Mathew Omonigho Omeike
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Ultimate Boundedness Results for a Certain Third Order Nonlinear Matrix Differential Equations

M. O. OMEIKE

*Department of Mathematical Sciences, University of Agriculture,
Abeokuta, Nigeria
e-mail: moomeike@yahoo.com*

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Abstract

This paper extends some known results on the boundedness of solutions and the existence of periodic solutions of certain vector equations to matrix equations.

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1 Introduction

Let $\mathcal{M}$ denote the space of all real $n \times n$ matrices, $\mathbb{R}^n$ the real n-dimensional Euclidean space and $\mathbb{R}$ the real line $-\infty < t < \infty$. We shall be concerned here with certain properties of solutions of differential equations of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.1}$$

where $X : \mathbb{R} \to \mathcal{M}$ is the unknown, $A, B \in \mathcal{M}$ are constants, $H : \mathcal{M} \to \mathcal{M}$ and $P : \mathbb{R} \times \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. The specific properties we shall be interested in are the ultimate boundedness of all solutions and the existence of periodic solutions when P is periodic in t.

In [8], Tejumola establishes conditions under which all solutions of the matrix differential equation,

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}), \tag{1.2}$$

are stable, bounded and periodic (depending on the choice of P). These results are extended to the equation (1.1).

For the special case in which (1.1) is an n-vector equation (so that $X : \mathbb{R} \to \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$) a number of boundedness, stability and existence of periodic solutions results have been established by Ezeilo and Tejumola [4], Afuwape [1] , Meng [5] and others for a number of various vector third order differential equations. The conditions obtained in each of these previous investigations are generalizations of the well-known Routh–Hurwitz conditions

$$a > 0, \quad c > 0, \quad ab - c > 0 \tag{1.3}$$

for the stability of the trivial solution of the linear differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0 \tag{1.4}$$

with constant coefficients. Our present investigations are akin to those of Tejumola [8], Meng [5], Afuwape [1] and we shall provide extensions of their results to matrix differential equations of the form (1.1).

2 Notations and definitions

Some standard matrix notation will be used. For any $X \in \mathcal{M}$, X^T and x_{ij}, $i, j = 1, 2, \ldots, n$ denote the transpose and the elements of X respectively while $(x_{ij})(y_{ij})$ will sometimes denote the product matrix XY of the matrices $X, Y \in \mathcal{M}$. $X_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ and $X^j = (x_{1j}, x_{2j}, \ldots, x_{nj})$ stand for the i-th row and j-th column of X respectively and $\underline{X} = (X_1, X_2, \ldots, X_n)$ is the n^2 column vector consisting of the n rows of X.

We shall denote by $JH(X)$ the $n^2 \times n^2$ generalised Jacobian matrix associated with the function $H : \mathcal{M} \to \mathcal{M}$ and evaluated at X : that is, $JH(X)$ is the matrix associated with the Jacobian determinant $\frac{\partial(H_1, H_2, \ldots, H_n)}{\partial(X_1, X_2, \ldots, X_n)}$. Corresponding to the constant matrix $A \in \mathcal{M}$ we define an $n^2 \times n^2$ matrix $\tilde{A}$ consisting of n^2 diagonal $n \times n$ matrix$(a_{ij}I_n)(I_n$ being the unit $n \times n$ matrix) and such that $(a_{ij}I_n)$ belongs to the i-th n row and j-th n column (that is, counting n at a time) of $\tilde{A}$. In the special case $n = 2$, $\tilde{A}$ is the 4×4 matrix

$$\begin{pmatrix} a_{11}I_2 & a_{12}I_2 \\ a_{21}I_2 & a_{22}I_2 \end{pmatrix}.$$

Next we introduce an inner product $\langle ., . \rangle$ and a norm $\| \cdot \|$ on $\mathcal{M}$ as follows. For arbitrary $X, Y \in \mathcal{M}$, $\langle X, Y \rangle = \text{trace } XY^T$. It is easy to check that $\langle X, Y \rangle = \langle Y, X \rangle$ and that $\|X - Y\|^2 = \langle X - Y, X - Y \rangle$ defines a norm of $\mathcal{M}$. Indeed,

$\|X\| = |\underline{X}|_{n^2}$ where $|\cdot|_{n^2}$ denotes the usual Euclidean norm in $\mathbb{R}^{n^2}$ and $\underline{X} \in \mathbb{R}^{n^2}$ is as defined above.

Lastly the symbol δ, with or without subscripts, denote finite positive constants whose magnitudes depend only on A, B, H and P. Any δ, with a subscript, retains a fixed identity throughout while the unnumbered ones are not necessarily the same each time they occur.

3 Statement of results

It will be assumed throughout the sequel that $H \in C'(\mathcal{M})$ and that $P \in C(\mathbb{R} \times \mathcal{M} \times \mathcal{M} \times \mathcal{M})$. Further, H and P satisfy conditions for the existence of solutions of (1.1) for any set of preassigned initial conditions.

Theorem 1 *Let $H(0) = 0$ and suppose that*

(i) the Jacobian matrix $JH(X)$ of $H(X)$ is symmetric and furthermore that the eigenvalues $\lambda_i(JH(X))$ of $JH(X)$, $(i = 1, 2, \ldots, n^2)$ satisfy for $X \in \mathcal{M}$,

$$0 < \delta_h \leq \lambda_i(JH(X)) \leq \Delta_h \tag{3.1}$$

where δ_h, Δ_h are finite constants;

(ii) the matrices $\tilde{A}, \tilde{B}, JH(X)$ are associative and commute pairwise. The eigenvalues $\lambda_i(\tilde{A})$ of $\tilde{A}$ and $\lambda_i(\tilde{B})$ of $\tilde{B}$ $(i = 1, 2, \ldots, n^2)$ satisfy

$$0 < \delta_a \leq \lambda_i(\tilde{A}) \leq \Delta_a \tag{3.2}$$

$$0 < \delta_b < \lambda_i(\tilde{B}) \leq \Delta_b \tag{3.3}$$

where $\delta_a, \delta_b, \Delta_a, \Delta_b$ are finite constants. Furthermore,

$$\Delta_h \leq k\delta_a\delta_b, \tag{3.4}$$

where

$$k = \min\left\{\frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2}; \frac{\alpha(1-\beta)\delta_a}{2(\delta_a+2\alpha)^2}\right\} \tag{3.5}$$

$\alpha > 0$, $0 < \beta < 1$ are some constants,

(iii) P satisfies

$$\|P(t, X, Y, Z)\| \leq \delta_0 + \delta_1(\|X\| + \|Y\| + \|Z\|) \tag{3.6}$$

for all arbitrary $X, Y, Z \in \mathcal{M}$, where $\delta_0 \geq 0$, $\delta_1 \geq 0$ are constants and δ_1 is sufficiently small.

Then every solution $X(t)$ of (1.1) satisfies

$$\|X(t)\| \leq \Delta_1, \quad \|\dot{X}(t)\| \leq \Delta_1, \quad \|\ddot{X}(t)\| \leq \Delta_1 \tag{3.7}$$

for all t sufficiently large, where Δ_1 is a constant the magnitude of which depends only on δ_0, δ_1, A, B, H and P.

This result provides an extension of a result of Afuwape [1], and Meng [5] for an n-vector.

Theorem 2 *Suppose, further to the conditions of Theorem 1, that P satisfies $P(t, X, Y, Z) = P(t + \omega, X, Y, Z)$ uniformly for all $X, Y, Z \in \mathcal{M}$. Then (1.1) admits of at least one periodic solution with period ω.*

4 Some preliminaries

The following results will be basic to the proofs of Theorems 1 and 2.

Lemma 1 [8] *Let $H(0) = 0$ and assume that the matrices $\tilde{A}$ and $JH(X)$ are symmetric and commute for all $X \in \mathcal{M}$. Then*

$$\langle H(X), AX\rangle = \int_0^1 \underline{X}^T \tilde{A} JH(\sigma X)\underline{X} d\sigma.$$

Lemma 2 [1] *Let D be a real symmetric $\ell \times \ell$ matrix, then for any $X \in \mathbb{R}^\ell$ we have*

$$\delta_d \|X\|^2 \leq \langle DX, X\rangle \leq \Delta_d \|X\|^2,$$

where δ_d, Δ_d are the least and greatest eigenvalues of D, respectively.

Lemma 3 [1] *Let Q, D be any two real $\ell \times \ell$ commuting symmetric matrices. Then*

(i) the eigenvalues $\lambda_i(QD)$ $(i = 1, 2, \ldots, \ell)$ of the product matrix QD are all real and satisfy

$$\max_{i\leq j,k\leq \ell} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1\leq j,k\leq \ell} \lambda_j(Q)\lambda_k(D);$$

(ii) the eigenvalues $\lambda_i(Q + D)$ $(i = 1, 2, \ldots, \ell)$ of the sum of matrices Q and D are real and satisfy

$$\left\{\max_{i\leq j\leq \ell} \lambda_j(Q) + \max_{1\leq k\leq \ell} \lambda_k(D)\right\} \geq \lambda_i(Q+D) \geq \left\{\min_{1\leq j\leq \ell} \lambda_j(Q) + \min_{1\leq k\leq \ell} \lambda_k(D)\right\}.$$

Proof of Theorem 1 Let us for convenience, replace Eq.(1.1) by the equivalent system form

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= -AZ - BY - H(X) + P(t, X, Y, Z). \end{aligned} \tag{4.1}$$

Our main tool in the proof is the scalar Lyapunov function

$$V : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathbb{R}$$

adapted from [5] and defined for any function $X, Y, Z \in \mathcal{M}$ by

$$\begin{aligned} 2V = \big\{ &\langle \beta(1-\beta)BX, BX\rangle + \langle 2\alpha A^{-1}BY, Y\rangle + \langle \beta BY, Y\rangle \\ &+ \langle \alpha A^{-1}Z, Z\rangle + \langle \alpha(Z+AY), Y + A^{-1}Z\rangle \\ &\langle Z + AY + (1-\beta)BX, Z + AY + (1-\beta)BX\rangle \big\} \end{aligned} \tag{4.2}$$

where $\alpha > 0$, $0 < \beta < 1$ are some constants. For each term of this function it is clear that

$$\beta(1-\beta)\delta_b\|X\|^2 \leq \langle \beta(1-\beta)BX, BX\rangle = \beta(1-\beta)\sum_{i=1}^{n}|BX^i|_n^2 \leq \beta(1-\beta)\Delta_b\|X\|^2, \tag{4.3a}$$

$$2\alpha\Delta_a^{-1}\delta_b\|Y\|^2 \leq \langle 2\alpha A^{-1}BY, Y\rangle = 2\alpha\sum_{i=1}^{n}|A^{-1}BY^i|_n^2 \leq 2\alpha\delta_a^{-1}\Delta_b\|Y\|^2. \tag{4.3b}$$

In a similar manner,

$$\beta\delta_b\|Y\| \leq \langle \beta BY, Y\rangle = \beta\sum_{i=1}^{n}|BY^i|_n^2 \leq \beta\Delta_b\|Y\|^2, \tag{4.3c}$$

$$\alpha\Delta_a^{-1}\|Z\|^2 \leq \langle \alpha A^{-1}Z, Z\rangle \leq \alpha\delta_a^{-1}\|Z\|^2, \tag{4.3d}$$

$$0 \leq \langle \alpha(Z+AY), Y + A^{-1}Z\rangle \leq \nu(\|Y\|^2 + \|Z\|^2), \tag{4.3e}$$

and

$$\begin{aligned} &0 \leq \langle Z + AY + (1-\beta)BX, Z + AY + (1-\beta)BX\rangle \\ &= \sum_{i=1}^{n}|Z^i + AY^i + (1-\beta)BX^i|_n^2 \leq \mu(\|Z\|^2 + \|Y\|^2 + \|X\|^2), \end{aligned} \tag{4.3f}$$

for some positive constants ν, μ. The estimates above are valid since

$$\sum_{i=1}^{n}|X^i|_n^2 = \sum_{i=1}^{n}|X_i|_n^2 = |\underline{X}|_{n^2}^2 \qquad \text{for any } X \in \mathcal{M}.$$

Combining these estimates (4.3a–4.3f) in (4.2) we obtain that

$$\delta_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq 2V \leq \delta_3(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \tag{4.4}$$

$$\delta_2 = \min\{\beta(1-\beta)\delta_b; 2\alpha\Delta_a^{-1}\delta_b + \beta\delta_b; \alpha\Delta_a^{-1}\}$$

and

$$\delta_3 = \max\{\beta(1-\beta)\Delta_b + \mu;\ 2\alpha\delta_a^{-1}\Delta_b + \beta\Delta_b + \nu + \mu;\ \alpha\delta_a^{-1} + \nu + \mu\}.$$

From (4.4), we have that $V(X, Y, Z) \to \infty$ as $\|X\|^2 + \|Y\|^2 + \|Z\|^2 \to \infty$.

To prove our result, it suffices to prove that there exists a constant $\Delta_1 \geq 1$ such that

$$\|X\|^2 + \|Y\|^2 + \|Z\|^2 \leq \Delta_1, \quad \text{for } t \geq T(X_0, Y_0, Z_0), \tag{4.5}$$

for any solution (X, Y, Z) for (4.1), $(X_0 = X(0),\ Y_0 = Y(0),\ Z_0 = Z(0))$.

Let (X, Y, Z) be any solution of (4.1), then the total derivative of V with respect to t along this solution path is

$$\dot{V} = -U_1 - U_2 - U_3 + U_4 \tag{4.6}$$

where

$$U_1 = \left\langle \frac{1-\beta}{2} BX, H(X) \right\rangle + \langle \beta ABY, Y \rangle + \left\langle \frac{\alpha}{2} Z, Z \right\rangle$$

$$U_2 = \left\langle \frac{1-\beta}{2} BY, H(X) \right\rangle + \langle \alpha BY, Y \rangle + \langle (A + \alpha I)Y, H(X) \rangle$$

$$U_3 = \left\langle \frac{1-\beta}{4} BX, H(X) \right\rangle + \left\langle \frac{\alpha}{2} Z, Z \right\rangle + \langle (I + 2\alpha A^{-1})Z, H(X) \rangle$$

$$U_4 = \left\langle (1-\beta)BX + (A + \alpha I)Y + (I + 2\alpha A^{-1})Z, P(t, X, Y, Z) \right\rangle.$$

To arrive at (4.5), we first prove the following:

Lemma 4 *Subject to a conveniently chosen value for k in (3.5), we have for all X, Y, Z*

$$U_j \geq 0, \quad (j = 2, 3).$$

Proof For strictly positive constants k_1, k_2 conveniently chosen later, we have

$$\begin{aligned} \langle (\alpha I + A)Y, H(X) \rangle &= \left\| k_1 (\alpha I + A)^{1/2} Y + 2^{-1} k_1^{-1} (\alpha I + A)^{1/2} H(X) \right\|^2 \\ &- \langle k_1^2 (\alpha I + A)Y, Y \rangle - 4^{-1} k_1^{-2} \langle (\alpha I + A) H(X), H(X) \rangle \end{aligned} \tag{4.7a}$$

and

$$\begin{aligned} &\langle (I + 2\alpha A^{-1})Z, H(X) \rangle = \\ &= \left\| k_2 \left(I + 2\alpha A^{-1}\right)^{1/2} Z + 2^{-1} k_2^{-1} \left(I + 2\alpha A^{-1}\right)^{1/2} H(X) \right\|^2 \\ &- \langle k_2^2 (I + 2\alpha A^{-1})Z, Z \rangle - \left\langle 4^{-1} k_2^{-2} (I + 2\alpha A^{-1}) H(X), H(X) \right\rangle, \end{aligned} \tag{4.7b}$$

thus,

$$\begin{aligned} U_2 &= \|k_1(\alpha I + A)^{1/2} Y + 2^{-1} k_1^{-1} (\alpha I + A)^{1/2} H(X)\|^2 \\ &\langle 4^{-1}(1-\beta)BX - 4^{-1} k_1^{-1} (\alpha I + A) H(X), H(X) \rangle + \langle (\alpha B - k_1^2 (\alpha I + A) Y, Y \rangle \end{aligned}$$

and

$$\begin{aligned}U_3 = &\|k_2(I+2\alpha A^{-1})^{1/2}Z + 2^{-1}k_2^{-1}(I+2\alpha A^{-1})^{1/2}H(X)\|^2\\&+ \langle (1-\beta)4^{-1}BX - 4^{-1}k_2^{-2}(I+2\alpha A^{-1})H(X), H(X)\rangle\\&+ \left\langle \left[\frac{\alpha}{2}I - k_2^2(I+2\alpha A^{-1})\right]Z, Z\right\rangle.\end{aligned}$$

By Lemmas 1,2 and 3, we obtain

$$\begin{aligned}U_2 \geq &\left\{\int_0^1 \sigma \int_0^1 \underline{X}^T\left[\frac{1-\beta}{4}\tilde{B} - \frac{1}{4k_1^2}\left(\alpha\tilde{I}+\tilde{A}\right)JH(\sigma X)\right]JH(\tau\sigma X)\underline{X}d\tau d\sigma\right.\\&\left.+\underline{Y}^T\left[\alpha\tilde{B} - k_1^2(\alpha\tilde{I}+\tilde{A})\right]\underline{Y}\right\},\end{aligned} \tag{4.8a}$$

and

$$\begin{aligned}U_3 \geq &\left\{\int_0^1 \sigma \int_0^1 \underline{X}^T\left[\frac{1-\beta}{4}\tilde{B} - \frac{1}{4k_2^2}\left(\alpha\tilde{I}+2\alpha\tilde{A}^{-1}\right)JH(\sigma X)\right]JH(\tau\sigma X)\underline{X}d\tau d\sigma\right.\\&\left.+\underline{Z}^T\left[\frac{\alpha}{2}\tilde{I} - k_2^2(\tilde{I}+2\alpha\tilde{A}^{-1})\right]\underline{Z}\right\}.\end{aligned} \tag{4.8b}$$

Furthermore, by using Lemmas 2 and 3, we obtain

$$U_2 \geq \left\{\delta_h\left[\frac{1-\beta}{4}\delta_b - \frac{1}{4k_1^2}(\alpha+\Delta_a)\Delta_h\right]\|X\|^2 + \left[\alpha\delta_b - k_1^2(\alpha+\Delta_a)\right]\|Y\|^2\right\}, \tag{4.8c}$$

and

$$U_3 \geq \left\{\delta_h\left[\frac{1-\beta}{4}\delta_b - \frac{1}{4k_2^2}(1+2\alpha\delta_a^{-1})\Delta_h\right]\|X\|^2 + \left[\frac{\alpha}{2} - k_2^2(1+2\alpha\delta_a^{-1})\right]\|Z\|^2\right\}, \tag{4.8d}$$

Thus, using (3.1), (3.2), (3.3) we obtain, for all $X, Y \in \mathcal{M}$,

$$U_2 \geq 0 \tag{4.9a}$$

if $k_1^2 \leq \frac{\alpha\delta_b}{\alpha+\Delta_a}$ with

$$\Delta_h \leq \frac{k_1^2(1-\beta)\delta_b}{(\alpha+\Delta_a)} \leq \frac{\alpha(1-\beta)\delta_b^2}{(\alpha+\Delta_a)^2} \tag{4.10a}$$

and for all X, Z in $\mathcal{M}$,

$$U_3 \geq 0 \tag{4.9b}$$

if $k_2^2 \leq \frac{\alpha\delta_a}{2(\delta+2\alpha)}$ with

$$\Delta_h \leq \frac{k_2^2(1-\beta)\delta_a\delta_b}{(2\alpha+\delta_a)} \leq \frac{\alpha(1-\beta)\delta_a^2\delta_b}{2(2\alpha+\delta_a)^2}. \tag{4.10b}$$

Combining all the inequalities in (4.9) and (4.10), we have inequalities (3.4) with (3.5) satisfied. Thus, for all $X, Y, Z \in \mathcal{M}$, $U_2 \geq 0$ and $U_3 \geq 0$. This completes the proof of Lemma 4. □

Finally, we are left with estimates for U_1 and U_4. From (4.6), we clearly have

$$U_1 = \frac{1-\beta}{2}\int_0^1 \underline{X}^T \tilde{B} J H(\sigma X)\underline{X}\, d\sigma + \beta \underline{Y}^T \tilde{A}\tilde{B}\underline{Y} + \frac{\alpha}{2}\underline{Z}^T\underline{Z}$$

$$\geq \frac{1-\beta}{2}\delta_b\delta_h\|X\|^2 + \beta\delta_a\delta_b\|Y\|^2 + \frac{\alpha}{2}\|Z\|^2 \geq \delta_4(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \quad (4.11)$$

where

$$\delta_4 = \min\left\{\frac{\delta_b}{2}\delta_h(1-\beta); \beta\delta_a\delta_b; \frac{\alpha}{2}\right\}.$$

Since $P(t, X, Y, Z)$ satisfies (3.6), by Schwarz's inequality, we obtain

$$\begin{aligned}|U_4| &\leq \{(1-\beta)\Delta_b\|X\| + (\alpha + \Delta_a)\|Y\| + (1 + 2\alpha\delta_a^{-1})\|Z\|\}\|P(t, X, Y, Z)\| \\ &\leq \delta_5(\|X\| + \|Y\| + \|Z\|)[\delta_0 + \delta_1(\|X\| + \|Y\| + \|Z\|)] \\ &\leq 3\delta_1\delta_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + 3^{1/2}\delta_0\delta_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2},\end{aligned} \quad (4.12)$$

where

$$\delta_5 = \max\{(1-\beta)\Delta_b; (\alpha + \Delta_a); (1 + 2\alpha\delta_a^{-1})\}.$$

Combining inequalities (4.9), (4.11) and (4.12) in (4.6), we obtain

$$\dot{V} \leq -2\delta_6(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \delta_7(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2}, \quad (4.13)$$

where

$$\delta_6 = \frac{1}{2}(\delta_4 - 3\delta_1\delta_5) \quad \text{and} \quad \delta_7 = 3^{1/2}\delta_0\delta_5.$$

Thus, with $\delta_1 < 3^{-1}\delta_5^{-1}\delta_4$, we have that $\delta_6 > 0$.

If we choose

$$(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{1/2} \geq \delta_8 = 2\delta_7\delta_6^{-1},$$

inequality (4.13) implies that

$$\dot{V} \leq -\delta_6(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \quad (4.14)$$

Then there exists δ_9 such that

$$\dot{V} \leq -1 \quad \text{if} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 \geq \delta_9^2.$$

The remainder of the proof of Theorem 1 may now be obtained by use of the estimates (4.4) and (4.14) and an obvious adaptation of the Yoshizawa type reasoning employed in [5].

Proof of Theorem 2 The proof of this theorem follows as in the proof of [5, Theorem 3].

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