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Some Stability and Boundedness Results for the Solutions of Certain Fourth Order Differential Equations

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Abstract

Sufficient conditions are established for the asymptotic stability of the zero solution of the equation (1.1) with $p \equiv 0$ and the boundedness of all solutions of the equation (1.1) with $p \neq 0$. Our result includes and improves several results in the literature ([4], [5], [8]).

Key words: Differential equations of fourth order, boundedness, stability, Lyapunov functions.

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1 Introduction

In the current paper, we consider the nonlinear differential equation of the form

$$x^{(4)} + a(\ddot{x}, \dddot{x})\ \dddot{x} + b(x, \dot{x})\ \ddot{x} + c(\dot{x}) + d(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}). \tag{1.1}$$

It can be written in the phase variables form

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = z, \quad \dot{z} = u, \\ \dot{u} &= -a(z, u)u - b(x, y)z - c(y) - d(x) + p(t, x, y, z, u), \end{aligned} \tag{1.2}$$

in which the functions a, b, c, d and p depend only on the arguments displayed and the dots denote differentiation with respect to t. The functions a, b, c, d and p are continuous for all values of their respective arguments.The derivatives $\frac{\partial a(z,u)}{\partial u} \equiv a_u(z,u)$, $\frac{\partial b(x,y)}{\partial x} \equiv b_x(x,y)$, $\frac{dc}{dy} \equiv c'(y)$, and $\frac{dd}{dx} \equiv d'(x)$ exist and are continuous. Moreover, the existence and the uniqueness of the solutions of (1.1) will be assumed.

It is well known that the stability and boundedness of solutions of ordinary differential equations are very important problems in the theory and applications of differential equations. So far, perhaps, the most effective method to study the stability and boundedness of solutions of nonlinear differential equations is still the Lyapunov's direct (or second) method. In the relevant literature, for the fourth order nonlinear differential equations, many stability and boundedness results have been established by using this method. We refer to [1-8] and the references cited there for some of those topics. In [5], Ponzo discussed the stability of solutions of the equation (1.1) in the case $p(t, x, \dot{x}, \ddot{x}, \dddot{x}) = 0$. Nearly four decades later, Hu [4] proved that the result of Ponzo [5] was not true in general, except the special case $b(x, y) \equiv$ constant and $d(x) \equiv cx$ (c is a constant) in (1.1). Recently, in [8], Wu and Xiong also investigated the asymptotic stability of the zero solution of the differential equations described as follows:

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + f(x) = 0$$

and

$$x^{(4)} + a_1\dddot{x} + f(x,\dot{x})\,\ddot{x} + a_3\dot{x} + a_4 x = 0,$$

in which a_1, a_2, a_3 and a_4 are constants. The motivation for the present work has come from the papers of Ponzo [5], Hu [4], Wu and Xiong [8] and the papers mentioned above. Our aim is to obtain similar results and improve some results in the papers stated above. It should also be noted that the domain of attraction of the zero solution $x = 0$ of the equation (1.1) (for $p \equiv 0$) in the following first result is not going to be determined here.

2 The stability and the boundedness results of solutions of (1.2)

In what follows we shall use the following notations:

$$a_1(z,0) := \begin{cases} \dfrac{1}{z}\displaystyle\int_0^z a(z,0)dz, & z \neq 0 \\ a(0,0), & z = 0 \end{cases}$$

and

$$c_1(y) := \begin{cases} \frac{c(y)}{y}, & y \neq 0 \\ c'(0), & y = 0. \end{cases}$$

For the case $P \equiv 0$ in (1.1) the following result is established.

Theorem 1 *Further to the basic assumptions on the functions a, b, c and d assume that the following conditions are satisfied ($\alpha, \beta, \mu, \gamma, \delta, \eta, \varepsilon$ and ε_1—some positive constants):*

(i) $0 \le a(z,u) - \alpha \le \varepsilon_1$ *for all z and u.*

(ii) $c_1(y) \ge \beta$ *for all* $y \ne 0, c(0) = 0$.

(iii) $0 \le b(x,y) - \mu \le \sqrt{\frac{\delta\varepsilon_1}{4\beta}}$ *and*

$$y \int_0^y b_x(x,y) y \, dy \le -\left(\frac{\beta^2}{\alpha\gamma}\right) y^2$$

for all x and y.

(iv) $d(x)x > 0$ *for all* $x \ne 0$, $0 \le \gamma - d'(x) \le \frac{\sqrt{\delta}}{2}$ *for all x, and* $d(0) = 0$.

(v) $\alpha\beta\mu - \beta c'(y) - \alpha\gamma a(z,u) \ge \delta$ *for all y, z and u.*

(vi) $c'(y) - c_1(y) \le \eta < \frac{2\delta\gamma}{\alpha\beta^2}$ *for all* $y \ne 0$, *and* $a_1(z,u) - a(z,u) \le \varepsilon < \frac{2\delta}{\alpha^2\beta}$ *for all* $z \ne 0$ *and u.*

(vii) $\gamma y a_u(z,u) + \beta z a_u(z,u) \ge 0$ *for all y, z and u.*

Then the trivial solution of the system (1.2) is asymptotically stable.

Remark 1 From the conditions (ii) and (v) of Theorem 1 we can obtain

$$a(z,u) < \frac{\beta\mu}{\gamma} \quad \text{and} \quad c'(y) < \alpha\mu.$$

Remark 2 When $a(\ddot{x}, \dddot{x}) = \alpha$, $b(x, \dot{x}) = \mu$, $c(\dot{x}) = \beta\dot{x}$ and $d(x) = \gamma x$, equation (1.1) reduces to the linear constant coefficient differential equation and conditions (i)–(vii) of Theorem 1 reduce to the corresponding Routh–Hurwitz criterion.

Remark 3 Theorem 1 includes and revises the result of Ponzo [5], and also includes and improves the result of Hu [4] except the restrictions on $a(z,u)$, $b(x,y)$ and $d(x)$, that is, $a(z,u) \le \alpha + \varepsilon_1$,

$$b(x,y) \le \mu + \sqrt{\frac{\delta\varepsilon_1}{2\beta}}, \qquad y \int_0^y b_x(x,y) y \, dy \le -\left(\beta^2 \alpha^{-1} \gamma^{-1}\right) y^2$$

and $\gamma - d'(x) \le \frac{\sqrt{\delta}}{2}$, and the results of Wu and Xiong [8] except the same restrictions on $b(x,y)$.

In the case $p \ne 0$ we have the following result

Theorem 2 *Suppose the following conditions are satisfied:*

(i) *conditions (i)–(vii) of Theorem 1 hold,*

(ii) $|p(t,x,y,z,u)| \le (A+|y|+|z|+|u|)q(t)$, *where* $q(t)$ *is a non-negative continuous function of* t, *and satisfies*

$$\int_0^t q(s)\,ds \le B < \infty$$

for all $t \ge 0$, A *and* B *are some positive constants.*

Then for any given finite constants x_0, y_0, z_0 *and* u_0, *there exists a constant* $K = K(x_0, y_0, z_0, u_0)$, *such that any solution* $(x(t), y(t), z(t), u(t))$ *of the system (1.2) determined by*

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \quad u(0) = u_0$$

satisfies for all $t \ge 0$,

$$|x(t)| \le K, \quad |y(t)| \le K, \quad |z(t)| \le K, \quad |u(t)| \le K.$$

If p *is a bounded function, then the constant* K *above can be fixed independent of* x_0, y_0, z_0 *and* u_0, *as will be seen from our the following result.*

Theorem 3 *Assume that the conditions (i)–(vii) of Theorem 1 hold, and that* $p(t,x,y,z,u)$ *satisfies*

$$|p(t,x,y,z,u)| \le A < \infty$$

for all values of t, x, y, z *and* u, *where* A *is a positive constant. Then there exists a constant* K_1 *whose magnitude depends* $\alpha, \beta, \mu, \gamma, \delta, \eta, \varepsilon$ *and* ε_1 *as well as on the functions* a, b, c *and* d *such that every solution* $(x(t), y(t), z(t), u(t))$ *of the system (1.2) ultimately satisfies*

$$|x(t)| \le K_1, \quad |y(t)| \le K_1, \quad |z(t)| \le K_1, \quad |u(t)| \le K_1.$$

Remark 4 Theorem 2 and Theorem 3 based on the results in ([4], [5], [8]) give additional results to those obtained in ([4], [5], [8]).

The proofs of Theorem 1 and Theorem 2 depend on some certain fundamental properties of a continuously differentiable Lyapunov function $V = V(x,y,z,u)$ defined by:

$$\begin{aligned} V = {} & \alpha\gamma \int_0^x d(x)\,dx + \alpha\gamma \int_0^y b(x,y)y\,dy - \left(\frac{\beta\gamma}{2}\right) y^2 + \alpha\beta \int_0^y c(y)\,dy \\ & + \left(\frac{\beta\mu}{2}\right) z^2 + \alpha\beta \int_0^z a(z,0)z\,dz - \left(\frac{\alpha\gamma}{2}\right) z^2 + \left(\frac{\beta}{2}\right) u^2 + \alpha\beta d(x)y \\ & + \beta d(x)z + \beta c(y)z + \alpha\gamma y \int_0^z a(z,0)\,dz + \alpha\gamma yu + \alpha\beta zu. \end{aligned} \tag{2.1}$$

The first property of V is stated in the following.

Lemma 1 *Assume that the conditions of Theorem 1 hold. Then*

(I) $V(x,y,z,u)=0$ *at* $x^2+y^2+z^2+u^2=0.$ (2.2)

(II) $V(x,y,z,w)>0$ *if* $x^2+y^2+z^2+u^2>0;$ (2.3)
$\dot{V}\mid_{(1.2)}\leq 0$ *for all* $t\geq 0.$ (2.4)

(III) *Any of the positive semi-trajectory of the system (1.2) is bounded.*

(IV) *The set* $M=\{(x,y,z,u):\dot{V}=0,\ (x,y,z,u)\in R^4\}$, *except* $(x,y,z,u)=0$, *does not contain the entire positive semi trajectory of the solution of the system (1.2).*

Proof *Part (I):* $V(0,0,0,0)=0$, since $c(0)=d(0)=0$. Hence (2.2) is verified.
Rewrite the function $V(x,y,z,u)$ as follows:

$$\begin{aligned}V =& \left(\frac{\alpha\beta}{2c_1(y)}\right)\left[d(x)+c(y)+\frac{c_1(y)z}{\alpha}\right]^2\\&+\left(\frac{\alpha\beta}{2a_1(z,0)}\right)\left[u+za_1(z,0)+\frac{\gamma}{\beta}ya_1(z,0)\right]^2\\&+\left(\frac{\beta\mu}{2}\right)z^2-\left(\frac{\beta c_1(y)}{2\alpha}\right)z^2-\left(\frac{\alpha\gamma}{2}\right)z^2\\&+\alpha\gamma\int_0^y b(x,y)y\,dy-\left(\frac{\beta\gamma}{2}\right)y^2-\left(\frac{\alpha\gamma^2a_1(z,0)}{2\beta}\right)y^2\\&+\left(\frac{\beta}{2}\right)\left[1-\frac{\alpha}{a_1(z,0)}\right]u^2+\sum_{i=1}^{3}W_i,\end{aligned}\tag{2.5}$$

where

$$W_1=\alpha\gamma\int_0^x d(x)dx-\frac{\alpha\beta d^2(x)}{2c_1(y)},$$

$$W_2=\alpha\beta\int_0^y c(y)\,dy-\frac{\alpha\beta c^2(y)}{2c_1(y)},$$

$$W_3=\alpha\beta\int_0^z a(z,0)z\,dz-\frac{\alpha\beta a_1(z,0)}{2}z^2.$$

Part (II): Now we verify (2.3). To do this we have four cases.
(a) Let $y\neq 0$, $z\neq 0$. From (iv) of Theorem 1 it follows that

$$W_1\geq \alpha\gamma\int_0^x d(x)\,dx-\frac{\alpha d^2(x)}{2}\geq \alpha\int_0^x d(x)[\gamma-d'(x)]\,dx\geq 0.$$

Now note that

$$yc(y)\equiv\int_0^y c(y)\,dy+\int_0^y c'(y)y\,dy.$$

Therefore,

$$W_2 = \alpha\beta \int_0^y c(y)\,dy - \frac{\alpha\beta c(y)}{2} = \frac{\alpha\beta}{2}\int_0^y [c_1(y) - c'(y)]y\,dy \geq -\left(\frac{\alpha\beta\eta}{4}\right)y^2$$

by (vi). From the identity

$$\int_0^z za(z,0)\,dz \equiv z\int_0^z a(z,0)\,dz - \int_0^z za_1(z,0)\,dz$$

we find

$$\begin{aligned} W_3 &= \alpha\beta\int_0^z a(z,0)z\,dz - \frac{\alpha\beta}{2}z\int_0^z a(z,0)\,dz \\ &= \frac{\alpha\beta}{2}\int_0^z [a(z,0) - a_1(z,0)]z\,dz \geq -\left(\frac{\alpha\beta\varepsilon}{4}\right)z^2 \end{aligned}$$

by (vi) of Theorem 1. On gathering the estimates for W_1, W_2 and W_3 into (2.5), we have that

$$\begin{aligned} V \geq\ & \alpha\int_0^x d(x)[\gamma - d'(x)]\,dx + \left(\frac{\alpha\beta}{2a_1(z,0)}\right)\left[u + za_1(z,0) + \frac{\gamma}{\beta}ya_1(z,0)\right]^2 \\ & + \left(\frac{\alpha\beta}{2c_1(y)}\right)\left[d(x) + c(y) + \frac{c_1(y)z}{\alpha}\right]^2 + \left(\frac{\beta\mu}{2}\right)z^2 \\ & - \left(\frac{1}{2\alpha}\right)\left[\beta c_1(y) + \alpha^2\gamma + \frac{\alpha^2\beta\varepsilon}{2}\right]z^2 + \alpha\gamma\int_0^y b(x,y)y\,dy \\ & - \left(\frac{\beta\gamma}{2}\right)y^2 - \left(\frac{\gamma}{2\beta}\right)\left[\alpha\gamma a_1(z,0) + \frac{\alpha\beta^2\eta}{2\gamma}\right]y^2 \\ & + \left(\frac{\beta}{2}\right)\left[1 - \frac{\alpha}{a_1(z,0)}\right]u^2. \end{aligned} \tag{2.6}$$

Now consider the terms

$$W_4 = \left(\frac{\beta\mu}{2}\right)z^2 - \left(\frac{1}{2\alpha}\right)\left[\beta c_1(y) + \alpha^2\gamma + \frac{\alpha^2\beta\varepsilon}{2}\right]z^2$$

and

$$W_5 = \alpha\gamma\int_0^y b(x,y)y\,dy - \left(\frac{\beta\gamma}{2}\right)y^2 - \left(\frac{\gamma}{2\beta}\right)\left[\alpha\gamma a_1(z,0) + \frac{\alpha\beta^2\eta}{2\gamma}\right]y^2$$

which are contained in (2.6).

By using the assumptions (i), (v), (vi) of Theorem 1 and the mean value theorem (for derivative), we find

$$\begin{aligned} W_4 &= \left(\frac{1}{2\alpha}\right)\left[\alpha\beta\mu - \beta c'(\theta_1 y) - \alpha^2\gamma - \frac{\alpha^2\beta\varepsilon}{2}\right]z^2 \\ &\geq \left(\frac{1}{2\alpha}\right)\left[\alpha\beta\mu - \beta c'(\theta_1 y) - \alpha\gamma a(z,u) - \frac{\alpha^2\beta\varepsilon}{2}\right]z^2 \\ &\geq \left(\frac{1}{2\alpha}\right)\left[\delta - \frac{\alpha^2\beta\varepsilon}{2}\right]z^2 > 0, \end{aligned}$$

where $0 \le \theta_1 \le 1$. Similarly, from (iii), (v), (vi) of Theorem 1 and the mean value theorem (for integral), we obtain

$$\begin{aligned} W_5 &\ge \left(\frac{\gamma}{2\beta}\right)\left[\alpha\beta\mu - \beta^2 - \alpha\gamma a_1(z,0) - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 \\ &= \left(\frac{\gamma}{2\beta}\right)\left[\alpha\beta\mu - \beta^2 - \alpha\gamma a(\theta_2 z,0) - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 \\ &\ge \left(\frac{\gamma}{2\beta}\right)\left[\delta - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 > 0, \end{aligned}$$

where $0 \le \theta_2 \le 1$. On substituting the estimate for W_4 and W_5 into (2.6) we have

$$\begin{aligned} V \ge \alpha &\int_0^x d(x)[\gamma - d'(x)]\,dx + \left(\frac{\alpha\beta}{2a_1(z,0)}\right)\left[u + za_1(z,0) + \frac{\gamma}{\beta} y a_1(z,0)\right]^2 \\ &+ \left(\frac{\alpha\beta}{2c_1(y)}\right)\left[d(x) + c(y) + \frac{c_1(y)z}{\alpha}\right]^2 + \left(\frac{1}{2\alpha}\right)\left[\delta - \frac{\alpha^2\beta\varepsilon}{2}\right] z^2 \\ &+ \left(\frac{\gamma}{2\beta}\right)\left[\delta - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 + \left(\frac{\beta}{2}\right)\left[1 - \frac{\alpha}{a_1(z,0)}\right] u^2 > 0. \end{aligned}$$

(b) Let $y^2 + z^2 = 0$. Then it follows from (2.5) that

$$V \ge \alpha\gamma \int_0^x d(x)\,dx + \left(\frac{\beta}{2}\right) u^2 > 0 \quad \text{if } x^2 + u^2 > 0.$$

(c) Let $y \ne 0$, $z = 0$. Similarly, it is easy to see that

$$\begin{aligned} V \ge \alpha &\int_0^x d(x)[\gamma - d'(x)]\,dx \\ &+ \left(\frac{\alpha\beta}{2a_1(0,0)}\right)\left[u + \frac{\gamma}{\beta} y a_1(0,0)\right]^2 + \left(\frac{\alpha\beta}{2c_1(y)}\right)[d(x) + c(y)]^2 \\ &+ \left(\frac{\gamma}{2\beta}\right)\left[\delta - \frac{\alpha\beta^2\eta}{2\gamma}\right] y^2 + \left(\frac{\beta}{2}\right)\left[1 - \frac{\alpha}{a_1(0,0)}\right] u^2 > 0. \end{aligned}$$

(d) Let $y = 0$, $z \ne 0$. It is clear from (a) that

$$\begin{aligned} V \ge \alpha &\int_0^x d(x)[\gamma - d'(x)]\,dx \\ &+ \left(\frac{\alpha\beta}{2a_1(z,0)}\right)[u + za_1(z,0)]^2 + \left(\frac{\alpha\beta}{2c_1(0)}\right)\left[d(x) + \frac{c_1(0)z}{\alpha}\right]^2 \\ &+ \left(\frac{1}{2\alpha}\right)\left[\delta - \frac{\alpha^2\beta\varepsilon}{2}\right] z^2 + \left(\frac{\beta}{2}\right)\left[1 - \frac{\alpha}{a_1(z,0)}\right] u^2 > 0 \end{aligned}$$

by (2.5). Because of the estimates given by (a)–(d) we get the desired result (2.3).

From (2.1) and (1.2) it is trivial that the time derivative of V as follows:

$$\begin{aligned}\dot{V} &= -\alpha\beta\left[\frac{c(y)}{y}\frac{\gamma}{\beta} - d'(x)\right]y^2\\ &\quad - \left[\alpha\beta b(x,y) - \beta c'(y) - \alpha\gamma\left(\frac{1}{z}\right)\int_0^z a(z,0)\,dz\right]z^2\\ &\quad - \beta\left[a(z,u)-\alpha\right]u^2 - \beta\left[b(x,y)-\mu\right]zu - \beta\left[\gamma - d'(x)\right]yz\\ &\quad + \alpha\gamma y\int_0^y b_x(x,y)y\,dy\\ &\quad - \alpha\gamma\left[a(z,u)-a(z,0)\right]yu - \alpha\beta\left[a(z,u)-a(z,0)\right]zu.\end{aligned}$$

Hence the assumptions (i)–(v) of Theorem 1 and the mean value theorem (for the integral) show that

$$\begin{aligned}\dot{V} &\le -\left[\alpha\beta\mu - \beta c'(y) - \alpha\gamma a(\theta_3 z,0)\right]z^2\\ &\quad - (\beta\varepsilon_1)u^2 - \beta\left[b(x,y)-\mu\right]zu - \beta\left[\gamma - d'(x)\right]yz\\ &\quad + \alpha\gamma y\int_0^y b_x(x,y)y\,dy\\ &\quad - \alpha\gamma\left[a(z,u)-a(z,0)\right]yu - \alpha\beta\left[a(z,u)-a(z,0)\right]zu, \quad (0\le\theta_3\le 1),\\ &\le -\left(\frac{3\beta\varepsilon_1}{4}\right)u^2 - \left(\frac{\delta}{2}\right)z^2 - \left(\frac{3\beta^2}{4}\right)y^2 - W_6 - W_7 - W_8, \qquad (2.7)\end{aligned}$$

where

$$\begin{aligned}W_6 &= \left(\frac{\delta}{4}\right)z^2 + \beta\left[b(x,y)-\mu\right]zu + \left(\frac{\beta\varepsilon_1}{4}\right)u^2,\\ W_7 &= \left(\frac{\beta^2}{4}\right)y^2 + \beta\left[\gamma - d'(x)\right]yz + \left(\frac{\delta}{4}\right)z^2,\\ W_8 &= \alpha\gamma\left[a(z,u)-a(z,0)\right]yu + \alpha\beta\left[a(z,u)-a(z,0)\right]zu.\end{aligned}$$

From (iii) of Theorem 1

$$W_6 \ge \left(\frac{\delta}{4}\right)z^2 - \beta\left[b(x,y)-\mu\right]|zu| + \left(\frac{\beta\varepsilon_1}{4}\right)u^2 = \left[\frac{\sqrt{\delta}}{2}z \pm \frac{\sqrt{\beta\varepsilon_1}}{2}u\right]^2 \ge 0.$$

Similarly, by (iv) of Theorem 1, we find

$$W_7 \ge \left(\frac{\beta^2}{4}\right)y^2 - \beta\left[\gamma - d'(x)\right]|yz| + \left(\frac{\delta}{4}\right)z^2 = \left[\frac{\beta}{2}y \pm \frac{\sqrt{\delta}}{2}z\right]^2 \ge 0.$$

The assumption (vii) of Theorem 1 (for $u \ne 0$) also shows that

$$W_8 = \alpha\left[\gamma y a_u(z,\theta_4 u) + \beta z a_u(z,\theta_4 u)\right]u^2 \ge 0, 0 \le \theta_4 \le 1,$$

but $W_8 = 0$, when $u = 0$. Hence $W_8 \ge 0$ for all y, z and u.

On combining the estimates for W_6, W_7 and W_8 into (2.7) we find

$$\dot{V} \leq -\left(\frac{3\beta\varepsilon_1}{4}\right)u^2 - \left(\frac{\delta}{2}\right)z^2 - \left(\frac{3\beta^2}{4}\right)y^2.$$

This completes the proof of Part (II).

The proofs of Part (III) and Part (IV) follow the lines indicated in [4], except some minor modification. And hence the proof is omitted.

This completes the proof of the lemma. □

The proof of Theorem 1 From Lemma 1, we see that the function $V(x, y, z, u)$ is a Lyapunov function for the system (1.2). Hence, the zero solution of the system (1.2) is asymptotically stable (see [8]).

This completes the proof. □

The proof of Theorem 2 The proof of this theorem is similar to that of Theorem 2 of Tunc [7] and hence is omitted.

Finally, the actual proof of Theorem 3 will rest mainly on the existence of a piecewise continuously differentiable function $V_1 = V_1(x, y, z, u)$ satisfying

$$V_1(x, y, z, u) \geq -D \quad \text{for all } (x, y, z, u), \tag{2.8}$$

$$V_1(x, y, z, u) \to \infty \quad \text{as } x^2 + y^2 + z^2 + u^2 \to \infty; \tag{2.9}$$

and also such that the limit

$$\dot{V}_1^+(t) = \limsup_{h\to 0+}\left[\frac{V_1(x(t+h), y(t+h), z(t+h), u(t+h)) - V_1(x(t), y(t), z(t), u(t))}{h}\right] \tag{2.10}$$

exists corresponding any solution $(x(t), y(t), z(t), u(t))$ of the system (1.2), and satisfies

$$\dot{V}_1^+(t) \leq -1 \quad \text{if } x^2(t) + y^2(t) + z^2(t) + u^2(t) \geq D_1,$$

where D and D_1 are certain positive constants to be determined in the proof.

Once the existence of such a V_1 is established an appeal to Yoshizawa's argument (see [2]) concludes the proof of Theorem 3.

We define the required V_1 as follows:

$$V_1 = V_0 + V, \tag{2.11}$$

where

$$V_0(x, u) := \begin{cases} x \operatorname{sgn} u, & \text{if } |u| \geq |x| \\ u \operatorname{sgn} x, & \text{if } |u| \leq |x| \end{cases} \tag{2.12}$$

and V is defined by (2.1).

The property of $\dot{V}_1^+$ is required and is stated in Lemma 2.

Lemma 2 *Subject to the conditions of Theorem 3, the function V_1 defined in (2.11) satisfies the properties in (2.8), (2.9) and (2.10).*

Proof Let (x, y, z, u) be any solution of the system (1.2). From (2.12) we obtain $|V_0(x,u)| \le |u|$ for all x and u. It follows that $|V_0(x,u)| \ge -|u|$ for all x and u. Now, V here is the same as the function V defined by (2.1). Since V is positive definite, then it has infinite inferior limit and infinitesimal upper limit, that is, there exists a positive constant τ such that

$$V(x,y,z,u) > \tau(x^2 + y^2 + z^2 + u^2).$$

From these estimates for V_0 and V we get the estimate for V_1 as

$$V_1 > \tau(x^2 + y^2 + z^2 + u^2) - 2|u| = \tau(x^2 + y^2 + z^2) + \tau\left(|u| - \frac{1}{\tau}\right)^2 - \frac{1}{\tau}.$$

So it is evident that (2.8) and (2.9) are verified, where $D = \frac{1}{\tau}$.

Next, in accordance with the representation $V_1 = V + V_0$ we have a representation $v_1 = v + v_0$. Hence, the function $v_1 = v_1(t)$ can be defined by $v_1(t) = V_1(x(t), y(t), z(t), u(t))$. Then, the existence of $\overset{+}{\dot{v}}_1$, that is,

$$\overset{+}{\dot{v}}_1(t) = \limsup_{h\to 0+}\left[\frac{v_1(t+h) - v_1(t)}{h}\right]$$

is quite immediate, since v has continuous first partial derivatives and v_0 is easily shown to be locally Lipschitizian in x and u so that the composite function $v_1 = v + v_0$ is at the least locally Lipschitizian in x, y, z and u. Subject to the assumptions of the theorem an easy calculation from (2.11) and (1.2) shows that

$$\overset{+}{\dot{v}}_1 = \dot{v} + \overset{+}{\dot{v}}_0 \le -\left(\frac{3\beta\varepsilon_1}{4}\right)u^2 - \left(\frac{\delta}{2}\right)z^2 - \left(\frac{3\beta^2}{4}\right)y^2 + D_2(|y| + |z| + |u|), \text{ if } |u| \ge |x|$$

or

$$\begin{aligned}\overset{+}{\dot{v}}_1 &= \dot{v} + \overset{+}{\dot{v}}_0 \le -\left(\frac{3\beta\varepsilon_1}{4}\right)u^2 - \left(\frac{\delta}{2}\right)z^2 - \left(\frac{3\beta^2}{4}\right)y^2 - d(x)sgnx + |c(y)| \\ &\quad + D_3(1 + |y| + |z| + |u|), \quad \text{if } |u| \le |x|.\end{aligned}$$

The following arguments are similar to those in [3] and hence we omit the details of the proof. The proof of this lemma is now complete. □

The proof of Theorem 3 By considering the results obtained in Lemma 2, the usual Yoshizawa-type argument (see the result established in [2]) applied to (2.8), (2.9) and (2.10) would then show that, for any solution (x, y, z, u) of the system (1.2), we have

$$|x(t)| \le K_1, \quad |y(t)| \le K_1, \quad |z(t)| \le K_1, \quad |u(t)| \le K_1,$$

for all sufficiently large t, which proves the theorem.

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