$ *is a solution of the problem*

$$\begin{cases} x''(t) - a^2(t)x(t) + \mu f(t,x(t),x'(t)) = 0 \\ x(0) = x(1), \quad x'(0) = x'(1) \end{cases} \tag{2.4}$$

if and only if x *satisfies the integral equation*

$$x(t) = \mu \int_0^1 G_1(t,s) f(s,x(s),x'(s))\, ds. \tag{2.5}$$

Lemma 2.5 *Suppose that* $a \in C[0,1], a(t) > 0$ *for* $t \in [0,1]$, $\sup_{t\in[0,1]} a(t) < \pi$ *and* $f : \mathbb{R}^3 \to \mathbb{R}$ *is continuous. Then* $x \in C^2[0,1]$ *is a solution of the problem*

$$\begin{cases} x''(t) + a^2(t)x(t) = \mu f(t,x(t),x'(t)) \\ x(0) = x(1), \quad x'(0) = x'(1) \end{cases} \tag{2.6}$$

if and only if x *satisfies the integral equation*

$$x(t) = \mu \int_0^1 G_2(t,s) f(s,x(s),x'(s))\, ds. \tag{2.7}$$

Lemma 2.6 *Let* $a \in C[0,1]$ *and* $a(t) > 0$ *for* $t \in [0,1]$. *Then*

$$\inf_{t,s\in[0,1]} G_1(t,s) = \inf_{t\in[0,1]} G_1(t,1), \quad \text{(see [7])} \tag{2.8}$$

$$\begin{cases} d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right| \geq G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s-0,s)\right| \\ \text{for } s,t \in [0,1] \text{ and} \\ d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right| \geq G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s+0,s)\right| \\ \text{for } s,t \in [0,1] \text{ where } \frac{\partial G_1}{\partial t}(s-0,s) \quad \left(\frac{\partial G_1}{\partial t}(s+0,s)\right) \\ \text{denote the left-hand (the right-hand) side derivative of } G_1 \\ \text{at the point } (s,s) \text{ and } d_0 \geq \frac{2\overline{M_1}' + \overline{M_1}}{\overline{m_1}}, \end{cases} \tag{2.9}$$

$$G_1(s,s)+\left|\frac{\partial G_1}{\partial t}(s-0,s)\right|\geq M_0\left(G_1(t,s)+\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right)\tag{2.10}$$

for $s,t\in[0,1]$ *and* $M_0\in\left(0,\frac{\overline{m_1}+\overline{m_1}'}{\overline{M_1}+\overline{M_1}'}\right]$,

$$G_1(s,s)+\left|\frac{\partial G_1}{\partial t}(s+0,s)\right|\geq M_0\left(G_1(t,s)+\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right),\tag{2.11}$$

where $s,t\in[0,1]$.

Lemma 2.7 *Let* $a\in C[0,1]$ *and* $a(t)>0$ *for* $t\in[0,1]$ *and* $\sup_{t\in[0,1]}a(t)<\pi$. *Then*

$$\sup_{t,s\in[0,1]}G_2(t,s)=\sup_{t\in[0,1]}G_2(t,1)\quad\text{(see [7])},\tag{2.12}$$

$$\overline{d_0}G_2(t,s)-\left|\frac{\partial G_2}{\partial t}(t,s)\right|\geq G_2(s,s)+\left|\frac{\partial G_2}{\partial t}(s-0,s)\right|\tag{2.13}$$

for $t,s\in[0,1]$ *and* $\overline{d_0}\geq\frac{2\overline{M_2}'+\overline{M_2}}{\overline{m_2}}$,

$$\overline{d_0}G_2(t,s)-\left|\frac{\partial G_2}{\partial t}(t,s)\right|\geq G_2(s,s)+\left|\frac{\partial G_2}{\partial t}(s+0,s)\right|,\tag{2.14}$$

where $s,t\in[0,1]$,

$$G_2(s,s)+\left|\frac{\partial G_2}{\partial t}(s-0,s)\right|\geq\overline{M_0}\left(G_2(t,s)+\left|\frac{\partial G_2}{\partial t}(t,s)\right|\right)\tag{2.15}$$

for $s,t\in[0,1]$, $\overline{M_0}\in\left(0,\frac{\overline{m_2}+\overline{m_2}'}{\overline{M_2}+\overline{M_2}'}\right]$ *and*

$$G_2(s,s)+\left|\frac{\partial G_2}{\partial t}(s+0,s)\right|\geq\overline{M_0}\left(G_2(t,s)+\left|\frac{\partial G_2}{\partial t}(t,s)\right|\right),\tag{2.16}$$

where $s,t\in[0,1]$.

It is not difficult to prove the following

Corollary 2.8 *Let* $a(t)\equiv k>0$ *for* $t\in[0,1]$. *Then*

$$(2.8)'\quad\begin{cases}\sup_{t,s\in[0,1]}G_1(t,s)=\frac{e^k+1}{2k(e^k-1)},\\ \inf_{t,s\in[0,1]}G_1(t,s)=\frac{e^{k/2}}{k(e^k-1)},\\ G_1(s,s)\geq G_1(t,s)\ \text{ for } s,t\in[0,1],\quad \sup_{t,s\in[0,1]}\left|\frac{\partial G_1}{\partial t}(t,s)\right|=\frac{1}{2},\\ \inf_{t,s\in[0,1]}\left|\frac{\partial G_1}{\partial t}(t,s)\right|=0,\quad \int_0^1 G_1(t,s)\,ds=\frac{1}{k^2}\ \text{for } t\in[0,1],\\ \sup_{t\in[0,1]}\int_0^1 G_1(t,s)\,ds+\sup_{t\in[0,1]}\int_0^1\left|\frac{\partial G_1}{\partial t}(t,s)\right|ds=m_1\leq\frac{1}{k^2}+\frac{1}{2},\end{cases}$$

$$(2.9)' \qquad \begin{cases} d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right| \geq G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s-0,s)\right| \\ \text{for } s,t \in [0,1] \text{ and} \\ d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right| \geq G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s+0,s)\right| \\ \text{for } s,t \in [0,1] \text{ and } d_0 \geq \frac{e^k+1+2k(e^k-1)}{2e^{k/2}}, \end{cases}$$

$$(2.10)' \qquad G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s-0,s)\right| \geq M_0 \left(G_1(t,s) + \left|\frac{\partial G_1}{\partial t}(t,s)\right|\right)$$

for $s,t \in [0,1]$ *and* $M_0 \in \left(0, \frac{2e^{k/2}}{e^k(1+k)+1-k}\right]$,

$$(2.11)' \qquad G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s+0,s)\right| \geq M_0 \left(G_1(t,s) + \left|\frac{\partial G_1}{\partial t}(t,s)\right|\right),$$

where $s,t \in [0,1]$.

Corollary 2.9 *Let* $a(t) \equiv k$ *for* $t \in [0,1]$ *and let* $0 < k < \pi$. *Then*

$$(2.12)' \qquad \begin{cases} \inf_{t,s\in[0,1]} G_2(t,s) = \frac{\cot k/2}{2k}, \\ \sup_{t,s\in[0,1]} G_2(t,s) = \frac{1}{2k\sin^{k/2}}, \quad \sup_{t,s\in[0,1]} \left|\frac{\partial G_2}{\partial t^{(t,s)}}(t,s)\right| = \frac{1}{2}, \\ \inf_{t,s\in[0,1]} \left|\frac{\partial G_2}{\partial t}(t,s)\right| = 0, \quad \int_0^1 G_2(t,s)\,ds = \frac{1}{k^2} \text{ for } t \in [0,1], \\ \sup_{t\in[0,1]} \int_0^1 G_2(t,s)\,ds + \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial G_2}{\partial t}(t,s)\right| ds \\ \quad = m_2 \leq \frac{1}{k^2} + \frac{1}{2}, \end{cases}$$

$$(2.13)' \qquad \overline{d_0} G_2(t,s) - \left|\frac{\partial G_2}{\partial t}(t,s)\right| \geq G_2(s,s) + \left|\frac{\partial G_2}{\partial t}(s-0,s)\right|$$

for $t,s \in [0,1]$ *and*

$$(2.14)' \qquad \overline{d_0} G_2(t,s) - \left|\frac{\partial G_2}{\partial t}(t,s)\right| \geq G_2(s,s) + \left|\frac{\partial G_2}{\partial t}(s+0,s)\right|,$$

where $s,t \in [0,1]$ *and* $\overline{d_0} \geq 2k\tan k/2 + 1$,

$$(2.15)' \qquad G_2(s,s) + \left|\frac{\partial G_2}{\partial t}(s-0,s)\right| \geq \overline{M_0} \left(G_2(t,s) + \left|\frac{\partial G_2}{\partial t}(t,s)\right|\right)$$

for $s,t \in [0,1]$, $\overline{M_0} \in \left(0, \frac{\cos k/2}{1+k\sin k/2}\right]$ *and*

$$(2.16)' \qquad G_2(s,s) + \left|\frac{\partial G_2}{\partial t}(s+0,s)\right| \geq \overline{M_0} \left(G_2(t,s) + \left|\frac{\partial G_2}{\partial t}(t,s)\right|\right),$$

where $s,t \in [0,1]$.

Throughout the paper

$\mathcal{D} = (-\infty, \infty) \times [0, \infty) \times (-\infty, \infty)$, $\tilde{\mathcal{D}} = (-\infty, \infty) \times (-\infty, 0] \times (-\infty, \infty)$,
$\mu > 0$, $a : (-\infty, \infty) \to (0, \infty)$ is continuous and 1-periodic, $L > 0$,

$$\phi_i(t) = \mu L \int_0^1 G_i(t, s) ds \quad \text{for } i = 1, 2, \ t \in [0, 1],$$

$\overline{\phi_i} : (-\infty, \infty) \to (-\infty, \infty)$, $\overline{\phi_i} \in P_1^2(\mathbb{R})$, $\overline{\phi_i}(t) = \phi_i(t)$ *for* $i = 1, 2$ *and* $t \in [0, 1]$,

$$m_i = \sup_{t \in [0,1]} \int_0^1 G_i(t, s)\, ds + \sup_{t \in [0,1]} \int_0^1 \left| \frac{\partial G_i}{\partial t}(t, s) \right| ds$$

for $i = 1, 2$.

3 Positive periodic solutions

In this section we present results on existence of positive 1-periodic solutions of the equations (1.1) and (1.2).

Theorem 3.1 *Suppose that*

$$(3.1) \qquad \begin{cases} f : \mathcal{D} \to (-\infty, \infty) \text{ is continuous,} \\ f(t + 1, v_0, v_1) = f(t, v_0, v_1) \text{ for all } (t, v_0, v_1) \in \mathcal{D}, \\ \text{there exists a constant } L > 0 \text{ with} \\ f(t, v_0, v_1) + L \geq 0 \text{ for all } (t, v_0, v_1) \in \mathcal{D}, \end{cases}$$

(3.2) *there exists a function* $\psi(u)$ *such that* $f(t, v_0, v_1) + L \leq \psi(v_0 + |v_1|)$ *on* $\mathcal{D}$, *where* $\psi : [0, \infty) \to [0, \infty)$ *is continuous and nondecreasing and* $\psi(u) > 0$ *for* $u > 0$,

$$(3.3) \qquad \begin{cases} \text{there exist } C_1 > 0 \text{ and } r > 0 \text{ such that } r \geq \mu L C_1 d_0, \\ \int_0^1 G_1(t, s) ds \leq M_0 C_1 \text{ for } t \in [0, 1] \text{ and } \frac{r}{\psi(r + \|\phi_1\|_1)} \geq \mu m_1, \end{cases}$$

where d_0, M_0 *and* m_1 *have properties (2.9)–(2.11),*

$$(3.4) \quad \begin{cases} f(t, v_0, v_1) + L \geq \tau(t) g(v_0) \text{ on } \mathcal{D}, \text{ where } \tau : (-\infty, \infty) \to [0, \infty) \\ \text{is continuous and 1-periodic and } g : [0, \infty) \to [0, \infty) \text{ is continuous,} \\ g(u) > 0 \text{ for } u > 0 \text{ and } g \text{ is nondecreacing,} \end{cases}$$

(3.5) *there exists* $R > 0$ *such that* $R > r$ *and*

$$d_0 R \leq \mu \int_0^1 \tau(s) \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left| \frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right) \right| \right] g\left(\frac{\varepsilon M_0 R}{d_0}\right) ds,$$

where $\varepsilon > 0$ *is any constant such that*

$$1 - \frac{\mu L C_1 d_0}{R} \geq \varepsilon.$$

Then (1.1) has a positive solution $x \in P_1^2(\mathbb{R})$.

Proof The proof of Theorem is similar to that of Theorem 2.1 in the paper [1]. To show (1.1) has a positive 1-periodic solution we will look at

$$x(t)=\mu\int_0^1 G_1(t,s)f_+^*(s,x(s)-\overline{\phi_1}(s),x'(s)-\overline{\phi_1'}(s))\,ds, \tag{3.6}$$

where

$$f_+^*(t,v_0,v_1)=\begin{cases} f(t,v_0,v_1)+L, & \text{if } (t,v_0,v_1)\in\mathcal{D}\\ f(t,0,v_1)+L, & \text{if } (t,v_0,v_1)\in\tilde{\mathcal{D}}.\end{cases}$$

We will show that there exists a solution x_1 to (3.6) with $x_1(t)\ge\overline{\phi_1}(t)$ for $t\in[0,1]$. If this is true then $u(t)=x_1(t)-\overline{\phi_1}(t)$ is a positive solution of (3.6) since for $t\in[0,1]$ we have

$$\begin{aligned} u(t)=\mu\int_0^1 G_1(t,s)[f_+^*(s,x_1(s)-\overline{\phi_1}(s),x_1'(s)-\overline{\phi_1'}(s))\,ds-\mu L\int_0^1 G_1(t,s)\,ds\\ =\mu\int_0^1 G_1(t,s)f(s,u(s),u'(s))\,ds.\end{aligned}$$

We concentrate our study on (3.6). Let $E=(P_1^1(\mathbb{R}),\|\cdot\|_1)$ and

$$K_1=\{u\in P_1^1(\mathbb{R}):\ \min_{t\in[0,1]}[d_0u(t)-|u'(t)|\ge M_0\|u\|_1\}.$$

Obviously K_1 is a cone of E. Let

$$\Omega_1=\{u\in P_1^1(\mathbb{R}):\ \|u\|_1<r\} \tag{3.7}$$

and

$$\Omega_2=\{u\in P_1^1(\mathbb{R}):\ \|u\|_1<R\}. \tag{3.8}$$

Now let $A_1:\ K_1\cap(\overline{\Omega_2}\setminus\Omega_1)\to P_1^1(\mathbb{R})$ be defined by

$$A_1\varphi=x_\varphi,\quad \text{where } \varphi\in K_1\cap(\overline{\Omega_2}\setminus\Omega_1)$$

and x_φ is the unique 1-periodic solution of the equation

$$x''(t)-a^2(t)x(t)+\mu f_+^*(t,\varphi(t)-\overline{\phi_1}(t),\varphi'(t)-\overline{\phi_1}'(t))=0. \tag{3.9}$$

First we show $A_1:\ K_1\cap(\overline{\Omega_2}\setminus\Omega_1)\to K_1$. If $\varphi\in K_1\cap(\overline{\Omega_2}\setminus\Omega_1)$ and $t\in[0,1]$, then by Lemma 2.4 we have

$$(A_1\varphi)(t)=\mu\int_0^1 G_1(t,s)f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds. \tag{3.10}$$

The relations (2.8)–(2.11) imply

$$\begin{cases} d_0(A_1\varphi)(t) - |(A_1\varphi)'(t)| = \\ \quad = \mu d_0 \int_0^1 G_1(t,s) f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds \\ \quad - \mu\Big|\Big(\int_0^1 G_1(t,s) f_+^*(s,\varphi(s)-\overline{\phi}_1(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds\Big)'\Big| \\ \quad \geq \mu\int_0^t \left[d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right|\right] f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds \\ \quad + \mu\int_t^1 \left[d_0 G_1(t,s) - \left|\frac{\partial G_1}{\partial t}(t,s)\right|\right] f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds \\ \quad \geq \mu\int_0^t \left(G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s+0,s)\right|\right) f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds \\ \quad + \mu\int_t^1 \left(G_1(s,s) + \left|\frac{\partial G_1}{\partial t}(s-0,s)\right|\right) f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds \\ \quad \geq \mu M_0\int_0^t \left(G_1(\overline{t},s) + \left|\frac{\partial G_1}{\partial t}(\overline{t},s)\right|\right) f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds \\ \quad + \int_t^1 \left(G_1(\overline{t},s) + \left|\frac{\partial G_1}{\partial t}(\overline{t},s)\right|\right) f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds \\ \quad \geq \mu M_0\left(\int_0^1 \left(G_1(\overline{t},s) + \left|\frac{\partial G_1}{\partial t}(\overline{t},s)\right|\right)\right) f_+^*(s,\varphi(s)-\overline{\phi_1}(s),\varphi'(s)-\overline{\phi_1}'(s))\,ds \\ \quad \geq M_0\left((A_1\varphi)(\overline{t}) + |(A_1\varphi)'(\overline{t})|\right), \quad \text{where } \overline{t}\in[0,1]. \end{cases}$$

Hence

$$d_0(A_1\varphi)(t) \geq d_0(A_1\varphi)(\overline{t}) - |(A_1\varphi)'(\overline{t})| \geq M_0\|A_1\varphi\|_1. \tag{3.11}$$

Consequently $A_1\varphi \in K_1$. So $A_1:\ K_1\cap(\overline{\Omega_2}\setminus\Omega_1)\to K_1$. We now show

$$\|A_1\varphi\|_1 \leq \|\varphi\|_1 \qquad \text{for } \ \varphi\in K_1\cap\partial\Omega_1. \tag{3.12}$$

To see this let $\varphi\in K_1\cap\partial\Omega_1$. Then

$$\|\varphi\|_1 = r \quad \text{and} \quad \varphi(t) \geq \frac{M_0 r}{d_0} \quad \text{for } t\in\mathbb{R}.$$

From (3.2)–(3.3) we have

$$(A_1\varphi)(t) + |(A_1\varphi)'(t)| \leq \mu\psi(r + \|\overline{\phi_1}\|_1)m_1 \leq r \leq \|\varphi\|_1.$$

So (3.12) holds. Next we show

$$\|A_1\varphi\|_1 \geq \|\varphi\|_1 \quad \text{for } \varphi\in K_1\cap\partial\Omega_2. \tag{3.13}$$

To see it let $\varphi\in K_1\cap\partial\Omega_2$. Then $\|\varphi\|_1 = R$ and $d_0\varphi(t)\geq RM_0$ for $t\in\mathbb{R}$. Let ε be as in (3.5). From (3.3) we have

$$\begin{aligned} \varphi(t) - \overline{\phi_1}(t) &= \varphi(t) - \mu L\int_0^1 G_1(t,s)ds \geq \varphi(t) - \frac{\mu L C_1 M_0 R d_0}{d_0 R} \\ &\geq \varphi(t)\left(1 - \frac{\mu L C_1 d_0}{R}\right) \geq \varepsilon\varphi(t) \geq \frac{\varepsilon R M_0}{d_0} > 0. \end{aligned}$$

This together with (3.4)–(3.5) yields

$$d_0\|A_1\varphi\|_1 \geq d_0(A_1\varphi)\left(\frac{1}{2}\right) - \left|(A\varphi)'\left(\frac{1}{2}\right)\right|$$
$$\geq \mu\int_0^1 \left(d_0 G_1\left(\frac{1}{2},s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2},s\right)\right|\right)\tau(s)g(\varphi(s)-\overline{\phi_1}(s))\,ds$$
$$\geq \mu\int_0^1 \tau(s)\left(d_0 G_1\left(\frac{1}{2},s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2},s\right)\right|\right) g\left(\frac{\varepsilon M_0 R}{d_0}\right) ds \geq d_0 R.$$

Hence we have (3.13). We will show that A_1 is continuous and compact. To see it let

$$G_1(t,s) = \begin{cases} a_1(s)y_1(t) + a_2(s)y_2(t), & 0 \leq t \leq s \leq 1 \\ b_1(s)y_1(t) + b_2(s)y_2(t), & 0 \leq s \leq t \leq 1 \end{cases}$$

where (y_1, y_2) is a fundamental system of equation (2.1) and $a_i, b_i : [0,1] \to \mathbb{R}$ are continuous for $i = 1, 2$. From relations (3.1)–(3.3) and properties of the function G_1 it follows that A_1 is a bounded and continuous operator. Notice that for $y \in K_1 \cap (\overline{\Omega_2} \setminus \Omega_1)$; $t_1, t_2 \in [0,1]$ and $t_1 < t_2$ that

$$|A_1 y)(t_2) - (A_1 y)(t_1)| \leq \int_0^1 |G_1(t_2,s) - G_1(t_1,s)|\psi(R + \|\overline{\phi_1}\|_1)\,ds$$

and

$$|(A_1 y)'(t_2) - (A_1 y)'(t_1)| \leq$$
$$\leq \int_0^{t_1} |b_1(s)(y_1'(t_2) - y_1'(t_1)) + b_2(s)(y_2'(t_2) - y_2'(t_1))|\psi(R + \|\overline{\phi_1}\|_1)\,ds$$
$$+ \int_{t_1}^{t_2} |b_1(s)y_1'(t_2) - a_1(s)y_1'(t_1) + b_2(s)y_2'(t_2) - a_2(s)y_2'(t_1)|\psi(R + \|\overline{\phi_1}\|_1)\,ds$$
$$+ \int_{t_2}^{1} |a_1(s)(y_1'(t_2) - y_1'(t_1)) + a_2(s)(y_2'(t_2) - y_2'(t_1))|\psi(R + \|\phi_1\|_1)\,ds$$
$$\leq \int_0^1 (|y_1'(t_2) - y_1'(t_1)| + |y_2'(t_2) - y_2'(t_1)|)h(s)\psi(R + \|\overline{\phi_1}\|_1)\,ds$$
$$+ 2\int_{t_1}^{t_2} (\|y_1\|_1 + \|y_2\|_1)h(s)\psi(R + \|\overline{\phi_1}\|_1)\,ds,$$

where $h(s) = |a_1(s)| + |a_2(s)| + |b_1(s)| + |b_2(s)|$.

Using the Arzela–Ascoli theorem we conclude that $A_1 : K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_1$ is compact. Theorem 1.1 implies A_1 has a fixed point $x \in K_1 \cap (\overline{\Omega_2} \setminus \Omega_1)$, i.e. $r \leq \|x\|_1 \leq R$ and $x(t) \geq \frac{M_0 r}{d_0}$ for $t \in \mathbb{R}$. This completes the proof of Theorem 3.1. □

Theorem 3.2 *Suppose that*

(3.14) $f : \mathcal{D} \to [0, \infty)$ *is continuous*

(3.15) $f(t+1, v_0, v_1) = f(t, v_0, v_1)$ *for all* $(t, v_0, v_1) \in \mathcal{D}$,

$$(3.16) \qquad \begin{cases} \text{there exist a function } \psi(u) \text{ such that} \\ f(t, v_0, v_1) \le \psi(v_0 + |v_1|) \text{ on } \mathcal{D}, \\ \text{where } \psi : [0, \infty) \to [0, \infty) \text{ is continuous and nondecreasing and} \\ \psi(u) > 0 \text{ for } u > 0, \end{cases}$$

(3.17) *there exists* r *such that* $r \ge \psi(r)\mu m_1$,

$$(3.18) \qquad \begin{cases} \text{there exist function } \tau \text{ and } g \text{ such that } f(t, v_0, v_1) \ge \tau(t) g(v_0) \\ \text{for all } (t, v_0, v_1) \in \mathcal{D}, \text{ where } g : [0, \infty) \to [0, \infty),\ g(u) > 0 \\ \text{for } u > 0, g \text{ is continuous and nondecreasing and} \\ \tau : (-\infty, \infty) \to [0, \infty) \text{ is continuous and 1-periodic,} \end{cases}$$

(3.19) *there exists* $R > 0$ *such that* $R > r$ *and*

$$d_0 R \le \mu \int_0^1 \tau(s) \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left| \frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right) \right| \right] g\left(\frac{M_0 R}{d_0}\right) ds.$$

Then (1.1) has a positive solution $x \in P_1^2(\mathbb{R})$.

Proof The proof of Theorem 3.2 is similar to that of Theorem 3.1. Let E, Ω_1, Ω_2 and K_1 be as in Theorem 3.1. Now let $\varphi \in K_1 \cap (\overline{\Omega_2} \setminus \Omega_1)$ and let x_φ be the unique 1-periodic solution of the equation (3.9) and let $A_2 : K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to P_1^1(\mathbb{R})$ be defined by $A_2\varphi = x_\varphi$. It is easy to check that $A_2 : K_1 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_1$, A_2 is continuous and compact, $\|A_2\varphi\|_1 \le \|\varphi\|_1$ for $\varphi \in K_1 \cap \partial\Omega_1$ and $\|A_2\varphi\| \ge \|\varphi\|_1$ for $\varphi \in K_1 \cap \partial\Omega_2$. Applying Theorem 1.1 we can show that the equation (1.1) has a positive solution $x \in P_1^2(\mathbb{R})$ which implies our assertion. □

Example 3.3 To illustrate the applicabillity of Theorem 3.2 we consider the following equation

$$x''(t) - x(t) + \mu(x(t) + |x'(t)|)^2 = 0. \tag{3.20}$$

Fix

$$a(t) \equiv 1, \quad \tau(t) = 1, \quad d_0 = \frac{3e-1}{2\sqrt{e}}, \quad M_0 = \frac{1}{\sqrt{e}}, \quad g(u) = \psi(u) = u^2,$$

We claim that (3.17) holds for $r \le \frac{2}{3\mu}$. To see this notice that $\mu m_1 \le \frac{3}{2}\mu$. Clearly

$$g\left(\frac{RM_0}{d_0}\right) = \frac{RM_0^2}{d_0^2} = \frac{4R^2}{(3e-1)^2}$$

and

$$\mu\int_0^1 \tau(s)\left[d_0G_1\left(\frac{1}{2},s\right)-\left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2},s\right)\right|\right]g\left(\frac{M_0R}{d_0}\right)ds$$
$$=\frac{4\mu R^2}{(3e-1)^2}\int_0^1\left[\frac{(3e-1)}{2\sqrt{e}}G_1\left(\frac{1}{2},s\right)-\left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2},s\right)\right|\right]ds\geq\frac{(3e-1)}{2\sqrt{e}}R$$

for sufficiently large R. Thus all conditions of Theorem 3.2 are satisfied and the equation (3.20) has a positive solution $x\in P_1^2(\mathbb{R})$.

It is not difficult to verify that $x(t)=\frac{1}{\mu}$ is a periodic and positive solution of the equation (3.20).

Theorem 3.4 *Assume conditions (3.1)–(3.2) and (3.4). Suppose that*

(3.21) $\quad 0<a(t)<\pi$ *for* $t\in[0,1]$,

$$(3.22)\qquad \begin{cases} \text{there exists } C_2>0 \text{ and } r>0 \text{ such that } r\geq\mu LC_2\overline{d}_0,\\ \int_0^1 G_2(t,s)\,ds\leq C_2\overline{M}_0 \text{ for } t\in[0,1] \text{ and } r\geq\psi(r+\|\overline{\phi}_2\|_1)\mu m_2,\\ \text{where } \overline{d_0} \text{ and } \overline{M}_0 \text{ have properties (2.13)–(2.16),}\end{cases}$$

$$(3.23)\qquad \begin{cases} \text{there exists } R>0 \text{ such that } R>r \text{ and}\\ \overline{d}_0R\leq\mu\int_0^1\tau(s)\left[\overline{d}_0G_2\left(\frac{1}{2},s\right)-\left|\frac{\partial G_2}{\partial t}\left(\frac{1}{2},s\right)\right|\right]g\left(\frac{\varepsilon\overline{M}_0R}{\overline{d}_0}\right)ds,\\ \text{where } \varepsilon>0 \text{ is any constant such that } 1-\frac{\mu LC_2\overline{d}_0}{R}\geq\varepsilon.\end{cases}$$

Then (1.2) has a positive solution $x\in P_1^2(\mathbb{R})$.

Proof Let E,Ω_1 and Ω_2 be as in Theorem 3.1. Let

$$K_2=\{u\in P_1^1(\mathbb{R}):\ \min_{t\in[0,1]}[\overline{d}_0u(t)-|u'(t)|]\geq\overline{M}_0\|u\|_1\}.$$

Then K_2 is a cone of E. Now let $\varphi\in K_2\cap(\overline{\Omega_2}\setminus\Omega_1)$ and let x_φ be the unique 1-periodic solution of the equation

$$x''(t)+a^2(t)x(t)=\mu f_+^*(t,\varphi(t)-\overline{\phi}_2(t),\varphi'(t)-\overline{\phi}_2'(t)),$$

where f_+^* is defined by (3.6). Finally let $A_3:K_2\cap(\overline{\Omega_2}\setminus\Omega_1)\to P_1^1(\mathbb{R})$ be defined by $A_3\varphi=x_\varphi$. It is not difficult to prove that $A_3:K_2\cap(\overline{\Omega_2}\setminus\Omega_1)\to K_2$, A_3 is continuous and compact. The similar arguments as in Theorem 3.1 gurantee that $\|A_3\varphi\|_1\leq\|\varphi\|_1$ for $\varphi\in K_2\cap\partial\Omega_1$ and $\|A_3\varphi\|_1\geq\|\varphi\|_1$ for $\varphi\in K_2\cap\partial\Omega_2$. Theorems 1.1 implies that A_3 has a fixed point $x\in K_2\cap(\overline{\Omega_2}\setminus\Omega_1)$ i.e. $x(t)\geq\frac{\overline{M}_0r}{\overline{d}_0}$ for $t\in\mathbb{R}$. This completes the proof of Theorem 3.4. □

In a similar way we can prove

Corollary 3.5 *Assume conditions (3.14)–(3.16) and (3.18). Suppose that*

(3.24) *there exists $r > 0$ such that $r \geq \psi(r)\mu m_2$,*

(3.25) *there exists $R > 0$ such that $R > r$ and*

$$\overline{d}_0 R \leq \mu \int_0^1 \tau(t) \left[\overline{d}_0 G_2 \left(\frac{1}{2}, s \right) - \left| \frac{\partial G_2}{\partial t} \left(\frac{1}{2}, s \right) \right| \right] g \left(\frac{M_0 R}{\overline{d}_0} \right) ds.$$

Then (1.2) *has a positive solution $x \in P_1^2(\mathbb{R})$.*

Example 3.6 We consider the equation

$$x''(t) + x(t) = \mu |\sin \pi t| [(x(t) + |x'(t)|)^2 - 1]. \tag{3.26}$$

It is not difficult to verify that the equation (3.26) for $0 < \mu \leq 1/5$ has a solution x such that $x(t) > 0$ for $t \in \mathbb{R}$ and $x \in P_1^2(\mathbb{R})$. To see this we apply Theorem 3.4 with $a(t) \equiv 1$, $L = 1$, $\tau(t) = |\sin \pi t|$, $\overline{d}_0 = 2(\tan\frac{1}{2} + 1)$, $\overline{M}_0 = \frac{\cos 1/2}{1+\sin 1/2}$, $g(u) = \psi(u) = u^2$, $\overline{\phi}_2 = \mu$, $C_2 = 2$, $r = 1$ and with sufficiently large R ($R > 1$).

4 Negative periodic solutions

In a similar way we can prove theorems on existence of negative periodic solutions of the equations (1.1) and (1.2).

Theorem 4.1 *Suppose that*

$$\begin{cases} f : \tilde{\mathcal{D}} \to (-\infty, \infty) \text{ is continuous,} \\ f(t+1, v_0, v_1) = f(t, v_0, v_1) \text{ for } (t, v_0, v_1) \in \tilde{\mathcal{D}}, \\ \text{there exists a constant } L > 0 \text{ with} \\ f(t, v_0, v_1) - L \leq 0 \text{ for } (t, v_0, v_1) \in \tilde{\mathcal{D}}, \end{cases} \tag{4.1}$$

$$\begin{cases} \text{there exists a function } \psi(u) \text{ such that} \\ -f(t, v_0, v_1) + L \leq \psi(|v_0| + |v_1|) \text{ for } (t, v_0, v_1) \in \tilde{\mathcal{D}}, \\ \text{where } \psi : [0, \infty) \to [0, \infty) \text{ is continuous} \\ \text{and nondecreasing and } \psi(u) > 0 \text{ for } u > 0, \end{cases} \tag{4.2}$$

(4.3) *$L - f(t, v_0, v_1) \geq \tau(t) g(|v_0|)$ for $(t, v_0, v_1) \in \tilde{\mathcal{D}}$, where τ and g have property (3.4),*

(4.4) *there exist $R > 0$ and $r > 0$ such that (3.3) and (3.5) hold.*

Then (1.1) has a negative solution $x \in P_1^2(\mathbb{R})$.

Proof Let

$$f_-^*(t,v_0,v_1)=\begin{cases} f(t,v_0,v_1)-L, & \text{if } (t,v_0,v_1)\in\tilde{\mathcal{D}}\\ f(t,0,v_1)-L, & \text{if } (t,v_0,v_1)\in\mathcal{D}.\end{cases}$$

We will show that there exists a solution x_2 to the following equation

$$x(t)=\mu\int_0^1 G_1(t,s)f_-^*(s,x(s)+\overline{\phi}_1(s),x'(s)+\overline{\phi}_1'(s))\,ds \tag{4.5}$$

with $x_2(t)+\overline{\phi}_1(t)<0$ for $t\in[0,1]$. If this is true, then $u(t)=x_2(t)+\overline{\phi}_1(t)$ is a negative solution of the equation (1.1) since for $t\in[0,1]$ we have

$$u(t)=\mu\int_0^1 G_1(t,s)f(s,u(s),u'(s))\,ds.$$

Let Ω_1,Ω_2 and E be as in Theorem 3.1. Now let

$$K_3=\{u\in P_1^1(\mathbb{R}):\ \max_{t\in[0,1]}[d_0u(t)+|u'(t)|]\le -M_0\|u\|_1\}.$$

Then K_3 is a cone of E. Let $\varphi\in K_3\cap(\overline{\Omega_2}\setminus\Omega_1)$ and let x_φ be the unique 1-periodic solution of the equation

$$x''(t)-a^2(t)x(t)+\mu f_-^*(t,\varphi(t)+\overline{\phi}_1(t),\varphi'(t)+\overline{\phi}_1'(t))=0.$$

Finally let $A_4: K_3\cap(\overline{\Omega_2}\setminus\Omega_1)\to P_1^1(\mathbb{R})$ be defined by $A_4\varphi=x_\varphi$. Then

$$(A_4\varphi)(t)=\mu\int_0^1 G_1(t,s)f_-^*(s,\varphi(s)+\overline{\phi}_1(s),\varphi'(s)+\overline{\phi}_1'(s))\,ds$$

for $t\in[0,1]$. By Lemma 2.6 we have.

$$\begin{cases} d_0(A_4\varphi)(t)+|(A_4\varphi)'(t)|\\ \quad\le\mu\int_0^1\left[d_0G_1(t,s)-\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right]f_-^*(s,\varphi(s)+\overline{\phi}_1(s),\varphi'(s)+\overline{\phi}_1'(s))\,ds\\ \quad=\mu\int_0^t\left[d_0G_1(t,s)-\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right]f_-^*(s,\varphi(s)+\phi_1(s),\varphi'(s)+\overline{\phi_1}'(s))\,ds\\ \quad\ +\mu\int_t^1\left[d_0G_1(t,s)-\left|\frac{\partial G_1}{\partial t}(t,s)\right|\right]f_-^*(s,\varphi(s)+\phi_1(s),\varphi'(s)+\overline{\phi_1}'(s))\,ds\\ \quad\le\mu\int_0^t\left[G_1(s,s)+\left|\frac{\partial G_1}{\partial t}(s+0,s)\right|\right]f_-^*(s,\varphi(s)+\overline{\phi_1}(s),\varphi'(s)+\overline{\phi_1}'(s))\,ds\\ \quad\ +\mu\int_t^1\left[G_1(s,s)+\left|\frac{\partial G_1}{\partial t}(s-0,s)\right|\right]f_-^*(s,\varphi(s)+\overline{\phi}_1(s),\varphi'(s)+\overline{\phi}_1'(s))\,ds.\end{cases}$$

Hence, by (2.10)–(2.11) we get

$$\begin{aligned}&d_0(A_4\varphi)(t)+|(A_4\varphi)'(t)|\\ &\le-\mu M_0\int_0^1\left[G_1(\overline{t},s)+\left|\frac{\partial G_1}{\partial t}(\overline{t},s)\right|\right](-f_-^*(s,\varphi(s)+\overline{\phi}_1(s),\varphi'(s)+\overline{\phi}_1'(s)))\,ds,\end{aligned}$$

where $\overline{t}\in[0,1]$. So

$$d_0(A_4\varphi)(t)+|(A_4\varphi)'(t)|\le -M_0\|A_4\varphi\|_1.$$

Consequently $A_4 : K_3 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_3$. Using arguments similar to those in the proof of Theorem 3.1 we conclude that A_4 is continuous and compact. Let $\varphi \in K_3 \cap \partial\Omega_1$. Then $\|A_4\varphi\|_1 \leq \|\varphi\|_1$. If $\varphi \in K_3 \cap \partial\Omega_2$, then $\|\varphi\|_1 = R$ and $d_0\varphi(t) \leq -RM_0$.

Now let ε be as in (3.5). Then by (3.3) we have

$$\begin{cases} \varphi(t) \leq \varphi(t) + \overline{\phi}_1(t) \leq \varphi(t) + \mu L \int_0^1 G_1(t,s)\,ds \leq \varphi(t) + \mu L M_0 C_1 \\ \quad \leq -\frac{RM_0}{d_0} + \frac{\mu L M_0 C_1 R d_0}{d_0 R} = -\frac{RM_0}{d_0}\left(1 - \frac{\mu L C_1 d_0}{R}\right) \leq -\frac{\varepsilon R M_0}{d_0} < 0. \end{cases}$$

(for $t \in [0,1]$). This together with (3.5) and (4.3) yields

$$\begin{cases} -d_0\|A_4\varphi\|_1 \leq d_0(A_4\varphi)\left(\frac{1}{2}\right) + \left|(A_4\varphi)'\left(\frac{1}{2}\right)\right| \\ \leq \mu \int_0^1 \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] \left[f(s, \varphi(s) + \overline{\phi}_1(s), \varphi'(s) + \overline{\phi}_1'(s)) - L\right] ds \\ \leq -\mu \int_0^1 \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] \tau(s) g(|\varphi(s) + \overline{\phi}_1(s)|)\,ds \\ \leq -\mu \int_0^1 \left[d_0 G_1\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_1}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] \tau(s) g\left(\frac{\varepsilon R M_0}{d_0}\right) ds \leq -d_0 R. \end{cases}$$

So $\|A_4\varphi\|_1 \geq R = \|\varphi\|_1$. By Theorem 1.1 the operator A_4 has at least one fixed point in the set $K_3 \cap (\overline{\Omega_2} \setminus \Omega_1)$, which means that (1.1) has a negative solution x such that $x \in P_1^2(\mathbb{R})$. This completes the proof of Theorem 4.1. □

By the same way we can prove the following

Corollary 4.2 *Suppose that*

(4.6) $f : \tilde{\mathcal{D}} \to (-\infty, 0]$ *is continuous*

(4.7) $f(t+1, v_0, v_1) = f(t, v_0, v_1)$ *for all* $(t, v_0, v_1) \in \tilde{\mathcal{D}}$,

(4.8) *there exists a function* ψ *such that*

$$|f(t, v_0, v_1)| \leq \psi(v_0 + |v_1|) \quad \text{on } \tilde{\mathcal{D}},$$

where $\psi : [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing and* $\psi(u) > 0$ *for* $u > 0$,

(4.9) *there exist functions* τ *and* g *such that*

$$-f(t, v_0, v_1) \geq \tau(t) g(|v_0|) \quad \text{for } (t, v_0, v_1) \in \tilde{\mathcal{D}},$$

where τ *and* g *have property (3.4),*

(4.10) *there exist constants* r *and* R *having properties (3.17) and (3.19).*

Then (1.1) has a negative solution $x \in P_1^2(\mathbb{R})$.

Theorem 4.3 *Assume that conditions (4.1)–(4.3), (3.21)–(3.23) are satisfied. Then (1.2) has a negative solution* $x \in P_1^2(\mathbb{R})$.

Proof The proof of Theorem 4.3 is similar to that of Theorem 4.1. Let Ω_1, Ω_2, f^*_- and E be as in Theorem 4.1. Let

$$K_4 = \Big\{u \in P^1_1(\mathbb{R}) : \max_{t\in[0,1]}[\overline{d}_0 u(t) + |u'(t)|] \le -\overline{M}_0\|u\|_1\Big\}.$$

Obviously K_4 is a cone of E. We will show there exists a solution x_3 of the equation

$$x(t) = \mu \int_0^1 G_2(t,s) f^*_-(s, x(s) + \overline{\phi}_2(s), x'(s) + \overline{\phi}'_2(s))\, ds$$

with $x_3(t) + \overline{\phi}_2(t) < 0$ for $t \in [0,1]$. Let $\varphi \in K_4 \cap (\overline{\Omega_2} \setminus \Omega_1)$ and let x_φ be the unique 1-periodic solution of the equation

$$x''(t) + a^2(t)x(t) = \mu f^*_-(t, \varphi(t) + \overline{\phi}_2(t), \varphi'(t) + \overline{\phi}'_2(t)).$$

Finally, let $A_5 : K_4 \cap (\overline{\Omega_2} \setminus \Omega_1) \to P^1_1(\mathbb{R})$ be defined by $A_5\varphi = x_\varphi$. Then

$$(A_5\varphi)(t) = \mu \int_0^1 G_2(t,s) f^*_-(s, \varphi(s) + \overline{\phi}_2(s), \varphi'(s) + \overline{\phi}'_2(s))\, ds$$

for $t \in [0,1]$. By Lemma 2.7 we have

$$\overline{d}_0(A_5\varphi)(t) + |(A_5\varphi)'(t)|$$
$$\le -\mu\overline{M}_0 \int_0^1 \left[G_2(\overline{t},s) + \left|\frac{\partial G_2}{\partial t}(\overline{t},s)\right|\right] \left(-f^*_-(s, \varphi(s) + \overline{\phi}_2(s), \varphi'(s) + \overline{\phi}_2(s))\right) ds,$$

where $\overline{t} \in [0,1]$. So

$$\overline{d}_0(A_5\varphi)(t) + |(A_5\varphi)'(t)| \le -\overline{M}_0\|A_5\varphi\|_1.$$

Consequently $A_5 : K_4 \cap (\overline{\Omega_2} \setminus \Omega_1) \to K_4$. Also A_5 is continuous and compact. Let $\varphi \in K_4 \cap \partial\Omega_1$. Then $\|A_5\varphi\|_1 \le \|\varphi\|_1$. If $\varphi \in K_4 \cap \partial\Omega_2$, then $\overline{d}_0\varphi(t) \le -R\overline{M}_0$ and

$$\varphi(t) \le \varphi(t) + \overline{\phi}_2(t) \le \frac{-R\overline{M}_0}{\overline{d}_0}\left(\frac{1 - \mu L C_2 \overline{d}_0}{R}\right) \le \frac{-\varepsilon R\overline{M}_0}{\overline{d}_0} < 0,$$

where ε is as in (3.22). This together with (4.3) yields

$$-\overline{d}_0\|A_5\varphi\|_1 \le \overline{d}_0(A_5\varphi)\left(\frac{1}{2}\right) + \left|(A_5\varphi)'\left(\frac{1}{2}\right)\right|$$
$$\le -\mu \int_0^1 \left[\overline{d}_0 G_2\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_2}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] [f(s, \varphi(s) + \overline{\phi}_2(s), \varphi'(s) + \overline{\phi}_2(s)) - L]\, ds$$
$$\le -\mu \int_0^1 \left[\overline{d}_0 G_2\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_2}{\partial t}\left(\frac{1}{2}, s\right)\right|\right] \tau(s) g\left(\frac{\varepsilon R\overline{M}_0}{\overline{d}_0}\right) ds \le -\overline{d}_0 R.$$

Thus $\|A_5\varphi\|_1 \ge \|\varphi\|_1$. By Theorem 1.1 the operator A_5 has at least on fixed point in the set $K_4 \cap (\overline{\Omega_2} \setminus \Omega_1)$ which means that (1.2) has a negative solution x such that $x \in P^2_1(\mathbb{R})$. This completes the proof of Theorem 4.3. □

In the simlar way we can prove the following

Corollary 4.4 *Assume that conditions (4.6)–(4.9), (3.24)–(3.25) are satisfied. Then (1.2) has a negative solution* $x \in P_1^2(\mathbb{R})$.

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