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Continuous Dependence of Inverse Fundamental Matrices of Generalized Linear Ordinary Differential Equations on a Parameter

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Abstract

The problem of continuous dependence for inverses of fundamental matrices in the case when uniform convergence is violated is presented here.

Key words: Generalized linear ordinary differential equations, fundamental matrix, adjoint equation, continuous dependence on a parameter, emphatic convergence.

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1 Introduction

In this work we are dealing with the problem of continuous dependence for inverses of fundamental matrices. We make use of the results from [A] and from [T1, chapter 3].

In the second section a survey of known results concerning systems of generalized linear ordinary differential equations, fundamental matrix and adjoint equation is given. Main results of [A] and [T1, chapter 3] are presented here, too.

Our main result is formulated in Theorem 4. The case when uniform convergence is violated is presented here.

1.1 Preliminaries

The following notations and definitions will be used throughout this text: $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ is the set of real numbers; $\mathbb{R}^{m\times n}$ is the space of real $m \times n$ matrices $B = (b_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ with the norm

$$|B| = \max_{j=1,\dots,n} \sum_{i=1}^{m} |b_{ij}|\,;$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ stands for the set of real column n-vectors $b = (b_i)_{i=1}^n$.

For a matrix $B \in \mathbb{R}^{n\times n}$, $\det B$ denotes the determinant of B. If $\det B \neq 0$, then the matrix inverse to B is denoted by B^{-1}. B^T is the matrix transposed to B. The symbol I stands for the identity matrix and 0 for the zero matrix.

If $a, b \in \mathbb{R}$ are such that $-\infty < a < b < +\infty$, then $[a, b]$ stands for the closed interval $\{x \in \mathbb{R};\ a \le x \le b\}$, (a, b) is its interior and $(a, b]$, $[a, b)$ are the corresponding half-closed intervals.

The sets $D = \{t_0, t_1, t_2, \dots, t_m\}$ of points in the closed interval $[a, b]$ such that $a = t_0 < t_1 < t_2 < \dots < t_m = b$ are called divisions of $[a, b]$. The set of all divisions of the interval $[a, b]$ is denoted by $\mathcal{D}[a, b]$.

Let $B : [a, b] \to \mathbb{R}^{m\times n}$ be a matrix valued function. Its variation $\operatorname{var}_a^b B$ on the interval $[a, b]$ is defined by

$$\operatorname{var}_a^b B = \sup_{D\in\mathcal{D}[a,b]} \sum_{i=1}^{m} |B(t_i) - B(t_{i-1})|\,.$$

If $\operatorname{var}_a^b B < +\infty$, we say that the function B is of bounded variation on the interval $[a, b]$. $\mathbf{BV}^{m\times n}[a, b]$ denotes the set of all $m \times n$ matrix valued functions of bounded variation on $[a, b]$. We will write $\mathbf{BV}^n[a, b]$ instead of $\mathbf{BV}^{n\times 1}[a, b]$. For further details concerning the space $\mathbf{BV}^{m\times n}[a, b]$, see e.g. [T2].

We will write briefly $B(t+) = \lim_{\tau\to t+} B(\tau)$, $B(s-) = \lim_{\tau\to s-} B(\tau)$ and $\Delta^+B(t) = B(t+) - B(t)$, $\Delta^-B(s) = B(s) - B(s-)$, $\Delta B(r) = B(r+) - B(r-)$ for $t \in [a, b)$, $s \in (a, b]$, $r \in (a, b)$.

If a sequence of $m \times n$ matrix valued functions $\{B_k(t)\}_{k=1}^\infty$ converges uniformly to a matrix valued function $B_0(t)$ on $[c, d] \subset [a, b]$, i.e.

$$\lim_{k\to\infty} \sup_{t\in[c,d]} |B_k(t) - B_0(t)| = 0,$$

we write

$$B_k \rightrightarrows B_0 \quad \text{on } [c, d].$$

We say that $\{B_k(t)\}_{k=1}^\infty$ converges locally uniformly to $B_0(t)$ on a set $M \subset [a, b]$, if $B_k \rightrightarrows B_0$ on each closed subinterval $J \subset M$.

We say that a proposition $P(n)$ holds for almost all (briefly a.a.) $n \in \mathbb{N}$ if it is true for all $n \in \mathbb{N} \setminus K$ where K is a finite set.

1.2 Kurzweil–Stieltjes integral

In this subsection we will recall the definition of the Kurzweil–Stieltjes integral (shortly KS-integral). We will work with the usual KS-integral which is equivalent to Perron–Stieltjes integral; cf. [STV, I.4.5], [T2, section 5].

Let $-\infty < a < b < +\infty$. For given $m \in \mathbb{N}$, a division $D = \{t_0, t_1, \ldots, t_m\} \in \mathcal{D}[a,b]$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m$, the couple $P = (D, \xi)$ is called a partition of $[a,b]$ if

$$t_{j-1} \le \xi_j \le t_j \quad \text{for all } j = 1, 2, \ldots, m.$$

The set of all partitions of the interval $[a,b]$ is denoted by $\mathcal{P}[a,b]$.

An arbitrary positive valued function $\delta : [a,b] \to (0, +\infty)$ is called a gauge on $[a,b]$. Given a gauge δ on $[a,b]$, the partition

$$P = (D, \xi) = \big(\{t_0, t_1, \ldots, t_m\},\, (\xi_1, \xi_2, \ldots, \xi_m)\big) \in \mathcal{P}[a,b]$$

is said to be δ-fine, if

$$[t_{j-1}, t_j] \subset \big(\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)\big) \quad \text{for all } j = 1, 2, \ldots, m.$$

The set of all δ-fine partitions of the interval $[a,b]$ is denoted by $\mathcal{A}(\delta; [a,b])$.

For functions $f, g : [a,b] \to \mathbb{R}$ and a partition $P \in \mathcal{P}[a,b]$,

$$P = \big(\{t_0, t_1, \ldots, t_m\},\, (\xi_1, \xi_2, \ldots, \xi_m)\big)$$

we define

$$S_P(f\, \Delta g) = \sum_{i=1}^{m} f(\xi_i)[g(t_i) - g(t_{i-1})].$$

We say, that $I \in \mathbb{R}$ is the KS-integral of f with respect to g from a to b if $\forall \varepsilon > 0\ \exists \delta : [a,b] \to (0, +\infty)\ \forall P \in \mathcal{A}(\delta; [a,b]) : |I - S_P(f\, \Delta g)| < \varepsilon$. In such a case we write $I = \int_a^b f \mathrm{d}g$ or $I = \int_a^b f(t) \mathrm{d}g(t)$.

It is known (cf. [T2, 5.20, 5.15]) that the KS-integral $\int_a^b f \mathrm{d}g$ exists, e.g. if $f \in \mathbf{BV}[a,b]$ and $g \in \mathbf{BV}[a,b]$. For the basic properties of the KS-integral, see [T2] and [STV].

If $F : [a,b] \to \mathbb{R}^{m \times n}$, $G : [a,b] \to \mathbb{R}^{n \times p}$ and $H : [a,b] \to \mathbb{R}^{p \times m}$ are matrix valued functions, then the symbols

$$\int_a^b F \mathrm{d}[G] \quad \text{and} \quad \int_a^b \mathrm{d}[H]\, F$$

stand for the matrices

$$\Big(\sum_{j=1}^{n} \int_a^b f_{ij}\, \mathrm{d}[g_{jk}]\Big)_{\substack{i=1,\ldots,m \\ k=1,\ldots,p}} \quad \text{and} \quad \Big(\sum_{i=1}^{m} \int_a^b f_{ki}\, \mathrm{d}[h_{ij}]\Big)_{\substack{k=1,\ldots,p \\ j=1,\ldots,n}},$$

whenever all the integrals appearing in the sums exist. Since the integral of a matrix valued function with respect to a matrix valued function is a matrix whose elements are sums of KS-integrals of real functions with respect to real functions, it is easy to reformulate all the statements from section 5 in [T2] for matrix valued functions (cf. [STV], I.4).

2 Generalized linear differential equations and the adjoint equation

Here we describe some fundamental properties of generalized linear differential equations, fundamental matrices and adjoint equations. More detailed information can be found in [STV]. We restrict ourselves to the interval $[0,1]$. The modification to the case of an arbitrary closed interval $[a,b] \subset \mathbb{R}$ in place of $[0,1]$ is evident.

2.1 Definition and basic properties

Assume that $A \in \mathbf{BV}^{n\times n}[0,1]$ and consider the equation

$$x(t) = x(s) + \int_s^t \mathrm{d}[A]\, x. \tag{2.1}$$

Let $[a,b] \subset [0,1]$. We say that a function $x : [a,b] \to \mathbb{R}^n$ is a solution of (2.1) on $[a,b]$ if there exists the KS-integral $\int_a^b \mathrm{d}[A]\, x \in \mathbb{R}^n$ and (2.1) holds for all $t, s \in [a,b]$.

Moreover, if $t_0 \in [a,b]$ and $\tilde{x} \in \mathbb{R}^n$ are given, we say that $x : [a,b] \to \mathbb{R}^n$ is a solution of the initial value problem (2.1), $x(t_0) = \tilde{x}$ on $[a,b]$ if it is a solution of (2.1) on $[a,b]$ and $x(t_0) = \tilde{x}$, i.e. if

$$x(t) = \tilde{x} + \int_{t_0}^t \mathrm{d}[A]\, x \tag{2.2}$$

for all $t \in [a,b]$.

Notice that, under the assumption $A \in \mathbf{BV}^{n\times n}[0,1]$, each solution of the equation (2.1) on $[0,1]$ is of bounded variation on $[0,1]$ (see [STV, III.1.3]).

Theorem 1 ([STV, III.1.4]) *Let $A \in \mathbf{BV}^{n\times n}[0,1]$. If $t_0 \in [0,1]$, then the initial value problem (2.2) possesses for any $\tilde{x} \in \mathbb{R}^n$ a unique solution $x(t)$ defined on $[0,1]$ if and only if $\det[\mathrm{I} - \Delta^- A(t)] \neq 0$ on $(t_0, 1]$ and $\det[\mathrm{I} + \Delta^+ A(t)] \neq 0$ on $[0, t_0)$.*

2.2 Fundamental matrix

Lemma 1 ([STV, III.2.10, III.2.11]) *For a given $A \in \mathbf{BV}^{n\times n}[0,1]$ such that*

$$\det[\mathrm{I} - \Delta^- A(t)] \neq 0 \text{ on } (0,1] \text{ and } \det[\mathrm{I} + \Delta^+ A(t)] \neq 0 \text{ on } [0,1) \tag{2.3}$$

there exists a unique $U : [0,1] \times [0,1] \to \mathbb{R}^{n\times n}$ such that

$$U(t,s) = \mathrm{I} + \int_s^t \mathrm{d}[A(r)]\, U(r,s)$$

for all $t, s \in [0,1]$.

Moreover, there exists a unique matrix valued function $X : [0,1] \to \mathbb{R}^{n\times n}$ *such that* $\det X(t) \neq 0$ *for* $t \in [0,1]$,

$$U(t,s) = X(t)\, X^{-1}(s) \quad \textit{for all } s,t \in [0,1] \tag{2.4}$$

and

$$X(t) = \mathrm{I} + \int_0^t \mathrm{d}[A]\, X,\ t \in [0,1]\,. \tag{2.5}$$

Furthermore, the inverse matrix $X^{-1}(t)$ *is of bounded variation on* $[0,1]$ *and it satisfies the relation*

$$X^{-1}(t) = X^{-1}(s) - X^{-1}(t)\, A(t) + X^{-1}(s)\, A(s) + \int_s^t \mathrm{d}[X^{-1}]\, A \tag{2.6}$$

for $t,s \in [0,1]$.

For a given $t_0 \in [0,1]$, the unique solution $x(t)$ of (2.2) on $[t_0,1]$ (see Theorem 1) is given by

$$x(t) = X(t)\, X^{-1}(t_0)\, \tilde{x}.$$

Definition 1 The matrix $X : [0,1] \to \mathbb{R}^{n\times n}$ given by Lemma 1 is called the fundamental matrix of the homogenous generalized linear differential equation (2.1) or briefly the fundamental matrix corresponding to the given matrix function A.

2.3 Adjoint equation

The equation (2.6), which is satisfied by the matrix function X^{-1}, is not a generalized linear differential equation of the type (2.1). This leads us to the consideration of adjoint equations, i.e. the equations of the form

$$y^T(t) = y^T(s) - y^T(t)\, A(t) + y^T(s)\, A(s) + \int_s^t \mathrm{d}[y^T]\, A\,. \tag{2.7}$$

Theorem 2 ([ST, 2.7]) *Let* $A \in \mathbf{BV}^{n\times n}[0,1]$ *satisfy* (2.3). *Then the initial value problem* (2.7), $y^T(1) = \tilde{y}^T$ *has for every* $\tilde{y} \in \mathbb{R}^n$ *a unique solution* $y : [0,1] \to \mathbb{R}^n$ *on* $[0,1]$. *This solution is of bounded variation on* $[0,1]$ *and is given on* $[0,1]$ *by*

$$y^T(s) = \tilde{y}^T X(1) X^{-1}(s). \tag{2.8}$$

Moreover, every solution $y^T(t)$ *of the equation (2.7) on* $[0,1]$ *possesses the onesided limits* $y^T(t+)$, $y^T(t-)$ *where the relations*

$$\begin{aligned} y^T(t+) &= y^T(t) - y^T(t+)\, \Delta^+ A(t) \quad \textit{for all } t \in [0,1)\,, \\ y^T(t-) &= y^T(t) + y^T(t-)\, \Delta^- A(t) \quad \textit{for all }, t \in (0,1] \end{aligned} \tag{2.9}$$

hold.

2.4 Convergence results for generalized linear ordinary differential equations

In [T1, Theorem 3.3.2] the continuous dependence of the fundamental matrix X of (2.1) on a parameter was described. Let us recall this result. To this aim we need the following notations.

Notation 1 Let a sequence $\{A_k\}_{k=1}^{\infty} \subset \mathbf{BV}^{n\times n}[0,1]$ and $A_0 \in \mathbf{BV}^{n\times n}[0,1]$. For a $k \in \mathbb{N}$ and an arbitrary closed interval $J = [\alpha, \beta] \subset [0,1]$, define

$$A_k^J(t) = A_k(t) - A_k(\alpha) \quad \text{for } k \in \mathbb{N}_0,\ t \in J.$$

Theorem 3 ([T1, Theorem 3.3.2]) *Let $A_k \in \mathbf{BV}^{n\times n}[0,1]$ for $k \in \mathbb{N}_0$ and $\det[\mathrm{I} - \Delta^- A_0(t)] \neq 0$ on $(0,1]$. Furthermore, assume that there is a finite set $D \subset [0,1]$ such that:*

$$A_k^J(s) \rightrightarrows A_0^J(s) \text{ on } J \text{ for any closed interval } J \subset [0,1] \setminus D, \tag{2.10}$$

$$\sup_{k\in\mathbb{N}} \operatorname{var} A_k < +\infty \text{ and } \det[\mathrm{I} - \Delta^- A_k(t)] \neq 0 \text{ for all } t \in D \text{ and for a.a. } k \in \mathbb{N}, \tag{2.11}$$

$$\left.\begin{aligned}
&\text{if } \tau \in D, \text{ then } \forall \xi \in \mathbb{R}^n \text{ and } \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that}\\
&\forall \delta' \in (0,\delta)\ \exists k_0 \in \mathbb{N} \text{such that the relations}\\
&\qquad |u_k(\tau) - u_k(\tau - \delta') - \Delta^- A_0(\tau)[\mathrm{I} - \Delta^- A_0(\tau)]^{-1}\xi| < \varepsilon,\\
&\qquad |v_k(\tau + \delta') - v_k(\tau) - \Delta^+ A_0(\tau)\xi| < \varepsilon\\
&\text{are satisfied } \forall k \geq k_0 \text{ and } \forall u_k, v_k \text{ such that}\\
&\qquad |\xi - u_k(\tau - \delta')| \leq \delta,\ |\xi - v_k(\tau)| \leq \delta \text{ and}\\
&\qquad u_k(t) = u_k(\tau - \delta') + \int_{\tau-\delta'}^{t} \mathrm{d}[A_k]\, u_k(s) \quad \text{on } [\tau - \delta', \tau],\\
&\qquad v_k(t) = v_k(\tau) + \int_{\tau}^{t} \mathrm{d}[A_k]\, v_k(s) \quad \text{on } [\tau, \tau + \delta'].
\end{aligned}\right\} \tag{2.12}$$

Then for a.a. $k \in \mathbb{N}$ the fundamental matrix X_k corresponding to A_k is defined on $[0,1]$ and

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \text{on } [0,1]. \tag{2.13}$$

A similar assertion concerning inverses of fundamental matrices will be proved in Theorem 4.

Remark 1 Theorem 3 is a slightly modified version of [T1, Theorem 3.3.2]. Notation is simplified and, in particular, from the proof given in [T1, Theorem 3.3.2] it follows that the assumption $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ on $(0,1]$ for all $k \in \mathbb{N}$ used in [T1] is not necessary and it can be replaced by a weaker one, i.e. $\det[\mathrm{I} - \Delta^- A_k(t)] \neq 0$ for all $t \in D$, for a.a. $k \in \mathbb{N}$.

Conditions (2.10)–(2.12) characterize the concept of emphatic convergence introduced by J. Kurzweil (cf. [K2, Definition 4.1]). For more details see [T1, Definition 3.2.8] or [S].

In the proof of Theorem 4 the following two lemmas are needed. The former one is from [A, Lemma 2]. The latter one is based on [T1, Theorem 3.2.5] and on [A, Lemma 2].

Lemma 2 ([A, Lemma 2]) *Let* $-\infty < a < b < +\infty$, $A_k \in \mathbf{BV}^{n\times n}[a,b]$ *for* $k \in \mathbb{N}_0$ *and let* $\det[\mathrm{I} + \Delta^+ A_0(t)] \neq 0$ *on* $[a,b)$ *and* $\det[\mathrm{I} - \Delta^- A_0(t)] \neq 0$ *on* $(a,b]$. *If* $X_k \rightrightarrows X_0$ *on* $[a,b]$, *then* $X_k^{-1} \rightrightarrows X_0^{-1}$ *on* $[a,b]$.

Lemma 3 *Let* $-\infty < a < b < +\infty$, $A_k \in \mathbf{BV}^{n\times n}[a,b]$ *for* $k \in \mathbb{N}_0$ *and* $\det[\mathrm{I} + \Delta^+ A_0(t)] \neq 0$ *on* $[a,b)$ *and* $\det[\mathrm{I} - \Delta^- A_0(t)] \neq 0$ *on* $(a,b]$. *Assume that the sequence* $\{A_k\}_{k=1}^{\infty}$ *satisfies the following two conditions*

(i) $\sup\limits_{k\in\mathbb{N}} \operatorname{var}_a^b A_k < +\infty$,

(ii) $[A_k(t) - A_k(a)] \rightrightarrows [A_0(t) - A_0(a)]$ *on* $[a,b]$.

Then for $k = 0$ *and for a.a.* $k \in \mathbb{N}$ *there exists the fundamental matrix* X_k *corresponding to* A_k *on* $[a,b]$ *and* $X_k^{-1} \rightrightarrows X_0^{-1}$ *on* $[a,b]$.

3 Main result

Theorem 3 deals with a sequence of fundamental matrices. According to definition, each fundamental matrix corresponding to a given matrix function A fulfills for all $s, t \in [0,1]$ the equation

$$X(t) = X(s) + \int_s^t \mathrm{d}[A]\, X.$$

This fact is essentially used in the proof of Theorem 4. Furthermore, we take into account that the inverse of fundamental matrix $X^{-1}(t)$ satisfies relation

$$X^{-1}(t) = X^{-1}(0) - X^{-1}(t)\, A(t) + X^{-1}(0)\, A(0) + \int_0^t \mathrm{d}[X^{-1}]\, A, \tag{3.14}$$

which is adjoint to (2.5), see (2.6) and (2.7).

We want to prove assertion analogous to Theorem 3 for inverses of fundamental matrices. To this aim it is necessary to suppose also the regularity of $[\mathrm{I} + \Delta^+ A_0(t)]$ for each $t \in [0,1)$ and the condition (3.15) which is a modification of (2.12) for relation (3.14). This is our main result.

Theorem 4 *Let the assumptions of Theorem 3 are satisfied. Furthermore assume that* $\det[\mathrm{I} + \Delta^+ A_0(t)] \neq 0$ *on* $[0,1)$ *and the following conditions hold:*

$$\left.\begin{aligned}
&\textit{if } \tau \in D, \textit{ then } \forall \eta \in \mathbb{R}^n \textit{ and } \forall \varepsilon > 0\ \exists \delta > 0 \\
&\quad \textit{such that } \forall \delta' \in (0,\delta)\ \exists k_0 \in \mathbb{N} \textit{such that the relations} \\
&\qquad |w_k^T(\tau) - w_k^T(\tau - \delta') + \eta^T\, \Delta^- A_0(\tau)| < \varepsilon\,, \\
&\qquad |z_k^T(\tau + \delta') - z_k^T(\tau) + \eta^T\, [\mathrm{I} + \Delta^+ A_0(\tau)]^{-1}\, \Delta^+ A_0(\tau)| < \varepsilon \\
&\quad \textit{are satisfied } \forall k \geq k_0 \textit{ and } \forall w_k,\, z_k \in \mathbb{R}^n \textit{ fulfilling (3.16), (3.17)} \\
&\quad \textit{and such that} \\
&\qquad |\eta^T - w_k^T(\tau - \delta')| \leq \delta,\ |\eta^T - z_k^T(\tau)| \leq \delta,
\end{aligned}\right\} \tag{3.15}$$

where

$$\begin{aligned} w_k^T(t) &= w_k^T(\tau - \delta') - w_k^T(t) A_k(t) + w_k^T(\tau - \delta') A_k(\tau - \delta') \\ &\quad + \int_{\tau-\delta'}^t \mathrm{d}[w_k^T]\, A_k \ \textit{ on } [\tau - \delta', \tau], \end{aligned} \tag{3.16}$$

$$z_k^T(t) = z_k^T(\tau) - z_k^T(t)\, A_k(t) + z_k^T(\tau) A_k(\tau) + \int_\tau^t \mathrm{d}[z_k^T]\, A_k \ \textit{ on } [\tau, \tau + \delta']. \tag{3.17}$$

Then for a.a. $k \in \mathbb{N}$ *the fundamental matrices* X_k *corresponding to* A_k *and their inverses* X_k^{-1} *are defined on* $[0,1]$,

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{on } [0,1] \tag{3.18}$$

and

$$\lim_{k\to\infty} X_k^{-1}(t) = X_0^{-1}(t) \quad \textit{on } [0,1]. \tag{3.19}$$

Moreover, (3.19) *holds locally uniformly on* $[0,1] \setminus D$.

Proof First notice that Lemma 3 implies that (3.19) holds locally uniformly on $[0,1] \setminus D$ and (3.18) immediately follows from Theorem 3.

Assume that $D = \{\tau\}$, where $\tau \in (0,1)$; i.e. D consists of one point $\tau \in (0,1)$ only and $m = 1$.

Recall that the existence of the fundamental matrices X_k for a.a. $k \in \mathbb{N}$ and (3.18) immediately follows from Theorem 3. Since each fundamental matrix is regular, we get the existence of X_k^{-1} for a.a. $k \in \mathbb{N}$. For $\tilde{y} \in \mathbb{R}^n$ and for a.a. $k \in \mathbb{N}_0$, denote by y_k^T the solution of the equation

$$y_k^T(t) = \tilde{y}^T - y_k^T(t)\, A_k(t) + \tilde{y}^T\, A_k(0) + \int_0^t \mathrm{d}[y_k^T]\, A_k \quad \text{on } [0,1]. \tag{3.20}$$

The rest of the proof splits into three steps. First, we prove that (3.19) is true for $t \in [0,\tau)$, then for $t = \tau$ and finally for $t \in (\tau, 1]$.

• Step 1. Let $\alpha \in (0,\tau)$ be given. Then by Lemma 3 the relation (3.19) holds uniformly on $[0,\alpha]$. Therefore (3.19) is true for any $t \in [0,\tau)$.

• Step 2. Now we will prove, that (3.19) is true also for $t=\tau$. For each $\delta^{'} \in (0,\tau)$ and $k \in \mathbb{N}$ we get using (2.9) the estimate

$$\begin{aligned}|y_0^T(\tau) - y_k^T(\tau)| &\le |y_0^T(\tau) + y_0^T(\tau-)\,\Delta^- A_0(\tau) - y_0^T(\tau-\delta^{'})| \\ &+ |y_0^T(\tau-\delta^{'}) - y_k^T(\tau-\delta^{'})| + |y_k^T(\tau-\delta^{'}) - y_0^T(\tau-)\,\Delta^- A_0(\tau) - y_k^T(\tau)| \\ &= |y_0^T(\tau-) - y_0^T(\tau-\delta^{'})| + |y_0^T(\tau-\delta^{'}) - y_k^T(\tau-\delta^{'})| \\ &+ |y_k^T(\tau) - y_k^T(\tau-\delta^{'}) + y_0^T(\tau-)\,\Delta^- A_0(\tau)|\,.\end{aligned}$$

Let $\varepsilon > 0$ be given. According to (3.15) we can choose $\delta \in (0,\varepsilon)$ in such a way that for all $\delta^{'} \in (0,\delta)$ there exists $k_1 \in \mathbb{N}$ with the property

$$|w_k^T(\tau) - w_k^T(\tau-\delta^{'}) + y_0^T(\tau-)\,\Delta^- A_0(\tau)| < \varepsilon \tag{3.21}$$

holds for any $k \ge k_1$ and for each solution $w_k^T(t)$ of (3.16) fulfilling

$$|y_0^T(\tau-) - w_k^T(\tau-\delta^{'})| \le \delta.$$

Set $w_k^T(t) = y_k^T(t)$ on $[\tau-\delta^{'},\tau]$. Choose $\delta^{'} \in (0,\delta)$ so that

$$|y_0^T(\tau-) - y_0^T(\tau-\delta^{'})| < \frac{\delta}{2}\,.$$

Considering that $y_k^T(t) \to y_0^T(t)$ on $[0,\tau)$ as $k \to \infty$ we get the existence of a $k_0 \in \mathbb{N}$, $k_0 \ge k_1$ such that $|y_0^T(\tau-\delta^{'}) - y_k^T(\tau-\delta^{'})| < \frac{\delta}{2}$ for all $k \ge k_0$. Therefore the estimate

$$|y_0^T(\tau-) - y_k^T(\tau-\delta^{'})| \le |y_0^T(\tau-) - y_0^T(\tau-\delta^{'})| + |y_0^T(\tau-\delta^{'}) - y_k^T(\tau-\delta^{'})| < \delta$$

is true for $k \ge k_0$. By (3.21) we have

$$|y_k^T(\tau) - y_k^T(\tau-\delta^{'}) + y_0^T(\tau-)\,\Delta^- A_0(\tau)| < \varepsilon.$$

To summarize, we have

$$|y_0^T(\tau) - y_k^T(\tau)| < \frac{\delta}{2} + \frac{\delta}{2} + \varepsilon < 2\,\varepsilon \quad \text{for all } k \ge k_0\,,$$

i.e. $y_k^T(\tau) \to y_0^T(\tau)$ for $k \to \infty$.

• Step 3. Proof of the convergence on $(\tau,1]$ consists of two parts. First, we show that there is a $\delta > 0$ such that $y_k^T(t) \to y_0^T(t)$ on $(\tau,\tau+\delta)$ as $k \to \infty$. Then we extend this result to the whole interval $(\tau,1]$. Let $\varepsilon > 0$ be given and let $\delta_0 \in (0,\varepsilon)$ be such that

$$|y_0^T(s) - y_0^T(\tau+)| < \varepsilon \quad \text{for all } s \in (\tau,\tau+\delta_0).$$

By the assumption (3.15), there exists $\delta \in (0,\delta_0)$ such that for all $\delta^{'} \in (0,\delta)$ there exists $k_1 = k_1(\delta^{'}) \in \mathbb{N}$ and such that

$$|z_k^T(\tau+\delta^{'}) - z_k^T(\tau) + y_0^T(\tau)\,[\mathrm{I} + \Delta^+ A_0(\tau)]^{-1}\,\Delta^+ A_0(\tau)| < \varepsilon \tag{3.22}$$

is true for each solution $z_k^T(t)$ of (3.17) with the property $|y_0^T(\tau) - z_k^T(\tau)| \le \delta$. Now the distance between $y_0^T(\tau+\delta^{'})$ and $y_k^T(\tau+\delta^{'})$ can be estimated. In view of (2.9) we get

$$\begin{aligned}|y_0^T(\tau+\delta^{'}) - y_k^T(\tau+\delta^{'})| &\le |y_0^T(\tau+\delta^{'}) - y_0^T(\tau) + y_0^T(\tau+)\,\Delta^+ A_0(\tau)| \\ &+ |y_0^T(\tau) - y_k^T(\tau)| + |y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A_0(\tau) - y_k^T(\tau+\delta^{'})| \\ = |y_0^T(\tau+\delta^{'}) - y_0^T(\tau+)| &+ |y_0^T(\tau) - y_k^T(\tau)| + |y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A_0(\tau) - y_k^T(\tau+\delta^{'})|.\end{aligned}$$

Considering that $y_k^T(\tau) \to y_0^T(\tau)$ for $k \to \infty$, we get the existence of $k_0 \in \mathbb{N}$, $k_0 \ge k_1$ such that $|y_0^T(\tau) - y_k^T(\tau)| < \delta$ for all $k \ge k_0$. Since $\tau + \delta^{'} \in (\tau, \tau+\delta_0)$, we have $|y_0^T(\tau+\delta^{'}) - y_0^T(\tau+)| < \varepsilon$. Setting $z_k^T(t) = y_k^T(t)$ on $[\tau, \tau+\delta^{'}]$, we get by (3.22) the relation

$$|y_k^T(\tau) - y_0^T(\tau+)\,\Delta^+ A_0(\tau) - y_k^T(\tau+\delta^{'})| < \varepsilon \quad \text{for all } k \ge k_0\,.$$

To summarize, for any $k \ge k_0$ the estimate

$$|y_0^T(\tau+\delta^{'}) - y_k^T(\tau+\delta^{'})| \le \varepsilon + \delta + \varepsilon < 3\,\varepsilon$$

is valid, as well. Therefore $y_k^T(t) \to y_0^T(t)$ on $(\tau, \tau+\delta)$ as $k \to \infty$. Now, choose an arbitrary σ in $(\tau, \tau+\delta)$. Making use of Lemma 3 with $[a,b] = [\sigma, 1]$ the proof of this step can be completed.

Having solution $y_k^T(t)$ to (3.20) for each $\tilde{y} \in \mathbb{R}^n$, we can determine the matrix function $X_k^{-1}(t)$ from $y_k^T(t)$ using (2.8). Indeed, since $X_k(1)$ is regular, we can choose $\tilde{y}^T$ in such a way that $y_k^T(t)$ is i-th row of $X_k^{-1}(t)$. This consideration completes the proof of the validity of (3.19) for any $t \in [0,1]$.

The extension to the case $m > 1$ is obvious. □

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