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Remarks on Existence of Positive Solutions of some Integral Equations

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Abstract

We study the existence of positive solutions of the integral equation

$$x(t) = \mu \int_0^1 k(t,s) f(s, x(s), x'(s), \dots, x^{(n-1)}(s))\, ds, \quad n \geq 2$$

in both $C^{n-1}[0,1]$ and $W^{n-1,p}[0,1]$ spaces, where $p \geq 1$ and $\mu > 0$. Throughout this paper k is nonnegative but the nonlinearity f may take negative values. The Krasnosielski fixed point theorem on cone is used.

Key words: Positive solutions, Fredholm integral equations, cone, boundary value problems, fixed point theorem.

2000 Mathematics Subject Classification: 34G20, 34K10, 34B10, 34B15

4 Introduction

In analyzing nonlinear phenomena many mathematical models give rise to problems for which only nonnegative solutions make sense. This paper deals with existence of positive solutions of the integral equations of the form

$$x(t) = \mu \int_0^1 k(t,s) f(s, x(s), s'(s), \dots, x^{(n-1)}(s))\, ds, \tag{1.1}$$

where $\mu > 0$ is a constant and $n \geq 2$.

Throughout this paper k is nonnegative but our nonlinearity f may take negative values. The literature on positive solutions is for the most part devoted to (1.1), when f takes nonnegative values and f is not dependent on derivatives of the function x (see [2]–[5]). Existence in this paper will be established using Krasnosielskii's fixed point theorem in a cone, which we state here for the convenience of the reader.

Theorem 4.1 (K. Deimling [4], D. Guo [5]). *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are bounded and open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ and let $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be continuous and completely continuous. In addition suppose either $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$ or $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$ hold. Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

5 Main results

In this section we present some results for the integral equation (1.1).

Throughout the paper

$$I = [0,1] \times [0,\infty) \times (-\infty,\infty)^{n-1}, \quad J = [0,\infty) \times (-\infty,\infty)^{n-1}$$

and

$$\|x\|_{n-1} = \sup_{t\in[0,1]} \left[|x(t)| + |x'(t)| + \ldots + |x^{(n-1)}(t)|\right],$$

where $x \in C^{n-1}[0,1]$.

Theorem 5.1 *Suppose the following conditions are satisfied:*

(2.1) $k : [0,1] \times [0,1] \to [0,\infty)$, $\frac{\partial^l k(t,s)}{\partial t^l}$ *$(l = 0,1,\ldots,n-2)$ exist and are continuous on* $[0,1] \times [0,1]$,

(2.2) there exists $\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}$ *for all $t \in [0,1]$ and a.e. $s \in [0,1]$,*

(2.3) there exist $k^* \in C[0,1], \overline{k}_i \in L^1[0,1]$ and $M > 0$ such that

- *(a) $k^*(t) > 0$ for a.e. $t \in [0,1]$,*
- *(b) $\overline{k}_i(s) \geq 0$ and $\int_0^1 \overline{k}_i(s)ds > 0$ for $i = 0,1,\ldots,n-1$ and a.e. $s \in [0,1]$,*
- *(c) $Mk^*(t)\overline{k}_i(s) \leq \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| \leq \overline{k}_i(s)$ for $i = 0,1,\ldots,n-1$; $t \in [0,1]$ and a.e. $s \in [0,1]$,*

(2.4) the map $t \to \frac{\partial^{n-1}}{\partial t^{n-1}} k(t,s)$ is continuous from $[0,1]$ to $L^1[0,1]$,

(2.5) there exists a function $d \in C[0,1]$ *with* $d(t) > 0$ *for a.e.* $t \in [0,1]$ *such that*

$$k(t,s) - d(t)\left[\left|\frac{\partial k(t,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}\right|\right] \geq d(t)\left[k(t,s) + \left|\frac{\partial k(t,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}\right|\right]$$

for all $t \in [0,1]$ *and a.e.* $s \in [0,1]$,

(2.6) there exists a constant $\tilde{c} > 0$ *with*

$$\int_0^1 k(t,s)\,ds \leq \tilde{c} M d(t) k^*(t) \quad \text{for } t \in [0,1],$$

(2.7) $f : I \to (-\infty, \infty)$ *is continuous and there exists a constant* $L > 0$ *with*

$$f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq 0 \quad \text{for } (t, v_0, v_1, \ldots, v_{n-1}) \in I,$$

(2.8) there exists a function $\psi(u)$ *such that*

$$f(t, v_0, v_1, \ldots, v_{n-1}) + L \leq \psi(v_0 + |v_1| + \ldots + |v_{n-1}|)$$

on I, *where* $\psi : [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing and* $\psi(u) > 0$ *for* $u > 0$,

(2.9) there exists $r > 0$ *such that* $r \geq \mu L \tilde{c}$ *and*

$$\frac{r}{\psi(r + \|\phi\|_{n-1})} \geq \sum_{i=0}^{n-1} \mu \sup_{t \in [0,1]} \int_0^1 \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| ds,$$

where $\phi(t) = \mu L \int_0^1 k(t,s)\,ds$,

(2.10) $f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq g(v_0)$ *for* $(t, v_0, v_1, \ldots, v_{n-1}) \in I$ *with* $g : [0,\infty) \to [0,\infty)$ *continuous and nondecreasing and* $g(u) > 0$ *for* $u > 0$,

(2.11) there exists $R > 0$ *and* $t_0 \in [0,1]$ *such that* $R > r$, $k^*(t_0) > 0, d(t_0) > 0$ *and*

$$R \leq \mu \int_0^1 k(t_0,s) + \left[\left|\frac{\partial k(t_0,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t_0,s)}{\partial t^{n-1}}\right|\right] d(t_0) g(\varepsilon R M d(s) k^*(s))\,ds,$$

where $\varepsilon > 0$ *is any constant such that* $1 - \frac{\mu L \tilde{c}}{R} \geq \varepsilon$.

Then (1.1) has a nonnegative solution $x \in C^{n-1}[0,1]$ *with* $x(t) > 0$ *for a.e.* $t \in [0,1]$.

Proof The proof of Theorem 2.1 is similar to that of Theorem 2.1 in the paper [1]. To show (1.1) has a positive solution we will look at

$$x(t) = \mu \int_0^1 k(t,s) f^*(s, x(s) - \phi(s), s'(s) - \phi'(s), \ldots, x^{n-1}(s) - \phi^{(n-1)}(s))\, ds, \tag{2.12}$$

where

$$f^*(t, v_0, v_1, \ldots, v_{n-1}) = \begin{cases} f(t, v_0, v_1, \ldots, v_{n-1}) + L, & \text{if } (t, v_0, v_1, \ldots, v_{n-1}) \in I, \\ f(t, 0, v_1, \ldots, v_{n-1}) + L, & \text{if } (t, v_0, v_1, \ldots, v_{n-1}) \in \tilde{I}, \end{cases}$$

with $\tilde{I} = [0,1] \times (-\infty, 0) \times (-\infty, \infty)^{n-1}$.

We will show that there exists a solution x_1 to (2.12) with $x_1(t) \geq \phi(t)$ for $t \in [0,1]$. If this is true then $u(t) = x_1(t) - \phi(t)$ is a nonnegative solution of (1.1) since for $t \in [0,1]$ we have

$$\begin{aligned} u(t) &= \\ &= \mu \int_0^1 k(t,s) \Big[f^*(s, x(s) - \phi(s), x'(s) - \phi'(s), \ldots, x^{(n-1)}(s) - \phi^{(n-1)}(s)) \Big]\, ds \\ &\qquad - \mu L \int_0^1 k(t,s)\, ds \\ &= \mu \int_0^1 k(t,s) f(s, x_1(s) - \phi(s), x_1'(s) - \phi'(s), \ldots, x_1^{(n-1)}(s) - \phi^{(n-1)}(s))\, ds \\ &= \mu \int_0^1 k(t,s) f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))\, ds. \end{aligned}$$

We will concentrate our study on (2.12).

Let $E = (C^{(n-1)}[0,1], \|\cdot\|_{n-1})$ and

$$K = \{u \in C^{n-1}[0,1] : u(t) - d(t)\big[|u'(t)| + \ldots + |u^{(n-1)}(t)|\big] \geq M d(t) k^*(t) \|u\|_{n-1}.$$

Clearly K is cone of E. Let

$$\begin{aligned} \Omega_1 &= \{u \in C^{n-1}[0,1] :\ \|u\|_{n-1} < r\}, \\ \Omega_2 &= \{u \in C^{n-1}[0,1] :\ \|u\|_{n-1} < R\} \end{aligned}$$

and

$$\tilde{f}(s, x(s) - \phi(s)) = f^*(s, x(s) - \phi(s), x'(s) - \phi'(s), \ldots, x^{(n-1)} - \phi^{(n-1)}(s)),$$

where $x \in C^{n-1}[0,1]$. Now, let

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to C^{n-1}[0,1]$$

be defined by

$$(Ax)(t) = \mu \int_0^1 k(t,s) \tilde{f}(s, x(s) - \phi(s))\, ds.$$

First we show $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$. If $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $t \in [0,1]$, then relations (2.1), (2.5) imply

$$Ax(t) - d(t)[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|]$$
$$\geq \mu \int_0^1 k(t,s)\tilde{f}(s, x(s) - \phi(s))\, ds$$
$$- \mu\, d(t) \int_0^1 \left[\left|\tfrac{\partial k(t,s)}{\partial t}\right| + \ldots + \left|\tfrac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}\right| \right] \tilde{f}(s, x(s) - \phi(s))\, ds$$
$$\geq \mu d(t) \int_0^1 \left[k(t,s) + \left|\tfrac{\partial k}{\partial t}(t,s)\right| + \ldots + \left|\tfrac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}\right| \right] \tilde{f}(s, x(s) - \phi(s))\, ds$$

and this together with (2.3) yields

$$\|Ax\|_{n-1} \geq Ax(t) - d(t) \left[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)| \right]$$
$$\geq \mu d(t) \left(\sum_{i=0}^{n-1} M k^*(t) \int_0^1 \overline{k}_i(s) \tilde{f}(s, x(s) - \phi(s))\, ds \right). \tag{2.13}$$

On the other hand (2.3) implies

$$\|Ax\|_{n-1} \leq \sum_{i=0}^{n-1} \mu \int_0^1 \overline{k}_i(s) \tilde{f}(s, x(s) - \phi(s))\, ds. \tag{2.14}$$

Taking into account (2.13)–(2.14) we conclude that

$$Ax(t) - d(t)[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|] \geq M d(t) k^*(t) \|Ax\|_{n-1} \quad \text{for } t \in [0,1].$$

Consequently $Ax \in K$ so $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$. We now show

$$\|Ax\|_{n-1} \leq \|x\|_{n-1} \quad \text{for } x \in K \cap \partial\Omega_1. \tag{2.15}$$

To see this let $x \in K \cap \partial\Omega_1$. Then $\|x\|_{n-1} = r$ and $x(t) \geq M d(t) k^*(t) r$ for $t \in [0,1]$. For $t \in [0,1]$ we have

$$\sum_{i=0}^{n-1} |(Ax)^{(i)}(t)| \leq \sum_{i=0}^{n-1} \int_0^1 \left| \frac{\partial^i k(t,s)}{\partial t^i} \right| \tilde{f}(s, x(s) - \phi(s))\, ds.$$

This together with (2.8)–(2.9) yields

$$\|Ax\|_{n-1} \leq \mu \psi \left(\|x\|_{n-1} + \|\phi\|_{n-1} \right) \sum_{i=0}^{n-1} \sup_{t \in [0,1]} \int_0^1 \left| \frac{\partial^i k(t,s)}{\partial t^i} \right| ds$$
$$\leq \mu \psi (r + \|\phi\|_{n-1}) \sum_{i=0}^{n-1} \sup_{t \in [0,1]} \int_0^1 \left| \frac{\partial^i k(t,s)}{\partial t^i} \right| ds \leq r = \|x\|_{n-1}.$$

So (2.15) holds. Next we show

$$\|Ax\|_{n-1} \geq \|x\|_{n-1} \quad \text{for } x \in K \cap \partial\Omega_2. \tag{2.16}$$

To see it let $x \in K \cap \partial\Omega_2$. Then we get $\|x\|_{n-1} = R$ and $x(t) \geq RMd(t)k^*(t)$ for $t \in [0,1]$. Let ε be as in (2.11). For $t \in [0,1]$ we have from (2.6) that

$$\begin{aligned} x(t) - \phi(t) = x(t) - \mu L \int_0^1 k(t,s)\,ds \geq x(t) - \frac{\mu L\tilde{c} M d(t) k^*(t) R}{R} \\ \geq x(t)\left(1 - \frac{\mu L \tilde{c}}{R}\right) \geq x(t)\varepsilon \geq \varepsilon RMd(t)k^*(t) > 0 \end{aligned}$$

for a.e. $t \in [0,1]$. By (2.10)–(2.11) and (2.5) we have

$$\begin{aligned} \|Ax\|_{n-1} \geq Ax(t_0) - d(t_0)[|(Ax)'(t_0)| + \ldots + |(Ax)^{(n-1)}(t_0)|] \\ \geq \mu d(t_0) \int_0^1 \left[k(t_0,s) + \left|\frac{\partial k(t_0,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t_0,s)}{\partial t^{n-1}}\right| \right] g(\varepsilon RMd(s)k^*(s))\,ds \\ \geq R = \|x\|_{n-1}. \end{aligned}$$

Hence we obtain (2.14). By (2.3)–(2.4) and the Arzela–Ascoli theorem we conclude that $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is continuous and compact. Theorem 1.1 implies A has a fixed point $x_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e. $r \leq \|x_1\|_{n-1} \leq R$ and

$$x_1(t) \geq Md(t)k^*(t)r \quad \text{for } t \in [0,1]. \tag{2.18}$$

Taking into account relations (2.6), (2.9) and (2.18) we have

$$x_1(t) \geq Md(t)k^*(t)r \geq \mu L\tilde{c}Md(t)k^*(t) \geq \mu L \int_0^1 k(t,s)\,ds = \phi(t).$$

This completes the proof of Theorem 2.1. □

Example 5.1 To illustrate the applicability of Theorem 2.1 we consider the following boundary value problem

$$x''(t) + \mu((x(t) + |x'(t)|)^2 - 1) = 0, \quad x(0) = x'(0), \quad x(1) = -x'(1). \tag{2.19}$$

The problem (2.19) is equivalent to the problem of determinig the fixed point of the operator T of the form

$$T(x)(t) = \mu \int_0^1 k(t,s)[(x(s) + |x'(s)|)^2 - 1]\,ds,$$

where $k(t,s)$ is defined as follows

$$k(t,s) = \begin{cases} \frac{(2-t)(1+s)}{3}, & 0 \leq s \leq t \leq 1 \\ \frac{(2-s)(1+t)}{3}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Fix $t_0 = \frac{1}{2}$, $d(t) = M = \frac{1}{4}$, $k^*(t) = 1$, $\overline{k_0}(s) = \overline{k_1}(s) = \frac{4}{3}$, $L = 1$ and $\psi(u) = g(u) = u^2$ for $t \in [0,1]$ and $u \in [0,\infty)$. We claim (2.6) holds with $\tilde{c} = 10$, $\mu < \frac{1}{10}, R > 1$ and $\varepsilon = 1 - \frac{\mu L \tilde{c}}{R} = 1 - \frac{10\mu}{R}$. To see this notice for $t \in [0,1]$ that

$$\int_0^1 k(t,s)\,ds = \frac{1}{2}(1+t-t^2) \leq \frac{5}{8} \leq \tilde{c}Md(t)k^*(t) \leq \frac{\tilde{c}}{16}.$$

Clearly $g(\varepsilon RMd(s)k^*(s)) = \varepsilon^2 R^2 M^2 d^2(s) k^{*2}(s) = \frac{\varepsilon^2 R^2}{256}$ and

$$\begin{aligned}
&\mu d\left(\frac{1}{2}\right)\int_0^1 \left[k\left(\frac{1}{2},s\right) + \left|\frac{\partial k\left(\frac{1}{2},s\right)}{\partial t}\right|\right] g(\varepsilon RMd(s)k^*(s))\,ds \\
&\quad = \frac{\mu\varepsilon^2 R^2}{1024}\int_0^1 \left[k\left(\frac{1}{2},s\right) + \left|\frac{\partial k\left(\frac{1}{2},s\right)}{\partial t}\right|\right] ds \geq R
\end{aligned}$$

for sufficiently large R. Next we claim (2.9) holds. To see this notice for $t \in [0,1]$ that

$$\phi(t) = \mu L \int_0^1 k(t,s)ds = \frac{\mu}{2}(1+t-t^2)$$

and

$$\|\phi\|_1 = \frac{\mu}{2}\|1-t-t^2\|_1 = \frac{\mu}{2}\sup_{t\in[0,1]}[(1+t-t^2) + |1-2t|] = \mu$$

and

$$\mu\left[\sup_{t\in[0,1]}\int_0^1 k(t,s)ds + \sup_{t\in[0,1]}\int_0^1 \left|\frac{\partial k(t,s)}{\partial t}\right| ds\right] = \frac{9\mu}{8}.$$

Finally notice (2.9) is satisfied with $r = 10\mu$ since $\frac{9}{8}\mu \leq \frac{r}{\psi(r+\mu)} = \frac{10}{121\mu}$ for $\mu \leq \frac{\sqrt{80}}{33}$. Thus all assumptions of Theorem 2.1 are satisfied so existence of a positive solution of the problem (2.19) is guaranted.

It is possible to obtain another existence results for (1.1) if we change some conditions on the nonlinearity f and some of conditions on the kernel k. Before formulating a next theorem we will introduce some notation.

For $p \geq 1, L^p[0,1]$ is the Banach space of all real functions x such that $|x|^p$ is Lebesgue integrable on $[0,1]$ with the norm

$$\|x\|_p^* = \left(\int_0^1 |x(t)|^p\right)^{\frac{1}{p}}.$$

The symbol $W^{n-1,p}[0,1]$ $(n \geq 2)$ denotes the set of all functions x with $x^{(n-2)}$ absolutely continuous and $x^{(n-1)} \in L^p[0,1]$.

For $x \in W^{n-1,p}[0,1]$ we introduce the following norm

$$\|x\|_{n-1,p} = \sup_{t\in[0,1]}\left[\sum_{j=0}^{n-2}|x^{(j)}(t)|\right] + \|x^{(n-1)}\|_p^*.$$

The space $(W^{n-1,p}[0,1], \|\cdot\|_{n-1,p})$ is the Banach space.

We adopt the following convention $y(t+\tau) = 0$ if $t+\tau \notin [0,1]$ and $y \in L^p[0,1]$.

A function $f : I \to (-\infty, \infty)$ is a Carathéodory function provided:

If $f = f(t, z)$, then

(i) the map $z \to f(t, z)$ is continuous for almost all $t \in [0,1]$,

(ii) the map $t \to f(t, z)$ is measurable for all $z \in [0, \infty) \times (-\infty, \infty)^{n-1}$.

If f is a Carathéodory function, by a solution to (1.1) we will mean a function x which has an absolutely continuous $(n-2)$ st derivative such that x satisfies the integral equation (1.1) almost everywhere in $[0,1]$.

Theorem 5.2 *Assume that conditions (2.1)–(2.2) and (2.5) are satisfied and* p, q *are such that* $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose the following conditions are satisfied*

(2.20) *there exist* $k^* \in C[0,1], \overline{k}_i \in L^p[0,1]$, $\tilde{c} > 0$ *and* $M > 0$ *such that*

(a) $k^*(t) > 0$ *for a.e.* $t \in [0,1]$,

(b) $\overline{k}_i(s) \geq 0$ *and* $\int_0^1 \overline{k}_i(s)ds > 0$ *for* $i = 0, 1, \ldots, n-1$ *and a.e.* $s \in [0,1]$,

(c) $Mk^*(t)\overline{k}_i(s) \leq \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| \leq \overline{k}_i(s)$ *for* $i = 0, 1, \ldots, n-1,\ t \in [0,1]$ *and a.e.* $s \in [0,1]$,

(d) *the map* $(t,s) \to \frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}$ *is measurable,*

(e) $\int_0^1 k(t,s)\, ds \leq \tilde{c}Md(t)k^*(t)$ *for* $t \in [0,1]$.

(2.21) $f : I \to (-\infty, \infty)$ *is a Carathéodory function and there exist nonnegative functions* $p_j \in L^q[0,1]$ $(j = 0, 1, \ldots, n-1)$ *and constants* $L > 0$ *and* $p_n > 0$ *such that*

(a) $f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq 0$ *for a.e.* $t \in [0,1]$ *and all* $(v_0, v_1, \ldots, v_{n-1}) \in J$,

(b) $|f(t, v_0, v_1, \ldots, v_{n-1})| \leq \sum_{i=0}^{n-2} p_i(t)|v_i| + p_{n-1}(t) + p_n|v_{n-1}|^{\frac{p}{q}}$ *for a.e.* $t \in [0,1]$ *and all* $(v_0, v_1, \ldots, v_{n-1}) \in J$,

(c) $f(t, v_0, v_1, \ldots, v_{n-1}) + L \leq \psi(v_0 + |v_1| + \ldots + |v_{n-1}|)$ *for a.e.* $t \in [0,1]$ *and all* $(v_0, v_1, \ldots, v_{n-1}) \in J$, *where* $\psi : [0, \infty) \to [0, \infty)$ *is a continuous and nondecreasing with* $\psi(u) > 0$ *for* $u > 0$,

(2.22) $\|\psi(x + |x'| + \ldots + |x^{(n-1)}|)\|_q^* \leq \varphi(\|x\|_{n-1,p})$ *with* $\varphi : [0, \infty) \to [0, \infty)$ *continuous and nondecreasing and* $x \in W^{n-1,p}[0,1]$,

(2.23) $f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq g(v_0)$ *for a.e.* $t \in [0, \infty)$ *and all* $(v_0, v_1, \ldots, v_{n-1}) \in J$ *with* $g : [0, \infty) \to [0, \infty)$ *continuous and nondecreasing and* $g(u) > 0$ *for* $u > 0$,

(2.24) there exists $r > 0$ such that $r \geq \mu L\tilde{c}$ and

$$\frac{r}{\varphi(r + \|\phi\|_{n-1,p})} \geq \mu(b + \|\overline{k}_{n-1}\|_p^*),$$

where

$$b = \sum_{i=0}^{n-2} \sup_{t\in[0,1]} \left\| \frac{\partial^i k(t,\cdot)}{\partial t^i} \right\|_p^*$$

and ϕ is defined by (2.9),

(2.25) there exist $R > 0$ and $t_0 \in [0,1]$ such that $R > r, k^*(t_0) > 0$, $d(t_0) > 0$ and

$$R \leq \mu \int_0^1 \left[k(t_0,s) + \left| \tfrac{\partial k(t_0,s)}{\partial t} \right| + \ldots + \left| \tfrac{\partial^{n-1} k(t_0,s)}{\partial t^{n-1}} \right| \right] d(t_0) g(\varepsilon RMd(s)k^*(s)) ds,$$

where ε is defined by (2.11).

Then (1.1) has a nonnegative solution $x \in W^{n-1,p}[0,1]$ with $x(t) > 0$ for a.e. $t \in [0,1]$.

Proof It is enough to show (2.12) has a solution $u \in W^{n-1,p}[0,1]$. Let $a(t) = Md(t)k^*(t)$ and let

$$\begin{aligned} K = \{u \in W^{n-1,p}[0,1] : u(t) - d(t) \left[|u'(t)| + \ldots + |u^{(n-1)}(t)| \right] \\ \geq a(t)\|u\|_{n-1,p} \quad \text{for a.e. } t \in [0,1]\}. \end{aligned}$$

Clearly K is a cone of $W^{n-1,p}[0,1]$.

Let

$$\begin{aligned} \Omega_1 &= \{x \in W^{n-1,p}[0,1] : \|x\|_{n-1,p} < r\}, \\ \Omega_2 &= \{x \in W^{n-1,p}[0,1] : \|x\|_{n-1,p} < R\} \end{aligned}$$

and

$$\tilde{f}(s, x(s) - \phi(s)) = f^*(s, x(s) - \phi(s), x'(s) - \phi'(s), \ldots, x^{(n-1)}(s) - \phi^{(n-1)}(s)),$$

where $x \in W^{n-1,p}[0,1]$ and f^* is defined by (2.12). We will show that there exist a solution $x_1 \in W^{n-1,p}[0,1]$ to the equation (2.12) with $x_1(t) \geq \phi(t)$ for $t \in [0,1]$.

Let $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to W^{n-1,p}[0,1]$ be defined by

$$Ax(t) = \mu \int_0^1 k(t,s)\tilde{f}(s, x(s) - \phi(s))\, ds.$$

Then

$$|(Ax)^{(n-1)}(t)| \leq \mu \int_0^1 \overline{k}_n(s)\tilde{f}(s, x(s) - \phi(s))\, ds \tag{2.27}$$

and

$$|Ax(t)| + |(Ax)'(t)| + \ldots + |(Ax)^{(n-2)}(t)| \le \mu \sum_{i=0}^{n-2} \int_0^1 \overline{k}_i(s)\tilde{f}(s, x(s) - \phi(s))\, ds. \tag{2.28}$$

From relations (2.27)–(2.28), (2.21)–(2.22) and Hölder's inequality it follows

$$\begin{aligned} \|Ax\|_{n-1,p} &\le \mu \sum_{i=0}^{n-1} \int_0^1 \overline{k}_i(s)\tilde{f}(s, x(s) - \phi(s))\, ds \\ &\le \mu \sum_{i=0}^{n-1} \varphi(\|x\|_{n-1,p} + \|\phi\|_{n-1,p})\|k_i\|_p^*. \end{aligned} \tag{2.29}$$

Note that A is well defined operator. Now we will prove

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K.$$

If $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $t \in [0,1]$, then (2.20), (2.5) and (2.29) imply

$$\begin{aligned} &Ax(t) - d(t)\left[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|\right] \\ &\ge \mu d(t) \int_0^1 \left[k(t,s) + \left|\frac{\partial k(t,s)}{\partial t}\right| + \ldots + \left|\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}\right|\right] \tilde{f}(s, x(s) - \phi(s))\, ds \\ &\ge \mu d(t) M k^*(t) \left(\sum_{i=0}^{n-1} \int_0^1 \overline{k}_i(s)\tilde{f}(s, x(s) - \phi(s))\right) ds \ge a(t)\|Ax\|_{n-1,p}. \end{aligned}$$

Thus $Ax \in K$ and $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$. Now we will prove that A is a continuous operator. It is enough to show that the Niemytzki operator $H : W^{n-1,p}[0,1] \to L^q[0,1]$ defined by

$$Hx(t) = f^*(t, x(t) - \phi(t), x'(t) - \phi'(t), \ldots, x^{(n-1)}(t) - \phi^{(n-1)}(t))$$

is continuous. The proof of the continuity of H is similar to the proof of Theorem 1.2 in [6]. Let $\{\overline{x}_\nu\}$ be a sequence of elements of $W^{n-1,p}[0,1]$ converging to $\overline{x}$ in $W^{n-1,p}[0,1]$. Then there exists a subsequence $\{x_{\nu_\lambda}^{(n-1)}(t)\}$ of the sequence $\{x_\nu^{(n-1)}(t)\}$ such that

$$\lim_{\lambda \to \infty} \overline{x}_{\nu_\lambda}^{(n-1)}(t) = \overline{x}^{(n-1)}(t) \quad \text{for a.e. } t \in [0,1].$$

Moreover, there exists a function $g \in L^p[0,1]$ with

$$|\overline{x}_{\nu_\lambda}^{(n-1)}(t)| \le g(t) \quad \text{for a.e. } t \in [0,1]$$

([6], Lemma 2.1). Hence by (2.21)(b) we conclude that there exists a function $h \in L^q[0,1]$ such that

$$\begin{gathered} |f^*(t, \overline{x}(t) - \phi(t), \overline{x}'(t) - \phi'(t), \ldots, \overline{x}^{(n-1)}(t) - \phi^{(n-1)}(t) \\ - f^*(t, \overline{x}_{\nu_\lambda}(t) - \phi(t), \overline{x}'_{\nu_\lambda}(t) - \phi'(t), \ldots, \\ \overline{x}_{\nu_\lambda}^{(n-1)}(t) - \phi^{(n-1)}(t))| \le h(t) \quad \text{for a.e. } t \in [0,1]. \end{gathered}$$

From the Lebesgue dominated convergence theorem it folows that the Niemytzki operator H is continuous at the point $\overline{x}$. We next show that A is completely continuous. Let Ω be a bounded set in $\left(W^{n-1,p}[0,1], \|\cdot\|_{n-1,p}\right)$. Then by virtue of (2.29) we have $A(\Omega)$ is bounded. We need to prove that $A(\Omega)$ is relatively compact. We will use the Arzela–Ascoli and the Riesz theorems. In fact, let $y_\nu \in A(\Omega)$ i.e.

$$y_\nu = A(x_\nu), \quad x_\nu \in \Omega.$$

Since $A(\Omega)$ is bounded in $(W^{n-1,p}[0,1], \|\cdot\|_{n-1,p})$ there exist subsequences $\{x_{\nu_\mu}^{(j)}\}$ and $\{y_{\nu_\mu}^{(j)}\}$ of sequences $\{x_\nu^{(j)}\}$ and $\{y_\nu^{(j)}\}$ uniformly convergent to $x^{(j)}$ and $y^{(j)}$ respectively for $j = 0, 1, \ldots, n-2$. Without loss of generality we can assume that the sequences $\{x_\nu^{(j)}\}$ and $\{y_\nu^{(j)}\}$ are uniformly convergent to $x^{(j)}$ and $y^{(j)}$. We will prove that there exists a subsequence $\{y_{\nu_\lambda}^{(n-1)}\}$ of the sequence $\{y_\nu^{(n-1)}\}$ such that

$$\lim_{\lambda\to\infty} \|y_{\nu_\lambda}^{(n-1)} - \overline{y}\|_p^* = 0, \quad \text{where } \overline{y} \in L^p[0,1].$$

Indeed, for fixed $\tau > 0$ we have by the Hölder inequality and the Fubini theorem that

$$\begin{aligned} &\int_0^1 \left|(Ax)^{(n-1)}(t+\tau) - (Ax)^{(n-1)}(t)\right|^p dt \le \\ &\le \mu^p \int_0^1 \left(\int_0^1 \left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t+\tau,s) - \frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|^p ds\right) dt \\ &\quad\times \int_0^1 \left(\int_0^1 |\tilde{f}(s, x(s) - \phi(s))|^q ds\right)^{\frac{p}{q}} dt \\ &\le \mu^p \varphi\left(\|x\|_{n-1,p} + \|\phi\|_{n-1,p}\right)^p \int_0^1 \left(\int_0^1 \left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t+\tau,s) - \frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|^p dt\right) ds. \end{aligned}$$

Now using the fact that translates of L^p are functions continuous in the norm we see that

$$\int_0^1 \left|(Ax)^{(n-1)}(t+\tau) - (Ax)^{(n-1)}(t)\right|^p dt \to 0$$

as $\tau \to 0$ uniformly. From the Riesz compactness theorem it folows that there exists a subsequence $\{y_{\nu_\lambda}^{(n-1)}\}$ of the sequence $\{y_\nu^{(n-1)}\}$ convergent in $L^p[0,1]$ to a function $\overline{y} \in L^p[0,1]$. It is easy to notice that

$$y^{(n-1)}(t) = \overline{y}(t) \quad \text{for a.e. } t \in [0,1].$$

So $A(\Omega)$ is relatively compact, i.e. A is completely continuous. Next we show that

$$\|Ax\|_{n-1,p} \le \|x\|_{n-1,p} \quad \text{for } x \in K \cap \partial\Omega_1. \tag{2.30}$$

Let $x \in K \cap \partial\Omega_1$, so $\|x\|_{n-1,p} = r$ and $x(t) \ge a(t)r$ for a.e. $t \in [0,1]$. The relations (2.21)–(2.22), (2.24), (2.27)–(2.29) yield

$$\sum_{j=0}^{n-2} \left|(Ax)^{(j)}(t)\right| \le \mu b \varphi(\|x\|_{n-1,p} + \|\phi\|_{n-1,p}) \tag{2.31}$$

and

$$\sum_{j=0}^{n-2} |(Ax)^{(j)}(t)| + \|Ax\|_p^* \leq \mu\varphi(\|x\|_{n-1,p} + \|\phi\|_{n-1,p})(b + \|\overline{k}_{n-1}\|_p^*) \leq r \quad (2.32)$$

By (2.31)–(2.32) and (2.24) we get

$$\|Ax\|_{n-1,p} \leq \|x\|_{n-1,p}.$$

So (2.30) holds. Using arguments similar to these in the proof of Theorem 2.1 we conclude that

$$\|Ax\|_{n-1,p} \geq \|x\|_{n-1,p} \quad \text{for } x \in K \cap \partial\Omega_2.$$

Theorem 1.1 implies A has a fixed point $x_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ i.e.

$$r \leq \|x_1\|_{n-1,p} \leq R \quad \text{and} \quad x_1(t) \geq a(t)r.$$

Thus for a.e. $t \in [0,1]$ we have $x_1(t) \geq a(t)r \geq \phi(t)$. This completes the proof of Theorem 2.3. □

References

[1] Agarwal, R. P., Grace, S. R., O'Regan, D.: *Existence of positive solutions of semipositone Fredholm integral equation.* Funkcialaj Equaciaj **45** (2002), 223–235.

[2] Agarwal, R. P., O'Regan, D.: Infinite Interval Problems For Differential, Difference and Integral Equations. *Kluwer Acad. Publishers, Dordrecht, Boston, London*, 2001.

[3] Agarwal, R. P., O'Regan, D., Wang, J. Y.: Positive Solutions of Differential, Difference and Integral Equations. *Kluwer Academic Publishers, Dordrecht, Boston, London*, 1999.

[4] Deimling, K.: Nonlinear Functional Analysis. *Springer, New York*, 1985.

[5] Guo, D., Lakshmikannthan, V.: Nonlinear Problems in Abstract Cones. *Academic Press, San Diego*, 1988.

[6] Galewski, A.: *On a certain generalization of the Krasnosielskii theorem.* J. Appl. Anal. **1** (2003), 139–147.