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Fixed Point Analysis for Non-oscillatory Solutions of Quasi Linear Ordinary Differential Equations

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Abstract

The paper deals with the quasi-linear ordinary differential equation $(r(t)\varphi(u'))' + g(t,u) = 0$ with $t \in [0,\infty)$. We treat the case when g is not necessarily monotone in its second argument and assume usual conditions on $r(t)$ and $\varphi(u)$. We find necessary and sufficient conditions for the existence of unbounded non-oscillatory solutions. By means of a fixed point technique we investigate their growth, proving the coexistence of solutions with different asymptotic behaviors. The results generalize previous ones due to *Elbert–Kusano*, [Acta Math. Hung. 1990]. In some special cases we are able to show the exact asymptotic growth of these solutions. We apply previous analysis for studying the non-oscillatory problem associated to the equation when $\varphi(u) = u$. Several examples are included.

Key words: Quasi-linear second order equations; unbounded, oscillatory and non-oscillatory solutions; fixed-point techniques.

2000 Mathematics Subject Classification: 34C10

1 Introduction

The paper deals with the quasi-linear ordinary differential equation

$$(r(t)\varphi(u'))' + g(t,u) = 0 \text{ on } [0,+\infty) \tag{1}$$

under the following assumptions concerning r, φ and g

$$\begin{aligned} & r \in C[0,+\infty), r(t) > 0 \text{ for } t \in [0,+\infty); \\ & \varphi \in C(\mathbb{R}), \text{ strictly increasing, surjective, } v\varphi(v) > 0 \text{ for } v \neq 0; \\ & \textstyle\int_0^\infty \varphi^{-1}(\frac{k}{r(s)})\, ds = \infty \text{ for } k \neq 0; \\ & g(t,u) \in C([0,+\infty) \times \mathbb{R}) \text{ with } ug(t,u) > 0 \text{ for } u \neq 0 \text{ and } t \geq 0. \end{aligned} \tag{2}$$

As usual by solution we shall mean a continuously differentiable function u such that $r(t)\varphi(u')$ has a continuous derivative satisfying (1). We recall that a solution of (1) is said to be oscillatory if it has an infinite sequence of zeros clustering at ∞, non-oscillatory otherwise. The oscillatory and non-oscillatory behavior of equation (1) is of special interest. On this purpose, it is important to find necessary and/or sufficient conditions for the existence of solutions with a prescribed asymptotic behavior. The following lemma gives the classification of all possible non-oscillatory solutions of (1) according to their asymptotic behavior. The result is due to Elbert–Kusano (see [6, Lemma 1]) and since its proof does not depend on the monotonicity of $g(t,\cdot)$, which is assumed in [6], it is also valid in this more general context.

Lemma 1 *[6, Lemma 1] Any non-oscillatory solution $u(t)$ of (1) is of one of the following types:*

I) $\lim_{t\to\infty} |u(t)| = \infty$ *and* $\lim_{t\to\infty} r(t)\varphi(u'(t)) = \text{const} \neq 0.$

II) $\lim_{t\to\infty} |u(t)| = \infty$ *and* $\lim_{t\to\infty} r(t)\varphi(u'(t)) = 0.$

III) $\lim_{t\to\infty} u(t) = \text{const} \neq 0$ *and* $\lim_{t\to\infty} r(t)\varphi(u'(t)) = 0.$

In Sections 2 and 3 we obtain sufficient and sometimes also necessary conditions for the existence of an unbounded non-oscillatory solution respectively of type I and II (see Theorems 1, 2 and 3). In Section 3 in some special cases, we also discuss the coexistence of type I and II solutions and prove the exact asymptotic behavior of a type II solution (see Proposition 2). Our main investigation technique combine a linearization device with Schauder–Tychonoff fixed point theorem. We compare our results with previous ones, in particular with those in [6] and furnish several examples. Finally, Section 4 deals with the special case

$$(r(t)u')' + g(t,u) = 0, \quad t \in [0,\infty) \tag{3}$$

occurring when $\varphi(u) = u$. Applying previous analysis we discuss the non-oscillatory properties of (3).

Equation (1) arises in several applications. We quote, as an example, the important study of the polar form of the semi-linear elliptic partial differential equation $\mathrm{div}(|Du|^{\alpha-2}Du) + q(t)f(u) = 0$. When $f(u) = |u|^{\gamma-1}u$, this reduces to the investigation of $(|u'|^{\alpha-1}u')' + q(t)|u|^{\gamma-1}u = 0$, including the half-linear equation ($\alpha = \gamma$) and the generalized Emden–Fowler equation ($\alpha = 1$, $q(t) = (t+1)^{-m}$). Therefore, a wide literature is available, concerning the existence and the asymptotic behavior of the solutions of (1) as well as their oscillatory properties. See e.g. [1]–[4], [6]–[11], and references therein contained. However, most of the quoted papers deals with the case when $g(t,u) = q(t)f(u)$ and very often it is assumed $f(u) = |u|^{\gamma-1}u$ for some $\gamma > 0$. In addition, also when $g(t,u)$ has not separable variables, as in [6] and [10], $g(t,\cdot)$ is always increasing. The main purpose of this paper is to investigate these matters in the case when g is not necessarily monotone in its second argument. More precisely, we often assume the existence of a constant $L > 0$ such that

$$|g(t,v)| \le L|g(t,u)| \quad \text{for } u \in \mathbb{R},\ v \in [\min\{0,u\}, \max\{0,u\}] \text{ and } t \ge 0. \tag{4}$$

Remark 1 Condition (4) states that $g(t,\pm u)$ give, for each t and u, an upper and a lower bound for the oscillations of $g(t,\cdot)$ in the interval $[-u,u]$. A typical situation occurs when

$$l_1|h(t,u)| \le |g(t,u)| \le l_2|h(t,u)| \quad \text{for } (t,u) \in [0,\infty) \times \mathbb{R}$$

for some positive constants l_1 and l_2, and $h(t,u) \in C([0,\infty) \times \mathbb{R})$, with $h(t,\cdot)$ increasing for $t \in [0,\infty)$ and $uh(t,u) > 0$ for $u \ne 0$. Indeed

$$|g(t,v)| \le l_2|h(t,v)| \le l_2|h(t,u)| \le \tfrac{l_2}{l_1}|g(t,u)| \quad \text{for } v \in [\min\{0,u\}, \max\{0,u\}],$$

that is (4) holds with $L = \frac{l_2}{l_1}$. In particular, every g increasing in its second argument satisfies (4) with $L = 1$.

Concerning φ, mainly investigated in previous papers is the case when $\varphi(u) = |u|^{\gamma-1}u$ for some positive α. Under this condition, (4) can be replaced by the weaker assumption (17) simply involving the asymptotic behavior of g. This is possible, in particular, when studying equation (3) where $\alpha = 1$.

2 Unbounded solutions of type I

This section deals with the existence of non-oscillatory type I unbounded solutions of equation (1). A related result on this topic is due to Elbert and Kusano [6] and it treats the case when $g(t,\cdot)$ is increasing in $\mathbb{R}$, for all $t \in [0,\infty)$. Assuming for $k \ne 0$

$$\lim_{\substack{h\to 0\\ hk>0}} \frac{\int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds}{\int_0^t \varphi^{-1}\left(\frac{k}{r(s)}\right) ds} = 0 \tag{5}$$

uniformly in $[t_0,\infty)$ for any $t_0 > 0$, they proved that the existence of constants $k \ne 0$ and $c > 0$ satisfying

$$\int_0^\infty \left| g(t, c\int_0^t \varphi^{-1}\left(\frac{k}{r(s)}\right) ds \right| dt < \infty \tag{6}$$

is a necessary and sufficient condition for the appearance of type I solutions. Condition (7) in [6] is indeed slightly different from (5), but one can easily see that they are equivalent. Theorem 1 is a generalization of [6, Theorem 1] since it shows that (6) is a necessary and sufficient condition for the existence of unbounded type I solutions also when g satisfies (4). On this purpose the following lemma is needed, explaining the role of assumption (5) (see also the discussion after the proof of Theorem 1).

Lemma 2 *Assume (4) and (5). Then (6), for some constants $k \neq 0$ and $c > 0$, is equivalent to*

$$\int_0^\infty \left| g\left(t, \int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds\right) \right| dt < \infty \tag{7}$$

for some $h \neq 0$.

Proof Trivially (7) yields (6) with $c = 1$. On the other hand, if (6) holds for some $k \neq 0$ and $c > 0$, then according to (5), we get the existence of h, with $hk > 0$, $|h| \leq |k|$, and $t_0 > 0$ such that

$$\left| \int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds \right| \leq c \left| \int_0^t \varphi^{-1}\left(\frac{k}{r(s)}\right) ds \right|$$

for each $t \geq t_0$ and (4) implies (7). □

Theorem 1 *Assume conditions (4) and (5). Then equation (1) has a non-oscillatory solution of type I if and only if (6) holds for some $k \neq 0$ and $c > 0$.*

Proof *Necessary condition.* Let $u(t)$ be a type I solution of equation (1), with

$$\lim_{t \to \infty} r(t)\varphi(u'(t)) = C \neq 0.$$

Take, in particular $C > 0$, implying $u(t)$ eventually positive; with a similar reasoning the case of an eventually negative $u(t)$ can be treated. Hence it is possible to find $\delta > 0$ and $t_0 \geq 0$ such that, for $t \geq t_0$, $u(t) > 0$ and

$$u'(t) > \varphi^{-1}\left(\frac{C - \delta}{r(t)}\right) > 0.$$

Given a sufficiently small $c \in (0, 1]$ such that $u(t_0) \geq c \int_0^{t_0} \varphi^{-1}(\frac{C-\delta}{r(s)})\, ds$, we get, for all $t \geq t_0$,

$$0 \leq c \int_0^t \varphi^{-1}\left(\frac{C - \delta}{r(s)}\right) ds \leq u(t).$$

Then, according to (4), it holds

$$g\left(t, c \int_0^t \varphi^{-1}\left(\frac{C - \delta}{r(s)}\right) ds\right) \leq Lg(t, u(t)),$$

and being
$$\int_0^\infty g(t,u(t))dt = r(0)\varphi(u'(0)) - C,$$
condition (6) holds.

Sufficient condition. Let (6) holds for some constants $k \neq 0$ and $c > 0$. Then, according to Lemma 2, (7) is valid for some $h \neq 0$ with $hk > 0$. With no loss of generality we can assume $k > 0$, so also $h > 0$ and the absolute value in (7) can be removed. Given $l \in (0,h)$, in view of the monotonicity of φ, applying (4) we obtain
$$\int_0^\infty \max_{\int_0^t \varphi^{-1}(\frac{l}{r(s)})ds \le u \le \int_0^t \varphi^{-1}(\frac{h}{r(s)})\,ds} g(t,u)\,dt$$
$$\le L\int_0^\infty g\Big(t, \int_0^t \varphi^{-1}\Big(\frac{h}{r(s)}\Big)\,ds\Big)\,dt < \infty$$
so we can take $t_0 > 0$ such that
$$\int_{t_0}^\infty \max_{\int_0^t \varphi^{-1}(\frac{l}{r(s)})ds \le u \le \int_0^t \varphi^{-1}(\frac{h}{r(s)})\,ds} g(t,u)\,dt \le h - l. \tag{8}$$

Let $C[t_0,\infty)$ be the Fréchet space of all continuous functions $x : [t_0,\infty) \to \mathbb{R}$ with the topology of the uniform convergence on compact subintervals of $[t_0,\infty)$. Let Ω be the closed, convex and bounded subset of $C[t_0,\infty)$ defined as
$$\Omega = \Big\{w \in C[t_0,\infty) : \int_0^t \varphi^{-1}\Big(\frac{l}{r(s)}\Big)\,ds \le w(t) \le \lambda + \int_{t_0}^t \varphi^{-1}\Big(\frac{h}{r(s)}\Big)\,ds,\ \forall t \ge t_0\Big\},$$
where $\lambda = \int_0^{t_0} \varphi^{-1}(\frac{l}{r(s)})\,ds$. For every $w \in \Omega$, consider the Cauchy problem
$$\begin{aligned} &(r(t)\varphi(u'))' + g(t,w) = 0,\\ &u(t_0) = \lambda, \quad u'(t_0) = \varphi^{-1}\Big(\tfrac{h}{r(t_0)}\Big). \end{aligned} \tag{9}$$

Since (9) is uniquely solvable, we can define the operator
$$\begin{aligned} T : \Omega &\to C[t_0,\infty)\\ w &\to T(w)(t) = \lambda + \int_{t_0}^t \varphi^{-1}\Big(\frac{h - \int_{t_0}^s g(\eta, w(\eta))d\eta}{r(s)}\Big)\,ds \end{aligned}$$
which associates to any $w \in \Omega$ the unique solution $T(w)$ of problem (9). Now we use the Schauder–Tychonoff fixed point theorem to prove that T has a fixed point. First we show that $T(\Omega) \subseteq \Omega$. In fact, according to the monotonicity of φ and the sign condition (2) on g, one has
$$T(w)(t) \le \lambda + \int_{t_0}^t \varphi^{-1}\Big(\frac{h}{r(s)}\Big)ds \quad \text{for all } t \ge t_0.$$

On the other hand (8) implies

$$T(w)(t) \geq \int_0^t \varphi^{-1}\left(\frac{l}{r(s)}\right) ds \quad \text{for any } t \geq t_0. \tag{10}$$

Now we prove the continuity of T. Let $\{w_n\}$ be a sequence of functions of Ω converging to w, in the topology of $C[t_0, \infty)$, as $n \to \infty$. The continuity of g and φ, the Lebesgue dominated convergence theorem and (8) imply that $T(w_n) \to T(w)$ in $C[t_0, \infty)$ as $n \to \infty$. It remains to prove the relative compactness of T. First notice that $T(\Omega) \subseteq \Omega$, which is bounded in $C[t_0, \infty)$. Moreover

$$(T(w))'(t) = \varphi^{-1}\left(\frac{h - \int_{t_0}^t g(\eta, w(\eta))d\eta}{r(t)}\right);$$

thus, in view of the positivity of g and (8), we get

$$\varphi^{-1}\left(\frac{l}{r(t)}\right) \leq (T(w))'(t) \leq \varphi^{-1}\left(\frac{h}{r(t)}\right) \tag{11}$$

for every $t \geq t_0$ and every $w \in \Omega$. Therefore, the functions in Ω are equicontinuous at each $t \geq t_0$ and Ascoli–Arzelá theorem implies the relative compactness of T. Hence Schauder–Tychonoff theorem can be applied; it guarantees the existence of a function $u \in \Omega$ which remains fixed in T, e.g. of a solution of (1) which is unbounded, in view of (10) and (2). Moreover, from (11) and the monotonicity of φ, u satisfies

$$\lim_{t \to +\infty} r(t)\varphi(u'(t)) = C \in [l, h]. \qquad \square$$

Looking at the proof of Theorem 1, it is clear that (6) is a very natural necessary condition for the existence of type I non-oscillatory solutions of (1). It also follows that (7) is a quite obvious sufficient condition, when employing a fixed point technique for the investigation of type I solutions. As showed in Lemma 2, whenever g satisfies (4) then assumptions (6) and (7) are equivalent, under condition (5). This is the only reason why we introduced (5).

Remark 2 Several results in this framework (see e.g. [5] and [9]) deal with the case when $\varphi(v) = v|v|^{\alpha-1}$ for some $\alpha > 0$. Notice that, for such φ, condition (5) is trivially fulfilled; indeed $\varphi^{-1}(v) = v|v|^{\frac{1}{\alpha}-1}$, hence

$$\lim_{\substack{h\to 0\\ hk>0}} \frac{\int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds}{\int_0^t \varphi^{-1}\left(\frac{k}{r(s)}\right) ds} = \lim_{\substack{h\to 0\\ hk>0}} \left(\frac{h}{k}\right)^{\frac{1}{\alpha}} = 0$$

and it is uniform on $[0, \infty)$, for all $k \neq 0$. Moreover it is easy to see that (6) yields (7) with $h = c^\alpha k$. Therefore, (6) and (7) are always equivalent without any additional requirement on g.

Other results (see e.g. [7, 8, 10, 12]) concern the case when $r(t) \equiv 1$. Also under this condition (5) is satisfied, because

$$\lim_{\substack{h\to 0\\ hk>0}} \frac{\int_0^t \varphi^{-1}\big(\frac{h}{r(s)}\big)\, ds}{\int_0^t \varphi^{-1}\big(\frac{k}{r(s)}\big)\, ds} = \lim_{\substack{h\to 0\\ hk>0}} \frac{\varphi^{-1}(h)}{\varphi^{-1}(k)} = 0$$

uniformly on $[0,\infty)$, for all $k \neq 0$.

In the following example we propose a pair of functions $(\varphi(u), r(t))$ which does not satisfy condition (5).

Example 1 Let $\varphi(u) = (\mathrm{e}^{|u|} - 1)\mathrm{sgn} u$ and $r \in C^1[0,\infty)$ such that $r(t) > 0$ for all t and $r(t) \to 0$ as $t \to \infty$. Being $\varphi^{-1}(v) = \log(1 + |v|)\mathrm{sgn} v$, all the assumptions in (2) concerning $\varphi(u)$ and $r(t)$ hold. Moreover, it is easy to see that

$$\lim_{t\to\infty} \frac{\int_0^t \log\big(1 + \frac{|h|}{r(s)}\big)\, ds}{\int_0^t \log\big(1 + \frac{|k|}{r(s)}\big)\, ds} = 1$$

for every choice of h and k with $hk > 0$ and this prevent to condition (5) to be satisfied.

Example 2 Consider the following equation

$$\Big(\frac{u'|u'|^{\alpha-1}}{(1+t)^{\beta}}\Big)' + q(t)u|u|^{\gamma-1}(a + b\sin^2|u|) = 0, \tag{12}$$

with $\alpha, \gamma, a > 0, \beta \in \mathbb{R}$ and $b \geq 0$. Since, for any $k \neq 0$,

$$\varphi^{-1}\Big(\frac{k}{r(t)}\Big) = k|k|^{\frac{1}{\alpha}-1}(1+t)^{\beta/\alpha}$$

we assume $\beta \geq -\alpha$ for guaranteeing condition (2). In this case, for $(t,u) \in [0,\infty) \times \mathbb{R}$, it holds $aq(t)|u|^{\gamma} \leq |g(t,u)| \leq (a+b)q(t)|u|^{\gamma}$. Thus, in view of Remark 1, (4) is satisfied, taking $L = 1 + \frac{b}{a}$. Moreover (6) is equivalent to the convergence of $\int_0^{\infty} q(t)\big[(1+t)^{\frac{\beta}{\alpha}+1} - 1\big]^{\gamma} dt$. Therefore, according to Theorem 1, the existence of a non-oscillatory unbounded solution of equation (12) is equivalent to the following condition

$$\int_0^{\infty} q(t)t^{(\frac{\beta}{\alpha}+1)\gamma}\, dt < \infty. \tag{13}$$

A special case occurs when $\alpha, a = 1$, $\beta, b = 0$ and $q(t) = (1+t)^{-m}$ for some real m. Indeed (12) reduces to the well known generalized Emden–Fowler equation

$$u'' + \frac{1}{(1+t)^m} u|u|^{\gamma-1} = 0. \tag{14}$$

We shall treat again equations (12) and (14) in the end of next sections.

Looking at the proof of Theorem 1, it is easy to deduce that the following stronger sufficient condition (15) is valid, for the existence of a non-oscillatory type I solution. Condition (15) does not require any assumption on φ or g, but it is equivalent to (6) when assuming (4) and (5). The following result holds; we omit its proof, since it is very similar to the sufficient part of Theorem 1.

Proposition 1 *Assume there exists $h < k$ with $hk > 0$ such that*

$$\int_0^\infty \left| \max_{\int_0^t \varphi^{-1}(\frac{h}{r(s)})ds \le u \le \int_0^t \varphi^{-1}(\frac{k}{r(s)})\, ds} g(t,u) \right| dt < \infty. \tag{15}$$

Then equation (1) has a non-oscillatory solution of type I.

Example 3 Consider the equation

$$(r(t)u'(t))' + \frac{\mathrm{e}^{u^2+u^4\sin^2 u}}{(1+t)^2}\operatorname{sign} u = 0, \tag{16}$$

where $r(t)$ satisfies conditions (2) and it is such that $\int_0^t \frac{1}{r(s)}\, ds$ goes to ∞, when $t \to \infty$, as $(\operatorname{loglog} t)^\mu$ for some $0 < \mu < 1/4$. Given $t \ge 0$ and an arbitrary value $l \in (0,1)$, it is easy to see that

$$\limsup_{u\to\infty} \frac{g(t,lu)}{g(t,u)} = \limsup_{n\to\infty} \mathrm{e}^{n^2\pi^2(l^2-1+n^2\pi^2l^4\sin^2 n\pi l)} = \infty.$$

Therefore condition (4) is not valid and Theorem 1 can not be applied. Take $\beta > 0$ and $T > 0$ satisfying

$$\int_0^t \frac{1}{r(s)}\, ds \le \beta(\log\log\, t)^\mu \quad \text{for all } t \ge T.$$

Given $p \ne 0$, $t \ge T$ and $0 \le u \le |p| \int_0^t \frac{1}{r(s)}\, ds$ it holds

$$0 \le |g(t,u)| \le \frac{\mathrm{e}^{1+2u^4}}{(1+t)^2} \le \frac{\mathrm{e}\,(\log t)^{2\beta^4|p|^4}}{(1+t)^2};$$

this implies (15). According to Proposition 1, also equation (16) has a non-oscillatory solution of type I.

We now consider the special case when φ is a power and prove that not only (5) can be omitted, as showed in Remark 2, but that also (4) can be weakened to an assumption on the asymptotic behavior of g.

Theorem 2 *let $\varphi(v) = v|v|^{\alpha-1}$ for some $\alpha > 0$. Assume the existence of $L \ge 0$ and $m \in (0,1)$ such that*

$$\limsup_{t,|u|\to\infty} \frac{g(t,v)}{g(t,u)} = L \tag{17}$$

for all $v \in [min\{mu,u\}, max\{mu,u\}]$. Then equation (1) has a non-oscillatory solution of type I if and only if (6) holds for some $k \ne 0$ and $c > 0$.

Proof *Necessary condition.* We do not lose in generality when assuming the existence of an eventually positive type I solution of equation (1), i.e. with $\lim_{t\to\infty} r(t)(u'(t))^\alpha = C > 0$. Applying L'Hospital rule we get

$$\lim_{t\to\infty} \frac{u(t)}{\int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds} = C^{\frac{1}{\alpha}}.$$

Take $\delta > 0$ such that $\frac{C^{\frac{1}{\alpha}}-\delta}{C^{\frac{1}{\alpha}}+\delta} = m$ and $t_0 \geq 0$ satisfying

$$(C^{\frac{1}{\alpha}} - \delta) \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds \leq u(t) \leq (C^{\frac{1}{\alpha}} + \delta) \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds,$$

for all $t \geq t_0$, that is

$$mu(t) \leq (C^{\frac{1}{\alpha}} - \delta) \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds \leq u(t).$$

Then, according to (17), there exists $t_1 \geq t_0$ such that, for $t \geq t_1$, it holds

$$g\left(t, (C^{\frac{1}{\alpha}} - \delta) \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds\right) \leq 2Lg(t, u(t)),$$

and the conclusion follows as in the proof of Theorem 1.

Sufficient condition. According to Remark 2, (6) implies (7) with $h = c^\alpha k$. For the sake of simplicity let us assume $k > 0$. A similar reasoning holds when $k < 0$. According to (17) and the divergence of $\int_0^\infty (\frac{1}{r(t)})^{\frac{1}{\alpha}} dt$, it is then possible to find $t_0 \geq 0$ such that, for all $t \geq t_0$ and

$$v \in \left[mk^{\frac{1}{\alpha}} \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds,\ k^{\frac{1}{\alpha}} \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds\right],$$

it holds

$$0 \leq g(t, v) \leq 2Lg\left(t, k^{\frac{1}{\alpha}} \int_0^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds\right).$$

Therefore

$$\int_0^\infty \max_{mk^{\frac{1}{\alpha}} \int_0^t (\frac{1}{r(s)})^{\frac{1}{\alpha}} ds \leq u \leq k^{\frac{1}{\alpha}} \int_0^t (\frac{1}{r(s)})^{\frac{1}{\alpha}} ds} g(t, u)\, dt < \infty,$$

and the conclusion follows from Proposition 1. □

3 Unbounded solutions of type II

We investigate now the existence of type II unbounded solutions $u(t)$ of equation (1), i.e. such that $\lim_{t\to\infty}|u(t)| = \infty$ and $\lim_{t\to\infty} r(t)\varphi(u'(t)) = 0$. Theorem 3 gives a sufficient condition. In the special case (12) we then discuss, in Proposition 2, the existence of a type II solution with prescribed behavior at infinity.

Theorem 3 *Assume condition (4) and let (7) hold for some $h \neq 0$. If*

$$\int_0^\infty \left|\varphi^{-1}\left(\frac{1}{Lr(t)}\int_t^\infty g(s,d)\,ds\right)\right| dt = \infty, \tag{18}$$

for all d satisfying $dh > 0$, then equation (1) has a non-oscillatory solution of type II.

Proof Notice that, with no loss of generality we can assume $h > 0$, implying that also the value d appearing in (18) must be positive. According to (7), it is possible to find $t_0 \geq 0$ such that

$$\int_{t_0}^\infty g\Big(t, \int_0^t \varphi^{-1}\Big(\frac{h}{r(s)}\Big)\,ds\Big)\,dt \leq \frac{h}{L}.$$

Let us denote $d = \int_0^{t_0} \varphi^{-1}\left(\frac{h}{r(s)}\right) ds$. As a consequence of (4) and (7) it follows

$$\int_{t_0}^\infty \max_{d\leq u\leq \int_0^t \varphi^{-1}(\frac{h}{r(s)})\,ds} g(t,u)\,dt \leq L\int_{t_0}^\infty g\left(t, \int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds\right) dt \leq h. \tag{19}$$

Given the usual Fréchet space of continuous functions $C[t_0,\infty)$, let Ω be its closed, convex and bounded subset defined as follows

$$\Omega = \Big\{w \in C[t_0,+\infty) : d \leq w(t) \leq \int_0^t \varphi^{-1}\left(\frac{h}{r(s)}\right) ds,\ \forall t \geq t_0\Big\}.$$

Since for every $w \in \Omega$, $\int_{t_0}^\infty g(s,w(s))\,ds < \infty$, it is possible to define the operator

$$\begin{aligned} T : \Omega &\to C[t_0,\infty) \\ w &\to T(w)(t) = d + \int_{t_0}^t \varphi^{-1}\left(\frac{\int_s^\infty g(\eta,w(\eta))d\eta}{r(s)}\right) ds \end{aligned}$$

associating to w the unique solution of the Cauchy problem

$$\begin{aligned} &(r(t)\varphi(u'))' + g(t,w) = 0, \\ &u(t_0) = d, \quad u'(t_0) = \varphi^{-1}\left(\frac{\int_{t_0}^\infty g(s,w(s))\,ds}{r(t_0)}\right). \end{aligned} \tag{20}$$

The monotonicity of φ, the sign condition on g and (19) easily yield that $T(\Omega) \subseteq \Omega$. Applying the Schauder–Tychonoff theorem as in the proof of Theorem 1, one can see that T has a fixed element $u(t)$, which is a solution of (1). Moreover, since $u(t) \geq d$ for all $t \geq t_0$, according to (4) and the definition of $T(u)$ it follows

$$u(t) \geq d + \int_{t_0}^{t} \varphi^{-1}\Big(\frac{1}{Lr(s)} \int_s^\infty g(\eta, d) d\eta\Big) ds;$$

hence condition (18) implies $u(t) \to \infty$ as $t \to \infty$. Finally, since $u(t)$ solves the Cauchy problem (20), it holds

$$r(t)\varphi\big(u'(t)\big) = \int_t^\infty g\big(s, u(s)\big) ds$$

and by (7) we obtain $r(t)\varphi\big(u'(t)\big) \to 0$ as $t \to \infty$. Consequently $u(t)$ is a type II non-oscillatory solution of equation (1) and the proof is complete. $\square$

Remark 3 In [6, Theorem 3], the case when $g(t, \cdot)$ is increasing for each $t \geq 0$ was studied. Assuming conditions (5), (6) and the natural reformulation of (18) in this context, i.e. with $L = 1$, the authors proved the existence of a type II unbounded solution of equation (1). We recall that condition (5) was introduced only to assure the equivalence between the necessary condition (6) and the sufficient condition (7) (see Lemma 2). However, since we are interested only in the sufficient condition, we don't need any assumption on φ and we directly assumed (7) instead of (6). Therefore, Theorem 3 is a generalization of the quoted result in [6], since it deals with a more general function g and does not require (5). In particular, Theorem 3 holds when $\varphi(u)$ and $r(t)$ behave as in Example 1.

The following part of this section is mainly devoted to equation (12), e.g.

$$\Big(\frac{u'|u'|^{\alpha-1}}{(1+t)^\beta}\Big)' + q(t)u|u|^{\gamma-1}(a + b\sin^2|u|) = 0,$$

with $\alpha, \gamma, a > 0$, $\beta \geq -\alpha$ and $b \geq 0$. In this case, condition (18) reduces to

$$\int_0^\infty \Big(\int_t^\infty q(s) ds\Big)^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}} dt = +\infty. \tag{21}$$

When $q(t) = \frac{1}{(1+t)^m}$, where m is an arbitrary constant, (13) holds if and only if $m > 1 + (1 + \frac{\beta}{\alpha})\gamma$ and (21) is satisfied if and only if $m \leq \alpha + \beta + 1$. Notice that this implies that assumptions (7) and (18) are not always consistent, as follows when $\gamma \geq \alpha$. On the contrary, when $0 < \gamma < \alpha$ and $1 + (1 + \frac{\beta}{\alpha})\gamma < m \leq 1 + \alpha + \beta$ both a type I and a type II unbounded solution exist. When $\alpha, a = 1$, $\beta, b = 0$, (12) reduces to the well known generalized Emden–Fowler equation (14). We recall that its possible solutions of type I are asymptotically linear functions, while the possible solutions of type II are asymptotically sub-linear functions. As a consequence of the analysis above conditions $0 < \gamma < 1$, $1 + \gamma < m \leq 2$ are sufficient for the contemporary presence, in equation (14), of a linear and a sub-linear unbounded solution. We stress that, while condition (6) is necessary for the existence of an unbounded type I solution of (1), neither (7) nor (18) are necessary for the existence of an unbounded type II solution of the same equation. In fact, consider the generalized Emden–Fowler equation with $m = 5/2$ and $\gamma = 2$. Then $\int_0^\infty q(t)t^2 dt = \infty$ and $\int_0^\infty q(t)t\,dt < \infty$ implying that both (7) and (18) are not satisfied; however this equation has the sub-linear solution $u(t) = \frac{\sqrt{t+1}}{4}$.

The following proposition shows that it is possible to determine the exact asymptotic behavior of a type II non-oscillatory solution. In order to simplify notation, we restrict our discussion to equation (12), though a similar investigation could be repeated for the general equation (1).

Proposition 2 *Consider equation (12) with $\alpha, a > 0$, $0 < \gamma < \alpha$, $\beta > -\alpha$, and $b \geq 0$. Given $\sigma \in (0, 1 + \frac{\beta}{\alpha})$, assume that*

$$\int_0^\infty q(t)t^{\sigma\gamma} dt < \infty \tag{22}$$

and

$$t^{1+\frac{\beta}{\alpha}-\sigma}\Big(\int_t^\infty q(s)s^{\sigma\gamma} ds\Big)^{\frac{1}{\alpha}} \to \Delta > 0 \text{ as } t \to \infty. \tag{23}$$

Then equation (12) admits a non-oscillatory solution of type II going at infinity like t^σ when $t \to \infty$.

Proof Let us introduce a continuous function $\vartheta_0 : [0, \infty) \to \mathbb{R}$ satisfying $\vartheta_0(t) = t$ for $t \in [0, 1]$, $\vartheta_0(t) = t^\sigma$ when $t \geq 2$ and $\vartheta_0(t) > 0$ for all $t \neq 0$. According to (22), it holds

$$\int_0^\infty q(t)\vartheta_0^\gamma(t)\,dt < \infty;$$

hence it is possible to define, for $t \geq 0$, the function

$$\psi(t) = \int_0^t (1+s)^{\frac{\beta}{\alpha}}\Big(\int_s^\infty q(\eta)\vartheta_0^\gamma(\eta)d\eta\Big)^{\frac{1}{\alpha}} ds.$$

As a consequence of (23), it follows, as $t \to \infty$

$$t^{1-\sigma}(1+t)^{\frac{\beta}{\alpha}}\Big(\int_s^\infty q(s)\vartheta_0^\gamma(s)ds\Big)^{\frac{1}{\alpha}} \to \Delta,$$

implying $\psi(t) \to \infty$, because $\sigma > 0$, and

$$\lim_{t\to\infty}\frac{\psi(t)}{\vartheta_0(t)} = \lim_{t\to\infty}\frac{t^{1-\sigma}(1+t)^{\frac{\beta}{\alpha}}\Big(\int_t^\infty q(s)s^{\sigma\gamma}ds\Big)^{\frac{1}{\alpha}}}{\sigma} = \frac{\Delta}{\sigma}.$$

Moreover it holds

$$\lim_{t\to 0^+}\frac{\psi(t)}{\vartheta_0(t)} = \Big(\int_0^\infty q(s)\vartheta_0^\gamma(s)ds\Big)^{\frac{1}{\alpha}} > 0.$$

We can then determine two positive constants $0 < m_1 < m_2$ such that $m_1\vartheta_0(t) \le \psi(t) \le m_2\vartheta_0(t)$ for all $t \ge 0$. Let

$$d = m_2^{\frac{\alpha}{\alpha-\gamma}}(a+b)^{\frac{1}{\alpha-\gamma}}, \qquad \delta = \frac{a^{\frac{1}{\alpha-\gamma}} m_1^{\frac{\alpha}{\alpha-\gamma}}}{d}$$

and put $\vartheta(t) = d\vartheta_0(t)$. Since $d > 0$ and $0 < \delta < 1$, we can then introduce the closed, convex and bounded set of functions $\Omega = \{w \in C[0,\infty) : \delta\vartheta(t) \le w(t) \le \vartheta(t), t \ge 0\}$. According to (22) the operator

$$\begin{aligned} T : \Omega &\to C[t_0,\infty) \\ w &\to T(w)(t) = \int_0^t (1+s)^{\frac{\beta}{\alpha}} \Big(\int_s^\infty q(\eta) w^\gamma(\eta)(a + b\sin^2 w(\eta)) d\eta \Big)^{\frac{1}{\alpha}} ds \end{aligned}$$

is well defined. Now we show that $T(\Omega) \subseteq \Omega$. In fact, given $w \in \Omega$, we have

$$T(w)(t) \le (a+b)^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}} \psi(t) \le (a+b)^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}-1} m_2 \vartheta(t).$$

Due to the definition of d it holds $(a+b)^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}-1} m_2 = 1$, implying $T(w)(t) \le \vartheta(t)$ for all $t \ge 0$. Moreover, since $a^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}-1} m_1 \delta^{\frac{\gamma}{\alpha}-1} = 1$, we get

$$T(w)(t) \ge \delta^{\frac{\gamma}{\alpha}} a^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}} \psi(t) \ge a^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}-1} m_1 \delta^{\frac{\gamma}{\alpha}-1} \delta\vartheta(t) = \delta\vartheta(t).$$

Hence $T(\Omega) \subseteq \Omega$.

As in the proof of Theorem 1, one can apply Schauder–Tychonoff theorem to T in order to show that it has a fixed element $u(t)$; then it is easy to see that $u(t)$ is a solution of equation (12). Finally, according to the definition of the set Ω, $u(t)$ is a type II unbounded solution of (12) satisfying $\frac{u(t)}{t^\sigma} \to l \in [d\delta, d]$ as $t \to \infty$. □

Notice that, since $\sigma \in (0, 1 + \frac{\beta}{\alpha})$, (13) implies (22). Consider again $q(t) = (1+t)^{-m}$. As already showed, equation (12) with $0 < \gamma < \alpha$ and $1 + (1 + \frac{\beta}{\alpha})\gamma < m \le 1 + \alpha + \beta$ has both a type I and a type II solution. Moreover, take $\sigma = \frac{\alpha+\beta+1-m}{\alpha-\gamma}$. Then $\sigma \in (0, 1 + \frac{\beta}{\alpha})$ and this implies $m - \sigma\gamma > 1$. Therefore, according to Proposition 2, (12) has a type II solution with asymptotic growth t^σ at infinity. In particular, the generalized Emden–Fowler equation (14), with $0 < \gamma < 1$ and $1 + \gamma < m < 2$, contemporarily admits a linear and a sub-linear unbounded solution and the latter one is asymptotic to $t^{\frac{2-m}{1-\gamma}}$.

4 Non-oscillatory theorems

In this section we restrict our attention to equation (3), obtained by (1) when assuming $\varphi(u) = u$. Concerning (3), we state a non-existence result of bounded oscillatory solutions and a non-oscillatory result. Both these problems were extensively investigated and also recent contributions appeared. We refer, in particular, to [3], [5], [10] and [12]. Nevertheless they all treat the cases when $g(t,\cdot)$ is monotone or $g(t,u) = q(t)f(u)$ often assuming $f(u) = |u|^{\gamma-1}u$ for some

$\gamma > 1$. Instead, in Theorems 4 and 5, $g(t,u)$ simply satisfies condition (17), hence no monotonicity is required on it. First notice that now conditions (6) and (7) are equivalent (see Remark 2) and they become

$$\int_0^\infty \left| g\Big(t, k\int_0^t \frac{1}{r(s)}\,ds\Big)\right| dt < \infty \tag{24}$$

with $k \neq 0$.

Theorem 4 *Assume condition (24) for some $k > 0$ and let (17) hold. Suppose further that for each $v > 0$ there exist $V \geq v$ and $T \geq 0$ satisfying*

$$\sup_{u\in[0,v]} \frac{g(t,u)}{u} \leq \inf_{u\geq V} \frac{g(t,u)}{u} \tag{25}$$

for each $t \in [T,\infty)$. Then equation (3) has no bounded oscillatory solutions.

Proof Let $y(t)$ be an oscillatory solution of (3) and suppose that there exists $t_0 \geq 0$ such that $y(t) \leq 0$ for all $t \geq t_0$. Take $\bar{t} \geq t_0$ satisfying $y(\bar{t}) = 0$; then also $y'(\bar{t}) = 0$ and integrating twice (3) in $[\bar{t}, t]$, by (2) we obtain

$$y(t) = -\int_{\bar{t}}^t \frac{1}{r(s)} \int_{\bar{t}}^s g\big(\sigma, y(\sigma)\big)\,d\sigma\,ds > 0, \quad \text{for all } t > \bar{t}$$

in contradiction with the sign of $y(t)$. Hence $y(t)$ has positive values for arbitrarily large t. Suppose now that $|y(t)| \leq v$ for some positive v and all $t \geq 0$; let V and T satisfying (25) and take t_1 and t_2, with $T \leq t_1 < t_2$, such that

$$y(t_1) = 0,\ y'(t_2) = 0,\ y'(t) > 0 \quad \text{for all } t_1 \leq t < t_2.$$

According to Theorem 2, we get the existence of an unbounded increasing solution $u(t)$ of (3) satisfying, with no loss of generality, $u(t) \geq V$ in $[t_1, t_2]$. Therefore we obtain, for $t \in [t_1, t_2]$,

$$\begin{aligned}\frac{d}{dt}\Big[r(t)u'(t)y(t) - r(t)y'(t)u(t)\Big] &= \big(r(t)u'(t)\big)'y(t) - \big(r(t)y'(t)\big)'u(t)\\ &= y(t)u(t)\left[\frac{g(t,y(t))}{y(t)} - \frac{g(t,u(t))}{u(t)}\right] \leq 0.\end{aligned}$$

On the other hand,

$$\int_{t_1}^{t_2} \frac{d}{ds}\Big[r(s)u'(s)y(s) - r(s)y'(s)u(s)\Big] ds \geq r(t_2)u'(t_2)y(t_2) + r(t_1)y'(t_1)V > 0,$$

which gives a contradiction. □

Remark 4 Similarly as in the previous theorem, the non-existence of bounded oscillatory solutions for (3) can be obtained when assuming (24) for some $k < 0$, (17) and the condition that for each $v < 0$ there exist $V \leq v$ and $T \geq 0$ such that

$$\sup_{u \in [v,0]} \frac{g(t,u)}{u} \geq \inf_{u \leq V} \frac{g(t,u)}{u}$$

for each $t \in [T, \infty)$.

Remark 5 Cecchi–Marini–Villari [3] obtained the non-existence of bounded oscillatory solutions in the case when $g(t,u) = q(t)f(u)$, assuming, instead of (25), the existence of $\theta \in [0,\infty)$ such that

$$\lim_{u \to 0^+} \frac{f(u)}{u} = \theta \quad \text{and} \quad \lim_{u \to +\infty} \frac{f(u)}{u} = \infty. \tag{26}$$

Notice that, in this case, (25) is equivalent to assume that for each $v > 0$ there exists $V \geq v$ such that

$$\sup_{u \in [0,v]} \frac{f(u)}{u} \leq \inf_{u \in [V,\infty)} \frac{f(u)}{u},$$

which is weaker than (26). In fact, (25) does not require the super-linearity of $\frac{f(u)}{u}$ at infinity, being, for example, fulfilled by any increasing $\frac{f(u)}{u}$.

Under stronger conditions on $r(t)$ and $g(t,u)$, now we give a non-oscillatory result for (3). On this purpose, given a solution $u(t)$ of (3), we introduce the function

$$V_u(t) = \frac{1}{2}\big(r(t)u'(t)\big)^2 + H(t,u(t)), \quad t \geq 0 \tag{27}$$

where

$$H(x,y) = r(x)\int_0^y g(x,s)\,ds, \quad x \geq 0,\ y \in \mathbb{R}.$$

The following estimate is satisfied.

Lemma 3 *Assume that $H_x(x,y)$ exists for $(x,y) \in [0,\infty) \times \mathbb{R}$ and satisfies*

$$H_x(x,y) \leq \rho(x)H(x,y), \quad x \geq 0 \tag{28}$$

where $\rho(t)$ is a non-negative locally integrable function. Then each solution $u(t)$ of (3) satisfies

$$V_u(t) \leq V_u(\tau)e^{\int_\tau^t \rho(s)ds}$$

for all $0 \leq \tau \leq t$.

Proof Given a solution $u(t)$ of (3), consider the function $V_u(t)$ defined in (27). By (28) we get

$$\frac{d}{dt}V_u(t)' \leq \rho(t)H(t,u(t)) \leq \rho(t)V_u(t)$$

for all $t \geq 0$ and the conclusion follows by dividing by $V_u(t)$ and integrating on $[\tau, t]$. $\square$

Remark 6 Notice that when $g(t,u) = q(t)f(u)$ with $q(t) > 0$ and $q(t)r(t)$ absolutely continuous on $[0,\infty)$, then condition (28) holds with

$$\rho(t) = \frac{\big((qr)'(t)\big)_+}{r(t)q(t)} = \frac{\max\{(qr)'(t),0\}}{r(t)q(t)}$$

and previous lemma can be found in [8].

Theorem 5 *Let (24) be satisfied for every $k > 0$. Assume conditions (17) and (28) with*

$$\int_0^\infty \rho(t)dt < \infty. \tag{29}$$

Suppose that there exist $a \geq 1$ and $T \geq 0$ such that (25) is satisfied, for all $v > 0$ and $t \in [T,\infty)$, with $V = av$. Then equation (3) has no oscillatory solutions.

Proof Assume, by contradiction, the existence of an oscillatory solution $y(t)$ of (3) and consider the function $V_y(t)$ defined in (27). According to (29) and Lemma 3, $V_y(t)$ is bounded on all $[0,\infty)$. Hence we get the existence of $k > 0$ such that $|r(t)y'(t)| \leq k$ for $t \geq 0$. As already showed in the proof of Theorem 4, it is possible to prove that $y(t)$ has positive values for arbitrarily large t and to find t_1 and t_2, with $T \leq t_1 \leq t_2$ such that $y(t_1) = 0$, $y'(t_2) = 0$ and $y'(t) > 0$ for all $t_1 \leq t < t_2$. Put $h = \frac{ak}{m}$. According to (24) and reasoning as in the proof of Theorem 2, from (17) we obtain

$$\int_0^\infty \max_{mh\int_0^t \frac{ds}{r(s)} \leq u \leq h\int_0^t \frac{ds}{r(s)}} g(t,u) < \infty.$$

Therefore we can find $t_0 \geq T$ satisfying

$$\int_{t_0}^\infty \max_{mh\int_0^t \frac{ds}{r(s)} \leq u \leq h\int_0^t \frac{ds}{r(s)}} g(t,u) < h(1-m).$$

Notice that it is not restrictive to assume $t_0 \leq t_1$. Reasoning as in Theorem 1, it then follows the existence of a solution $u(t)$ of (3) satisfying

$$u(t) \geq mh\int_{t_1}^t \frac{ds}{r(s)} \geq ay(t) \quad \text{for all } t \in [t_1,t_2].$$

Hence condition (25) can be applied, with $V = av$, implying

$$\frac{g(t,y(t))}{y(t)} - \frac{g(t,u(t))}{u(t)} \leq 0, \quad \text{for } t \in [t_1,t_2].$$

The contradiction then follows when reasoning as in the proof of Theorem 4. □

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