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Periodic BVP with ϕ-Laplacian and Impulses

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Abstract

The paper deals with the impulsive boundary value problem

$$\frac{d}{dt}[\phi(y'(t))] = f(t, y(t), y'(t)), \quad y(0) = y(T), \quad y'(0) = y'(T),$$
$$y(t_i+) = J_i(y(t_i)), \quad y'(t_i+) = M_i(y'(t_i)), \quad i = 1, \dots m.$$

The method of lower and upper solutions is directly applied to obtain the results for this problems whose right-hand sides either fulfil conditions of the sign type or satisfy one-sided growth conditions.

Key words: ϕ-Laplacian, impulses, lower and upper functions, periodic boundary value problem.

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0 Introduction

In this paper we study the existence of solutions to the following problem

$$\frac{d}{dt}[\phi(y'(t))] = f(t, y(t), y'(t)), \tag{0.1}$$

$$y(0) = y(T), \quad y'(0) = y'(T), \tag{0.2}$$

$$y(t_i+) = J_i(y(t_i)), \quad y'(t_i+) = M_i(y'(t_i)), \quad i = 1, \dots m, \tag{0.3}$$

where $f \in Car([0,T] \times R^2)$, ϕ is an increasing homeomorfismus, $\phi(R) = R$. $J_i \in C(R)$, $M_i \in C(R)$ and

$$y'(t_i) = y'(t_i-) = \lim_{t \to t_i-} y'(t), \quad y'(0) = y'(0+) = \lim_{t \to 0+} y'(t).$$

Let

$$\sigma_1(t_i) < x < \sigma_2(t_i) \Rightarrow J_i(\sigma_1(t_i)) < J_i(x) < J_i(\sigma_2(t_i)), \quad i = 1, \dots, m \tag{0.4}$$

hold. We will assume one of the following properties of M_i, either

$$M_i \text{ is increasing on } R, \ M_i(R) = R \quad i = 1, \dots, m, \tag{0.5}$$

or only

$$\begin{aligned} &y \le \sigma_1'(t_i) \Rightarrow M_i(y) \le M_i(\sigma_1'(t_i)), \\ &y \ge \sigma_2'(t_i) \Rightarrow M_i(y) \ge M_i(\sigma_2'(t_i)), \quad i = 1, \dots, m, \end{aligned} \tag{0.6}$$

In the mathematical literature we can find a lot of papers studying the equation (0.1) with various types of linear or nonlinear boundary conditions. Particularly, the existence results for such problems have been proved e.g. in [1–4].

On the other hand there are papers giving the existence theorems for impulsive problems to the second order differential equations $x'' = f(t, x, x')$. Some of them are based on the method of lower and upper functions ([5–14]). The aim of this paper is to join problems with ϕ-Laplacian and problems with impulses and to extend the method of lower and upper functions for the problem (0.1)–(0.3). Here, the method of lower and upper solutions is directly applied to obtain the results for problems (0.1)–(0.3) whose right-hand sides either fulfil conditions of the sign type or satisfy one-sided growth conditions.

The sections are organized as follows. In Section 1, we begin by definitions of solution and lower and upper functions of the problem (0.1)–(0.3). We state two existence theorems for the problem (0.1)–(0.3) with right-hand sides satisfying conditions of the sign type and one-sided growth conditions and show some applications of these theorems on the concrete problems. In Section 2, we state and prove the existence result for problems with bounded right-hand sides. This problem is reduced to a fixed point problem and using the Schauder fixed point theorem, we show its solvability. In Section 3, we use the previous result to prove the existence theorems which are stated in Section 1.

1 Formulation of the solution and main results

For a real valued function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup_{t \in [0,T]} \operatorname{ess} |u(t)|.$$

Let $m \in N$ and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$ be a division of the interval $J = [0,T]$. We denote $\Delta = \{t_1, t_2, \dots, t_m\}$ and define $C^1_\Delta(J)$, resp. $C_\Delta(J)$, as the set of functions $u : J \to R$,

$$u(t) = \begin{cases} u^{[0]}(t), & t \in [0, t_1], \\ u^{[1]}(t), & t \in (t_1, t_2], \\ \dots & \dots \\ u^{[m]}(t), & t \in (t_m, T], \end{cases}$$

where $u^{[i]} \in C^1[t_i, t_{i+1}]$, resp. $u^{[i]} \in C[t_i, t_{i+1}]$, for $i = 0, 1, \dots, m$. Moreover, $AC_\Delta(J)$ stands for the set of functions $u \in C_\Delta(J)$ being absolutely continuous on each subinterval (t_i, t_{i+1}), $i = 0, 1, \dots, m$. For $u \in C^1_\Delta(J)$ we write

$$\|u\|_{C^1_\Delta(J)} = \|u\|_\infty + \|u'\|_\infty.$$

Definition 1 A *solution of the problem* (0.1)–(0.3) is a function $y \in C^1_\Delta(J)$ such that $\phi(y') \in AC_\Delta(J)$, y fulfils equation (0.1) for a.e. $t \in J$, further satisfies the periodic conditions (0.2) and the impulsive conditions (0.3).

Definition 2 Functions $\sigma_1 \in C^1_\Delta(J)$, $\sigma_2 \in C^1_\Delta(J)$ are respectively called *lower and upper functions of the problem* (0.1)–(0.3), if $\phi(\sigma_1'), \phi(\sigma_2') \in AC_\Delta(J)$ and

$$(\phi(\sigma_1'(t)))' \ge f(t, \sigma_1(t), \sigma_1'(t)), \quad (\phi(\sigma_2'(t)))' \le f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for a.e. } t \in J,$$
$$\sigma_1(0) = \sigma_1(T), \quad \sigma_2(0) = \sigma_2(T),$$
$$\sigma_1'(0) \ge \sigma_1'(T), \quad \sigma_2'(0) \le \sigma_2'(T),$$
$$\sigma_1(t_i+) = J_i(\sigma_1(t_i)), \quad \sigma_2(t_i+) = J_i(\sigma_2(t_i)), \quad i = 1, \dots, m,$$
$$\sigma_1'(t_i+) \ge M_i(\sigma_1'(t_i)), \quad \sigma_2'(t_i+) \le M_i(\sigma_2'(t_i)), \quad i = 1, \dots, m.$$

Remark 1.1 If $M_i(0) = 0$ for $i = 1, \dots, m$ and $r_1 \in R$ is such that $J_i(r_1) = r_1$ for $i = 1, \dots, m$ and

$$f(t, r_1, 0) \le 0 \quad \text{for a.e. } t \in J,$$

then $\sigma_1(t) \equiv r_1$ on J is a lower function of the problem (0.1)–(0.3). Similarly, if $r_2 \in R$ is such that $J_i(r_2) = r_2$ for $i = 1, \dots, m$ and

$$f(t, r_2, 0) \ge 0 \quad \text{for a.e.} t \in J,$$

then $\sigma_2(t) \equiv r_2$ on J is an upper function of the problem (0.1)–(0.3).

The main results of this paper are contained in the following two theorems. In Theorem 1.1 we suppose that the right-hand side f of equation (0.1) fulfils conditions of the sign type.

Theorem 1.1 *Let lower and upper functions of the problem (0.1)–(0.3) exist and satisfy (0.4), (0.6) and $\sigma_1 \le \sigma_2$ on J. Let there exist functions $\varphi_1, \varphi_2 \in C_\Delta(J)$ such that $\phi(\varphi_1), \phi(\varphi_2) \in AC_\Delta(J)$ and*

$$\begin{aligned} &\varphi_1(0) \ge \varphi_1(T), \quad \varphi_2(0) \le \varphi_2(T), \\ &\varphi_1(t) \le \sigma_i'(t) \le \varphi_2(t), \ \textit{on } J, \ i = 1, 2, \\ &\varphi_1(t_j+) \ge M_j(\varphi_1(t_j)), \quad \varphi_2(t_j+) \le M_j(\varphi_2(t_j)), \quad j = 1, \dots, m. \end{aligned} \tag{1.7}$$

Furthermore, let φ_1, φ_2 satisfy inequalities

$$f(t, x, \varphi_1(t)) \le (\phi(\varphi_1(t)))', \quad f(t, x, \varphi_2(t)) \ge (\phi(\varphi_2(t)))' \tag{1.8}$$

for a.e. $t \in J$ and for all $x \in [\sigma_1(t), \sigma_2(t)]$.

Then the problem (0.1)–(0.3) has a solution $u \in C^1_\Delta(J)$ such that

$$\sigma_1 \le u \le \sigma_2, \quad \varphi_1 \le u' \le \varphi_2 \quad \text{on } J. \tag{1.9}$$

Remark 1.2 If $s_1 \le \sigma_j'(t)$ on J, $j = 1, 2$, is such that $M_i(s_1) = s_1$ for $i = 1, \dots, m$ and

$$f(t, x, s_1) \le 0 \quad \text{for a.e. } t \in J, \text{ for all } x \in [\sigma_1(t), \sigma_2(t)],$$

then $\varphi_1(t) \equiv s_1$ on J fulfils conditions of Theorem 1.1. If $s_2 \ge \sigma_j'(t)$ on J, $j = 1, 2$, is such that $M_i(s_2) = s_2$ for $i = 1, \dots, m$ and

$$f(t, x, s_2) \ge 0 \quad \text{for a.e. } t \in J, \text{ for all } x \in [\sigma_1(t), \sigma_2(t)],$$

then $\varphi_2(t) \equiv s_2$ on J fulfils conditions of Theorem 1.1.

Example 1.1

$$\begin{aligned}
&\tfrac{d}{dt}[\phi(x')] = t^p + x^q + (x')^r + \tfrac{\sqrt{T}}{\sqrt{t}}(x')^k, \quad x(0) = x(T), \quad x'(0) = x'(T),\\
&x(t_i+) = a_i(x(t_i))^2 + (1 - a_i(A+B))x(t_i) + ABa_i = J_i(x(t_i)),\\
&\qquad i = 1, \dots, m,\\
&x'(t_i+) = b_i(x'(t_i))^3 - b_i(D+C)(x'(t_i))^2 + (1 + b_iCD)x'(t_i) = M_i(x'(t_i)),\\
&\qquad i = 1, \dots, m,
\end{aligned} \tag{1.10}$$

$k > 0$ and $q > 0$ are odd, $p > 0$, $r > 0$, $A < 0$, $B > 0$, $C < 0$, $D > 0$. If $a_i \in [-\frac{1}{B-A}, \frac{1}{B-A}]$, $i = 1, \dots, m$, then J_i satisfy condition (0.4) for $i = 1, \dots, m$. If $b_i \in [0, \frac{4}{(D-C)^2}]$, $i = 1, \dots, m$, then M_i satisfy condition (0.6) for $i = 1, \dots, m$. $J_i(A) = A, J_i(B) = B, M_i(C) = C, M_i(D) = D$, $i = 1, \dots, m$.

If $A^q + T^p \le 0$ then $\sigma_1(t) \equiv A$ is a lower function of the problem (1.10). Function $\sigma_2(t) \equiv B$ is an upper function of the problem (1.10). Further, if $B^q + T^p \le -C^k - C^r$ and $|A|^q \le D^k + D^r$, then functions $\varphi_1(t) \equiv C$, $\varphi_2(t) \equiv D$ satisfy the conditions of Theorem 1.1, so there exists a solutions of the problem (1.10) fulfiling inequalities (1.9).

Example 1.2

$$\begin{aligned}
&\big((x')^3\big)' = \tfrac{1}{\sqrt{t}}(x'^k - \operatorname{sgn} x') + x^p + t^q, \quad k > 0,\ p > 0 \text{ are odd},\ q \ge 0,\\
&x(0) = x(3), \quad x'(0) = x'(3),\\
&x(1+) = x(1) + 1, \quad x'(1+) = x'(1) - 2,\\
&x(2+) = x(2) - 2, \quad x'(2+) = x'(2) + 2.
\end{aligned} \tag{1.11}$$

If we select functions σ_1 and σ_2 in the following way

$$\sigma_1 = \begin{cases} t+1-4\cdot 3^{\frac{q}{p}}, & t\in[0,1],\\ -t+4-4\cdot 3^{\frac{q}{p}}, & t\in(1,2],\\ t-2-4\cdot 3^{\frac{q}{p}}, & t\in(2,3], \end{cases}$$

$$\sigma_2 = \begin{cases} t-2+6\cdot 3^{\frac{q}{p}}, & t\in[0,1],\\ -t+1+6\cdot 3^{\frac{q}{p}}, & t\in(1,2],\\ t-5+6\cdot 3^{\frac{q}{p}}, & t\in(2,3], \end{cases}$$

then σ_1, σ_2 are respectively lower and upper functions of the problem (1.11). If we select functions φ_1 and φ_2 in this way

$$\varphi_1 = \begin{cases} -6^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}, & t\in[0,1],\\ -6^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}-2, & t\in(1,2],\\ -6^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}, & t\in(2,3], \end{cases}$$

$$\varphi_2 = \begin{cases} 4^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}+2, & t\in[0,1],\\ 4^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}, & t\in(1,2],\\ 4^{\frac{p+1}{k}}\cdot 3^{\frac{q}{pk}}+2, & t\in(2,3], \end{cases}$$

then these functions satisfy the conditions of Theorem 1.1, so there exists a solutions of the problem (1.11) fulfiling inequalities (1.9).

In Theorem 1.2 we impose one-sided conditions of the growth type on f.

Theorem 1.2 *Let σ_1, σ_2 be respectively lower and upper functions of the problem (0.1)–(0.3) and satisfy (0.4), (0.5) and $\sigma_1 \le \sigma_2$ on J. Assume that $k \in L(J)$ is nonnegative a.e. on $[0,T]$, $\omega \in C([0,\infty))$ is positive on $[0,\infty)$ and*

$$\int_{-\infty}^{\phi(-1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \qquad \int_{\phi(1)}^{\infty} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty$$

and

$$f(t,x,y) \le \omega(|y|)(k(t)+|y|) \text{ for a.e. } t\in J \text{ and every } (x,y)\in[\sigma_1(t),\sigma_2(t)]\times R. \tag{1.12}$$

Then the problem (0.1)–(0.3) has a solution u such that $\sigma_1 \le u \le \sigma_2$ on J.

Example 1.3

$$\begin{aligned} &\big(|x'|^{k-1}x'\big)' = \tfrac{1}{\sqrt{t}}(x'^k-1)+x^m+x'^{k+1}, \quad k>0 \text{ even},\ m>0 \text{ odd},\\ &x(0)=x(3), \quad x'(0)=x'(3),\\ &x(1+)=x(1)+1, \quad x'(1+)=x'(1)-2,\\ &x(2+)=x(2)-2, \quad x'(2+)=x'(2)+2. \end{aligned} \tag{1.13}$$

Define functions $\sigma_i : J \to R$, $i = 1,2$

$$\sigma_1(t) = \begin{cases} t-3 & \text{if } t \in [0,1], \\ -t & \text{if } t \in [1,2], \\ t-6 & \text{if } t \in [2,3], \end{cases} \qquad \sigma_2(t) = \begin{cases} t+1 & \text{if } t \in [0,1], \\ -t+4 & \text{if } t \in [1,2], \\ t-2 & \text{if } t \in [2,3]. \end{cases}$$

Then we have

$$\begin{aligned} f(t,\sigma_1,\sigma_1') &= \frac{1}{\sqrt{t}}\big(\sigma_1'^2 - 1\big) + \sigma_1^3 + \sigma_1'^3 \\ &= \left\{ \begin{array}{ll} \frac{1}{\sqrt{t}}(1-1) + (t-3)^m + 1 < 0 & \text{if } t \in [0,1] \\ \frac{1}{\sqrt{t}}(1-1) + (-t)^m - 1 < 0 & \text{if } t \in (1,2] \\ \frac{1}{\sqrt{t}}(1-1) + (t-6)^m + 1 < 0 & \text{if } t \in (2,3] \end{array} \right\} = \big(\phi(\sigma_1')\big)', \end{aligned}$$

$$\begin{aligned} f(t,\sigma_2,\sigma_2') &= \frac{1}{\sqrt{t}}\big(\sigma_2'^2 - 1\big) + \sigma_2^3 + \sigma_1'^3 \\ &= \left\{ \begin{array}{ll} \frac{1}{\sqrt{t}}(1-1) + (t+1)^m + 1 > 0 & \text{if } t \in [0,1] \\ \frac{1}{\sqrt{t}}(1-1) + (-t+4)^m - 1 > 0 & \text{if } t \in (1,2] \\ \frac{1}{\sqrt{t}}(1-1) + (t-2)^m + 1 > 0 & \text{if } t \in (2,3] \end{array} \right\} = \big(\phi(\sigma_2')\big)'. \end{aligned}$$

Functions σ_1, σ_2 are respectively lower and upper functions of the problem (1.13). The right-hand side of the equation does not fulfil conditions of the sign type, because $f(t,x,\varphi_1)$ is not bounded on $[0,1]$. Nevertheless, one-sided conditions of the growth type are valid.

$$\phi^{-1}(x) = |x|^{\frac{1}{k}} \operatorname{sgn} x, \qquad \omega(s) = 1 + s^k,$$

$$\int_1^{\infty} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \qquad \int_{-\infty}^{-1} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty,$$

$$\begin{aligned} f(t,x,y) &= \frac{1}{\sqrt{t}}\big(y^k - 1\big) + x^m + y^{k+1} \le \frac{1}{\sqrt{t}}(|y|^k + 1) + (\sigma_2^m(t) + |y|)(|y|^k + 1) \\ &\le (1 + |y|^k)\Big(\frac{1}{\sqrt{t}} + \sigma_2^m(t) + |y|\Big) = \omega(|y|)(k(t) + |y|). \end{aligned}$$

By means of Theorem 1.2, there exists a solution of the problem (1.13).

2 Existence result for bounded right-hand sides of equations

At the beginning of this section we introduce an auxiliary problem and find a priori estimates for its solution. The main result of this section is contained in Theorem 2.1. In the proof of this theorem we show that a solution of the auxiliary problem (2.6)–(2.9) exists and is also a solution of the problem (0.1)–(0.3).

Assume that there is $h \in L(J)$ such that $|f(t,x,y)| \le h(t)$ for a.e. $t \in J$, for all $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times R$. Define function $\varphi : J \times R \to R$

$$\varphi(t,x) = \begin{cases} \sigma_2(t) & \text{if } \ x > \sigma_2(t), \\ x & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_1(t) & \text{if } x < \sigma_1(t), \end{cases} \tag{2.1}$$

and further functions $\omega_i : J \times [0,1] \to R, \ i = 1, 2,$

$$\begin{aligned} \omega_1(t,\varepsilon) &= \sup\{|f(t,\sigma_1,\sigma_1') - f(t,\sigma_1,y)| : |y - \sigma_1'| \le \varepsilon\}, \\ \omega_2(t,\varepsilon) &= \sup\{|f(t,\sigma_2,\sigma_2') - f(t,\sigma_2,y)| : |y - \sigma_2'| \le \varepsilon\}. \end{aligned} \tag{2.2}$$

We see that $\omega_i \in Car(J \times [0,1])$ are nonnegative, nondecreasing in the second variable and $\omega_i(t,0) = 0$ for a.e. $t \in J$, $i = 1, 2$.

Now, define $F : J \times R^2 \to R$ such that

$$F(t,x,y) = \begin{cases} f(t,\sigma_2,y) + \omega_2(t, \frac{x-\sigma_2}{x-\sigma_2+1}) + \frac{x-\sigma_2}{x-\sigma_2+1} & \text{for } x > \sigma_2(t), \\ f(t,x,y) & \text{for } \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_1,y) - \omega_1(t, \frac{\sigma_1 - x}{\sigma_1 - x+1}) - \frac{\sigma_1 - x}{\sigma_1 - x + 1} & \text{for } x < \sigma_1(t). \end{cases} \tag{2.3}$$

This function is bounded by a Lebesgue integrable function H

$$|F(t,x,y)| \le H(t) \quad \text{for a.e. } t \in J, \text{ for all } (x,y) \in R^2. \tag{2.4}$$

Define a function $\beta : R \to R$

$$\beta(y) = \begin{cases} y & \text{if } |y| \le K, \\ K \cdot \operatorname{sign} y & \text{if } |y| > K \end{cases}$$

and

$$\begin{aligned} K = \max\Big\{ &\big|\phi^{-1}\big(- \max\{|\phi(-\tfrac{r}{\delta})|, |\phi(\tfrac{r}{\delta})|\} - \|H\|_{L(J)}\big)\big|, \\ &\big|\phi^{-1}\big(\max\{|\phi(-\tfrac{r}{\delta})|, |\phi(\tfrac{r}{\delta})|\} + \|H\|_{L(J)}\big)\big|\Big\} + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty, \end{aligned} \tag{2.5}$$

where

$$r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty, \quad \delta = \min_{j \in \{0,\dots,m\}} (t_{j+1} - t_j).$$

We consider the following modified problem

$$\frac{d}{dt}[\phi(x'(t))] = F(t, x(t), x'(t)), \tag{2.6}$$

$$\begin{aligned} x(0) &= \varphi(0, x(0) + x'(0) - x'(T)), \\ x(T) &= \varphi(0, x(0) + x'(0) - x'(T)), \end{aligned} \tag{2.7}$$

$$x(t_i+) = x(t_i) - \varphi(t_i, x(t_i)) + J_i(\varphi(t_i, x(t_i))) = \widetilde{J}_i(x(t_i)), \quad i = 1, \dots m, \tag{2.8}$$

$$\phi(x'(t_i+)) - \phi(x'(t_i)) = \phi\big(M_i(\beta(x'(t_i)))\big) - \phi\big(\beta(x'(t_i))\big), \quad i = 1, \dots m. \tag{2.9}$$

For this problem the following three lemmas rule

Lemma 2.1 *Let u be a solution of (2.6)–(2.9) and (0.4), (0.6) hold. Let σ_1, σ_2 be respectively lower and upper functions of (0.1)–(0.3) and $\sigma_1 \le \sigma_2$ on J. Then u satisfies*

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \textit{for all } t \in J. \tag{2.10}$$

Proof We show that $v(t) = \sigma_1(t) - u(t) \le 0$ for all $t \in J$. By (2.7), we have $v(0) = v(T) < 0$.

1. Assume, on the contrary, that there is $\alpha \in (0,T)\backslash\Delta$ such that

$$\max\{(\sigma_1 - u)(t) : t \in J\} = v(\alpha) > 0.$$

Then $(\sigma_1 - u)'(\alpha) = 0$. This guarantees the existence of $\delta > 0$ such that

$$(\sigma_1 - u)(t) > 0, \quad |v'(t)| < \frac{\sigma_1 - u}{\sigma_1 - u + 1} < 1 \quad \forall t \in (\alpha, \alpha+\delta) \subset (0,T)\backslash\Delta. \tag{2.11}$$

Using (2.3), (2.11) and the properties of σ_1, we get

$$\begin{gathered}[\phi(\sigma_1'(t))]' - [\phi(u'(t))]' \\ \ge f(t,\sigma_1(t),\sigma_1'(t)) - f(t,\sigma_1(t),u'(t)) + \omega_1\Big(t, \frac{\sigma_1(t)-u(t)}{\sigma_1(t)-u(t)+1}\Big) + \frac{\sigma_1(t)-u(t)}{\sigma_1(t)-u(t)+1} \\ > -|f(t,\sigma_1(t),\sigma_1'(t)) - f(t,\sigma_1(t),u'(t))| + \omega_1(t, |\sigma_1'(t)-u'(t)|) + |\sigma_1'(t)-u'(t)| > 0\end{gathered}$$

for a.e. $t \in (\alpha, \alpha+\delta)$.

Hence, $\phi(\sigma_1'(t)) - \phi(u'(t)) > \phi(\sigma_1'(\alpha)) - \phi(u'(\alpha)) = 0$ for all $t \in (\alpha, \alpha+\delta)$. Since ϕ is increasing, we get $u'(t) < \sigma_1'(t)$ for all $t \in (\alpha, \alpha+\delta)$. This contradicts that v has a maximum at α. We have showed that v does not have a positive maximum at any point of $(0,T)\backslash\Delta$.

2. If $v(t) > 0$ for some $t \in J$, there is a $t_j \in \Delta$ such that

$$\max\{v(t) : t \in [0,T]\} = v(t_j) > 0. \tag{2.12}$$

By (2.8) and the Definition 2 we get

$$v(t_j+) = \sigma_1(t_j+) - u(t_j+) = J_j(\sigma_1(t_j)) - u(t_j) + \sigma_1(t_j) - J_j(\sigma_1(t_j)) = v(t_j).$$

Then

$$v'(t_j+) \le 0. \tag{2.13}$$

Futhermore, taking into account (2.12), we have $v'(t_j) \ge 0$, and by Definition 2, the relations

$$\begin{gathered}\phi(\sigma_1'(t_j+)) \ge \phi\big(M_j(\sigma_1'(t_j))\big) \ge \phi\big(M_j(\beta(u'(t_j)))\big) \\ = \phi(u'(t_j+)) - \phi(u'(t_j)) + \phi(\beta(u'(t_j))) \ge \phi(u'(t_j+)) \\ \Rightarrow \phi(\sigma_1'(t_j+)) - \phi(u'(t_j+)) \ge 0\end{gathered}$$

follow. It means, since a function ϕ is increasing,

$$v'(t_j+) \ge 0. \tag{2.14}$$

Now, by (2.13), (2.14) we get $v'(t_j+) = 0$.

Thus, in view of the first part of the proof, there is $\delta > 0$ such that

$$v(t) > 0, \quad |v'(t)| < \frac{\sigma_1 - u}{\sigma_1 - u + 1} < 1 \quad \text{on } (t_j, t_j + \delta) \subset (0,T)\backslash\Delta$$

and we deduce that $v'(t) > 0$ for all $t \in (t_j, t_j + \delta)$, which contradicts (2.12). So, we have proved $\sigma_1(t) \le u(t)$ for all $t \in J$.

If we put $v(t) = u(t) - \sigma_2(t)$, we can prove $u(t) \le \sigma_2(t)$ on J by an analogous argument. □

Lemma 2.2 *Let u be a solution of (2.6)–(2.9) with a condition (0.6). Then u satisfies the periodic boundary conditions (0.2).*

Proof The first, we prove

$$\sigma_1(0) \le u(0) + u'(0) - u'(T) \le \sigma_2(0). \tag{2.15}$$

Suppose, on the contrary, that

$$u(0) + u'(0) - u'(T) > \sigma_2(0). \tag{2.16}$$

By the definition of the function φ it follows that $\varphi(0, u(0) + u'(0) - u'(T)) = \sigma_2(0)$. Then, by condition (2.7), we get $\sigma_2(0) = u(0)$. The inequality (2.16) implies that

$$u'(0) > u'(T). \tag{2.17}$$

The equality $\sigma_2(0) = u(0) = u(T) = \sigma_2(T)$ and (2.10)) yield $\sigma_2'(0) \ge u'(0)$ and $\sigma_2'(T) \le u'(T)$. This together with Definition 2, this head to

$$u'(0) \le \sigma_2'(0) \le \sigma_2'(T) \le u'(T),$$

contrary to (2.17). We can similary derive the inequality $\sigma_1(0) \le u(0) + u'(0) - u'(T)$.

So, if (2.15) is valid, then

$$u(0) = \varphi(0, u(0) + u'(0) - u'(T)) = u(0) + u'(0) - u'(T) \Rightarrow u'(0) = u'(T).$$

It means that a solution of (2.6)–(2.9) fulfils periodic boundary conditions. □

Lemma 2.3 *Let u be a solution of (2.6)–(2.9) with a condition (0.6). Then u satisfies the impulsive conditions (0.3).*

Proof By means of Lemma 2.1 the equality $\varphi(t_i, u(t_i)) = u(t_i)$ holds. Then the condition (2.8) implies $u(t_i+) = J_i(u(t_i))$ for all $i \in \{1, \dots, m\}$. We will prove the impulsive condition for u'.

We show that

$$\phi(M_j(u'(t_j))) = \phi\big(M_j(\beta(u'(t_j)))\big), \quad \phi(u'(t_j)) = \phi\big(\beta(u'(t_j))\big) \quad \forall t_j \in \Delta.$$

By the Mean Value Theorem there exists $\xi_j \in (t_j, t_{j+1})$, $j = 0, \dots, m$, such that

$$|u'(\xi_j)| = \frac{|u(t_{j+1}) - u(t_j)|}{t_{j+1} - t_j} \le \frac{r}{\delta}.$$

Then the equality

$$u'(t_j) = \phi^{-1}\big(\phi(u'(\xi_j)) + \int_{\xi_j}^{t_j} [\phi(u'(s))]' ds\big).$$

holds for all $j \in \{1, \dots, m\}$. With respect to (2.4), (2.5) and (2.6) we have

$$|u'(t_j)| \le K, \quad j = 1, \dots, m.$$

By (2.9), it means that u fulfils

$$\phi(u'(t_i+)) - \phi(u'(t_i)) = \phi(M_i(u'(t_i))) - \phi(u'(t_i)) \quad \forall i \in \{1, \dots, m\},$$

therefore $u'(t_i+) = M_i(u'(t_i))$ for all $i \in \{1, \dots, m\}$, which concludes the proof. □

Now, we will prove the main result of this section concerning the existence of a solution for problem (0.1)–(0.3) with a bounded right-hand side.

Theorem 2.1 *Let σ_1, σ_2 be respectively lower and upper functions of the problem (0.1)–(0.3) and $\sigma_1 \le \sigma_2$ on J.*

Assume that (0.4) and (0.6) hold. Further assume that there is $h \in L(J)$ such that $|f(t, x, y)| \le h(t)$ for a.e. $t \in J$, for all $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times R$. Then the problem (0.1)–(0.3) has a solution u fulfilling

$$\sigma_1 \le u \le \sigma_2 \quad \text{on } J. \tag{2.10}$$

Proof By means of the three previous lemmas it is sufficient to prove the existence of a solution of the auxiliary problem (2.6)–(2.9). Denote

$$\Psi_u(t) = \sum_{i=1}^{m} \chi_{(t_i,T]}(t) \big[\phi\big(M_i(\beta(u'(t_i)))\big) - \phi\big(\beta(u'(t_i))\big)\big] \quad \text{for } t \in J, \tag{2.18}$$

where $\chi_{(t_j,T]}(t)$ means the characteristic function of the interval $(t_j, T]$. For fixed $v \in C^1_\Delta(J)$ define $g_v : R \to R$ such that

$$g_v(x) = \int_0^T \phi^{-1}\big(x + \int_0^r F_v(s) ds + \Psi_u(r)\big) dr \quad \forall x \in R,$$

where $F_v(s) \equiv F(s, v(s), v'(s))$ for a.e. $s \in J$. Since ϕ^{-1} is continuous and increasing, g_v is continuous and increasing, too. We know that there is $H \in L(J)$ such that $|F_v(s)| \le H(s)$ for a.e. $s \in J$ and for all $v \in C^1_\Delta(J)$ and then

$$|\int_0^t F_v(s)| \le \|H\|_{L(J)} \quad \text{for all } t \in J \text{ and every } v \in C^1_\Delta(J). \tag{2.19}$$

By (2.18), there exists $\varrho > 0$ such that

$$\left|\Psi_u(t)\right| \le \varrho \quad \forall t \in J,\ u \in C^1_\Delta(J). \tag{2.20}$$

Since ϕ is increasing, for each $x \in R$ and for all $v \in C^1_\Delta(J)$

$$T\phi^{-1}(x - \|H\|_{L(J)} - \varrho) \le g_v(x) \le T\phi^{-1}(x + \|H\|_{L(J)} + \varrho).$$

holds. By this inequalities and by the fact that $\phi^{-1}(R) = R$, we have $g_v(R) = R$ for each $v \in C^1_\Delta(J)$. Therefore, for all $v \in C^1_\Delta(J)$ there exists a unique A_v satisfying

$$g_v(A_v) = \int_0^T \phi^{-1}\big(A_v + \int_0^r F_v(s)ds + \Psi_v(r)\big)dr = -\sum_{i=1}^m \big(\widetilde{J}_i(u(t_i)) - u(t_i)\big). \tag{2.21}$$

We show that there exists $N > 0$ such that $|A_v| \le N$ for every $v \in C^1_\Delta(J)$. The Mean Value Theorem for integrals implies that there is $\eta \in (0,T)$ such that

$$\begin{aligned} &\int_0^T \phi^{-1}\big(A_v + \int_0^r F_v(s)ds + \Psi_v(r)\big)dr \\ &= T\phi^{-1}\Big(A_v + \int_0^\eta F_v(s)ds + \Psi_v(\eta)\big) = -\sum_{i=1}^m \Big(\widetilde{J}_i(u(t_i)) - u(t_i)\Big) = C. \end{aligned}$$

Then $A_v = \phi\big(\frac{C}{T}\big) - \int_0^\eta F_v(s)ds - \Psi_v(\eta)$ and

$$\begin{aligned} |A_v| &= \Big|\phi\Big(\frac{C}{T}\Big) - \int_0^\eta F_v(s)ds - \Psi_v(\eta)\Big| \le \Big|\phi\Big(\frac{C}{T}\Big)\Big| + \int_0^\eta |F_v(s)|ds + \big|\Psi_v(\eta)\big| \\ &\le \Big|\phi\Big(\frac{C}{T}\Big)\Big| + \int_0^T H(s)ds + \varrho = \Big|\phi\Big(\frac{C}{T}\Big)\Big| + \|H\|_{L(J)} + \varrho. \end{aligned}$$

It means that

$$|A_v| \le \Big|\phi\Big(\frac{C}{T}\Big)\Big| + \|H\|_{L(J)} + \varrho = N \quad \text{for all } v \in C^1_\Delta(J). \tag{2.22}$$

Now define the following operator $\mathcal{T} : C^1_\Delta(J) \to C^1_\Delta(J)$ by the formula

$$\begin{aligned} (\mathcal{T}u)(t) \ = \ &\sum_{i=1}^m \chi_{(t_i,T]}(t)\big(\widetilde{J}_i(u(t_i)) - u(t_i)\big) + \varphi(0, u(0) + u'(0) - u'(T)) \\ &+ \int_0^t \phi^{-1}\big(A_u + \int_0^r F_u(s)ds + \Psi_u(r)\big)dr. \end{aligned} \tag{2.23}$$

Then for all $t \in J$ and all $u \in C^1_\Delta(J)$

$$(\mathcal{T}u)'(t) = \phi^{-1}\big(A_u + \int_0^t F_u(s)ds + \Psi_u(t)\big) \tag{2.24}$$

holds. If $u \in C^1_\Delta(J)$ is a fixed point of $\mathcal{T}$, then from equation (2.24), we obtain

$$\phi(u'(t)) = A_u + \int_0^t F_u(s)ds + \Psi_u(t) \text{ for all } t \in J \text{ and for every } u \in C^1_\Delta(J). \tag{2.25}$$

$F \in Car(J \times R^2)$ means that $F_u \in L(J)$, so we have $\phi(u') \in AC_\Delta(J)$. Differentiating in equation (2.25), we obtain that u satisfies equation (2.6). Using (2.21) we see that u satisfies conditions (2.7). From equation (2.25) we get for all $j \in \{1, \dots, m\}$ equalities

$$\phi(u'(t_j)) = A_u + \int_0^{t_j} F_u(s)ds + \sum_{i=1}^{j-1} \chi_{(t_i,T]}(t)\big[\phi\big(M_i(\beta(u'(t_i)))\big) - \phi\big(\beta(u'(t_i))\big)\big],$$

$$\phi(u'(t_j+)) = A_u + \int_0^{t_j} F_u(s)ds + \sum_{i=1}^{j} \chi_{(t_i,T]}(t)\big[\phi\big(M_i(\beta(u'(t_i)))\big) - \phi\big(\beta(u'(t_i))\big)\big].$$

From the difference of the left-hand and right-hand sides of these equalities we see that for all $t_j \in \Delta$ condition (2.9) follows. Moreover, from equation (2.23) we deduce

$$u(t_j+) = \widetilde{J}_j(u(t_j)) \quad \text{for every } j \in \{1, \dots, m\}.$$

Thus, if u is a fixed point of the operator $\mathcal{T}$ then u is a solution of (2.6)–(2.9).

Now, we will prove that the operator $\mathcal{T}$ has a fixed point $u \in C^1_\Delta(J)$. We start showing that the operator $\mathcal{T}$ is continuous in $C^1_\Delta(J)$. For $\{u_n\} \subset C^1_\Delta(J)$, we prove

$$u_n \to u \text{ in } C^1_\Delta(J) \implies \mathcal{T}u_n \to \mathcal{T}u \text{ in } C^1_\Delta(J).$$

Let A_n correnspond to u_n by equation (2.21), and similarly let A correnspond to u. We prove that $A_n \to A$. By the construction of A_n and A and by the Mean Value Theorem there exists $\xi_n \in (0, T)$ such that

$$\lim_{n\to\infty}\Big\{\int_0^T \phi^{-1}\big(A_n + \int_0^r F_{u_n}(s)ds + \Psi_{u_n}(r)\big)dr - \int_0^T \phi^{-1}\big(A + \int_0^r F_u(s)ds + \Psi_u(r)\big)dr\Big\}$$
$$= T\lim_{n\to\infty}\Big\{\phi^{-1}\big(A_n + \int_0^{\xi_n} F_{u_n}(s)ds + \Psi_{u_n}(\xi_n)\big) - \phi^{-1}\big(A + \int_0^{\xi_n} F_u(s)ds + \Psi_u(\xi_n)\big)\Big\} = 0. \tag{2.26}$$

Since ϕ is uniformly continuous in J, we have

$$\lim_{n\to\infty}\Big\{A_n + \int_0^{\xi_n} F_{u_n}(s)ds + \Psi_{u_n}(\xi_n) - A - \int_0^{\xi_n} F_u(s)ds - \Psi_u(\xi_n)\Big\} = 0.$$

By the continuity of ϕ and β in u it follows that $\|\Psi_{u_n} - \Psi_u\|_\infty \to 0$. Since $u_n \to u$ in $C^1_\Delta(J)$ and $F \in Car(J \times R^2)$, it holds that $F_{u_n} \to F_u$ a.e on J. By the Lebesgue theorem and from (2.19) we have

$$\lim_{n\to\infty}\int_0^{\xi_n}\big[F_{u_n}(s) - F_u(s)\big]ds = 0.$$

We conclude that $\lim_{n\to\infty} A_n = A$. Furthermore

$$A_n + \int_0^t F_{u_n}(s)ds + \Psi_{u_n}(t) \to A + \int_0^t F_u(s)ds + \Psi_u(t) \quad \text{for all } t \in J.$$

Now, since

$$\begin{aligned}&\Big|A_n + \int_0^t F_{u_n}(s)ds + \Psi_{u_n}(t) - A - \int_0^t F_u(s)ds - \Psi_u(t)\Big| \\ &\qquad \le |A_n - A| + \|F_{u_n} - F_u\|_{L(J)} + \|\Psi_{u_n} - \Psi_u\|_\infty,\end{aligned}$$

for all $t \in J$, the convergence is uniform. By the uniform continuity ϕ^{-1} on compact intervals, $(\mathcal{T}u_n)' \to (\mathcal{T}u)'$ uniformly on J.

Since φ is continuous

$$\varphi(0, u_n(0) + u_n'(0) - u_n'(T)) \to \varphi(0, u(0) + u'(0) - u'(T))$$

in R. Since $\widetilde{J}_i$ are continuous for all $i \in \Delta$

$$\sum_{i=1}^m \chi_{(t_i,T]}(\cdot)\big(\widetilde{J}_i(u_n(t_i)) - u_n(t_i)\big) \to \sum_{i=1}^m \chi_{(t_i,T]}(\cdot)\big(\widetilde{J}_i(u(t_i)) - u(t_i)\big)$$

uniformly on J. Thus $\mathcal{T}u_n \to \mathcal{T}u$ uniformly on J.

Now, we are going to prove a compactness of the operator $\mathcal{T}$. Let M be an arbitrary set in $C^1_\Delta(J)$ and $\{x_n\} \subset \overline{\mathcal{T}(M)}$ be an arbitrary sequence. We prove that we can choose a subsequence convergent in $C^1_\Delta(J)$ to the function $x \in \overline{\mathcal{T}(M)}$. Choose sequence $\{x_n\} \subset \overline{\mathcal{T}(M)}$. Then

$$x_n(t) = \begin{cases} x_n^{[0]}(t), & t \in [0, t_1], \\ x_n^{[1]}(t), & t \in (t_1, t_2], \\ \quad\cdots\cdots\cdots\cdots & \\ x_n^{[m]}(t), & t \in (t_m, T], \end{cases}$$

where $\{x_n^{[i]}\} \subset C^1[t_i, t_{i+1}], i = 0, \dots, m$. Consider $\{x_n^{[0]}\} \subset C^1[0, t_1]$. We will show that this sequence is bounded and $\{(x_n^{[0]})'\}$ is equicontinuous on $[0, t_1]$. Let $u_n \in M$ be such that $x_n = \mathcal{T}u_n$. Then by (2.19), (2.20) and (2.22)

$$\begin{aligned}\|x_n^{[0]}\|_{C^1[0,t_1]} &\le \sum_{i=1}^m |\widetilde{J}_i(u(t_i)) - u(t_i)| + \|\sigma_1\|_\infty + \|\sigma_2\|_\infty \\ &+ \int_0^t \Big|\phi^{-1}\big(A_{u_n} + \int_0^r F_{u_n}(s)ds + \Psi_{u_n}(r)\big)\Big|dr + \Big|\phi^{-1}\big(A_{u_n} + \int_0^t F_{u_n}(s)ds + \Psi_{u_n}(t)\big)\Big| \\ &\le \sum_{i=1}^m |\widetilde{J}_i(u(t_i)) - u(t_i)| + \|\sigma_1\|_\infty + \|\sigma_2\|_\infty \\ &+ (T+1)\max\{|\phi^{-1}(-N - \|H\|_{L(J)} - \varrho)|, |\phi^{-1}(N + \|H\|_{L(J)}) + \varrho)|\}.\end{aligned}$$

It means that $\{x_n^{[0]}\}$ is bounded.

On the basis of the absolute continuity of the Lebesgue integral the condition

$$\forall\varepsilon_1>0\ \exists\delta_1>0\ \forall\tau_1,\tau_2\in[0,t_1]\ \forall x_n\in\overline{\mathcal{T}(M)}:|\tau_1-\tau_2|<\delta_1$$

$$\Rightarrow\Big|A_{u_n}+\int_0^{\tau_1}F_{u_n}(s)ds+\Psi_{u_n}(t_1)-\Big(A_{u_n}+\int_0^{\tau_2}F_{u_n}(s)ds+\Psi_{u_n}(t_1)\Big)\Big|$$

$$=\Big|\int_{\tau_1}^{\tau_2}F_{u_n}(s)\,ds\Big|<\Big|\int_{\tau_1}^{\tau_2}H(s)\,ds\Big|<\varepsilon_1 \tag{2.27}$$

holds. By the uniform continuity of ϕ^{-1} we have

$$\forall\varepsilon>0\ \exists\varepsilon_2>0\ \forall\tau_1,\tau_2\in[0,t_1]\ \forall x_n\in\overline{\mathcal{T}(M)}:$$

$$\Big|A_{u_n}+\int_0^{\tau_1}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)-\Big(A_{u_n}+\int_0^{\tau_2}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)\Big|<\varepsilon_2$$

$$\Longrightarrow\Big|\phi^{-1}\Big(A_{u_n}+\int_0^{\tau_1}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)$$

$$-\phi^{-1}\Big(A_{u_n}+\int_0^{\tau_2}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)\Big|<\varepsilon.$$

If we choose δ_2 corrensponding to ε_2 by (2.27), then

$$\forall\varepsilon>0\ \exists\delta_2\ \forall\tau_1,\tau_2\in[0,t_1]\ \forall x_n\in\overline{\mathcal{T}(M)}:|\tau_1-\tau_2|<\delta_2$$

$$\Longrightarrow\big|(x_n^{[0]})'(\tau_1)-(x_n^{[0]})'(\tau_2)\big|=\Big|\phi^{-1}\Big(A_{u_n}+\int_0^{\tau_1}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)$$

$$-\phi^{-1}\Big(A_{u_n}+\int_0^{\tau_2}F_{u_n}(s)\,ds+\Psi_{u_n}(t_1)\Big)\Big|<\varepsilon.$$

It means that $\{(x_n^{[0]})'\}$ is equicontinuous. We can do similar considerations for the other sequences $\{x_n^{[i]}\}\subset C^1[t_i,t_{i+1}], i=1,\dots,m$. Now, we select $\{x_n^{[0]}\}\subset\{x_{k_n}^{[0]}\}$ convergent in $C^1[0,t_1]$, and corrensponding subsequences $\{x_{k_n}^{[i]}\}\subset\{x_n^{[i]}\}$, $i=1,\dots,m$. Having $\{x_{k_n}^{[1]}\}$ we can select convergent subsequence. Without loss of generality we denote it $\{x_{k_n}^{[1]}\}$ again, and choose corrensponding $\{x_{k_n}^{[i]}\}$, $i=0,2,\dots,m$. Continuing inductively we choose convergent $\{x_{l_n}^{[m]}\}\subset\{x_n^{[m]}\}$ and corrensponding sequences $\{x_{l_n}^{[i]}\}$, $i=0,\dots,m-1$. If we take

$$x_{l_n}(t)=\begin{cases}x_{l_n}^{[0]}(t), & t\in[0,t_1],\\ x_{l_n}^{[1]}(t), & t\in(t_1,t_2],\\ \quad\dots\dots\dots\dots & \\ x_{l_n}^{[m]}(t), & t\in(t_m,T],\end{cases}$$

we obtain the subsequence $\{x_{l_n}(t)\}\subset\{x_n(t)\}\subset\overline{\mathcal{T}(M)}$, such that $\{x_{l_n}(t)\}$ converges in $C^1_\Delta(J)$. It means that the operator $\mathcal{T}$ is compact.

For all $u\in C^1_\Delta(J)$ the following estimate holds

$$\|\mathcal{T}u\|_{C^1_\Delta(J)}\le\sum_{i=1}^m|\widetilde{J}_i(u(t_i))-u(t_i)|+\|\sigma_1\|_\infty+\|\sigma_2\|_\infty$$

$$+(T+1)\max\{|\phi^{-1}(-N-\|H\|_{L(J)}-\varrho)|,|\phi^{-1}(N+\|H\|_{L(J)})+\varrho)|\}=Q.$$

Define $\Omega = \{u \in C^1_\Delta(J) : \|u\|_{C^1_\Delta(J)} \leq Q\}$. Then Ω is a nonempty closed bounded and convex set. The operator $\mathcal{T}$ sends the set Ω into Ω, $\mathcal{T}$ is compact. By the Schauder fixed point theorem, operator $\mathcal{T}$ has a fixed point u. This fixed point is a solution of the problem (0.1)–(0.3). □

3 Proofs of main results

In this section we prove the existence results which are contained in Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 Define function $\psi(t, y) : J \times R \to R$

$$\psi(t,y) = \begin{cases} \varphi_2(t) & \text{if } \ y > \varphi_2(t), \\ y & \text{if } \varphi_1(t) \leq y \leq \varphi_2(t), \\ \varphi_1(t) & \text{if } y < \varphi_1(t). \end{cases} \tag{3.1}$$

Further define function $g : J \times R^2 \to R$ by the formula

$$g(t,u,v) = f(t,u,\psi(t,v)) + \frac{v - \psi(t,v)}{|v - \psi(t,v)| + 1}. \tag{3.2}$$

Then there exists $h_0 \in L(J)$

$$|g(t,x,y)| \leq h_0(t) \quad \text{for a.e. } t \in J, \text{ for all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times R.$$

Functions σ_1 and σ_2 are respectively lower and upper functions of the auxiliary problem

$$\frac{d}{dt}[\phi(x'(t))] = g(t, x(t), x'(t)), \tag{3.3}$$

$$x(0) = x(T), \quad \psi(0, x'(0)) = x'(T), \tag{3.4}$$

$$x(t_i+) = J_i(x(t_i)), \quad i \in \{1, \dots, m\}, \tag{3.5}$$

$$x'(t_i+) = x'(t_i) - \psi(t_i, x'(t_i)) + M_i(\psi(t_i, x'(t_i))) = \widetilde{M}_i(x'(t_i)), \ i \in \{1, \dots, m\}, \tag{3.6}$$

function $\widetilde{M}_i$ satisfies condition (0.6) for all $i \in \Delta$. Consider function φ defined by (2.1), further formulas (2.2) - (2.5) defined for function g. By means of the proof of Theorem 2.1 there exists a solution u of the following problem

$$\begin{aligned}
&\frac{d}{dt}[\phi(x'(t))] = F(t, x(t), x'(t)), \\
&x(0) = \varphi(0, x(0) + \psi(0, x'(0)) - x'(T)), \\
&x(T) = \varphi(0, x(0) + \psi(0, x'(0)) - x'(T)), \\
&x(t_i+) = x(t_i) - \varphi(t_i, x(t_i)) + J_i(\varphi(t_i, x(t_i))) = \widetilde{J}_i(x(t_i)), \quad i = 1, \dots m, \\
&\phi(x'(t_i+)) - \phi(x'(t_i)) = \phi\big(\widetilde{M}_i(\beta(x'(t_i)))\big) - \phi\big(\beta(x'(t_i))\big), \quad i = 1, \dots m.
\end{aligned}$$

with a property $\sigma_1 \leq u \leq \sigma_2$ on J.

In additions, function u is also solution of the problem (3.3)–(3.6). We will show that the following inequalities hold

$$\varphi_1 \le u' \le \varphi_2 \quad \text{on } J. \tag{3.7}$$

Since ϕ is increasing, it is enough to prove the inequality $\phi(\varphi_1) \le \phi(u') \le \phi(\varphi_2)$ on J.

1. Put $z = \phi(u') - \phi(\varphi_2)$ on J. Assume, that there is $\alpha \in (0,T)\backslash\Delta$ such that z has a positive local maximum at α, i.e. $z(\alpha) > 0$. Since u is a solution of the problem (3.3) - (3.6), there is $\delta > 0$ such that $z(t) > 0$ on $(\alpha, \alpha+\delta)$ and

$$\begin{aligned} z'(t) &= [\phi(u'(t))]' - [\phi(\varphi_2(t))]' = g(t, u(t), u'(t)) - [\phi(\varphi_2(t))]' \\ &\ge f(t, u(t), \varphi_2(t)) + \frac{u' - \varphi_2(t)}{u' - \varphi_2(t) + 1} - f(t, u(t), \varphi_2(t)) > 0 \end{aligned}$$

holds for a.e. $t \in (\alpha, \alpha+\delta)$ with respect to (1.8). Thus, for a.e. $t \in (\alpha, \alpha+\delta)$ we have $z'(t) > 0$. By integration of this inequality we get

$$\begin{aligned} 0 < \int_\alpha^t z'(s)ds &= \int_\alpha^t \left([\phi(u'(s))]' - [\phi(\varphi_2(s))]'\right) ds \\ = \phi(u'(t)) - \phi(\varphi_2(t)) &- (\phi(u'(\alpha)) - \phi(\varphi_2(\alpha))) = z(t) - z(\alpha). \end{aligned}$$

It means that $z(t) > z(\alpha)$ for all $t \in (\alpha, \alpha+\delta)$. It contradicts the assumption of the local maximum of z in α.

2. Assume that there is $t_j \in \Delta$ such that $z(t_j) > 0$. Then $u'(t_j) > \varphi_2(t_j)$. Since

$$(u' - \varphi_2)(t_j+) \ge u'(t_j) - \varphi_2(t_j) + M_j(\varphi_2(t_j)) - M_j(\varphi_2(t_j)) > 0,$$

the inequality $z(t_j+) > 0$ holds. Then there exists $\delta > 0$ such that

$$z(t) > 0 \ \text{ on } (t_j, t_j+\delta), \quad z'(t) > 0 \ \text{ for a.e. } t \in (t_j, t_j+\delta). \tag{3.8}$$

By the first part of the proof we have

$$z'(t) \ge 0 \quad \text{on } (t_j, t_{j+1}). \tag{3.9}$$

Now, by (3.8) and (3.9) we obtain

$$\max_{t \in (t_j, t_{j+1}]} z(t) = z(t_{j+1}) > 0.$$

Continuing inductively we get $z(T) = \phi(u'(T)) - \phi(\varphi_2(T)) > 0$. It means that $u'(T) > \varphi_2(T) \ge \varphi_2(0)$. It is contradiction because from (1.7) and (3.4) we get $u'(T) \le \varphi_2(0) \le \varphi_2(T)$. It means that the inequality $u' \le \varphi_2$ holds on J. By an analogous argument we can prove inequality $\varphi_1 \le u'$ using function $z(t) = \phi(\varphi_1(t)) - \phi(u'(t))$. So, u fulfils (3.7), consequently, u is a solution of (0.1)–(0.3) satisfying (1.9). □

Before proving Theorem 1.2, we prove the following lemma where we derive a priori estimates for derivatives of solutions.

Lemma 3.1 *Let σ_1, σ_2 be respectively lower and upper functions of the problem (0.1)–(0.3) and $\sigma_1 \le \sigma_2$ on J. Assume that (0.5) holds. Further assume that $k \in L(J)$ is nonnegative a.e. on $[0,T]$, $\omega \in C([0,\infty))$ is positive on $[0,\infty)$ and*

$$\int_{-\infty}^{\phi(-1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty, \qquad \int_{\phi(1)}^{\infty} \frac{ds}{\omega(|\phi^{-1}(s)|)} = \infty. \tag{3.10}$$

Then there exists $\mu_* > 0$ such that for each function $u \in C^1_\Delta(J)$ fulfiling (0.2), the conditions for derivative in (0.3) and inequalities

$$\sigma_1 \le u \le \sigma_2 \quad \text{on } J, \tag{3.11}$$

$$[\phi(u'(t))]' \le \omega(|u'(t)|)(k(t) + |u'(t)|) \quad \text{for a.e. } t \in J, \tag{3.12}$$

the following estimate holds $|u'(t)| < \mu_*$ for all $t \in J$.

Proof Put $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. By the Mean Value Theorem there is $\xi_i \in (t_i, t_{i+1})$ such that

$$|u'(\xi_i)| \le \frac{2r}{\delta} + 1 = r_1, \quad i = 0, 1, \dots, m, \tag{3.13}$$

where

$$\delta = \min_{i=0,1,\dots,m} (t_{i+1} - t_i).$$

The assumption (3.10) implies the existence of an increasing sequence $\{\mu_j\}_{j=1}^{2m+4} \in (r_1, \infty)$ such that

$$r_1 < M_j(\mu_j) < \mu_{j+1}, \quad -\mu_{m+4+j} < M^{-1}_{m+1-j}(-\mu_{m+3+j}) < -r_1$$

for $j = 1, \dots, m$ and satisfying

$$\int_{\phi(r_1)}^{\phi(\mu_1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(\mu_{m+1})}^{\phi(\mu_{m+2})} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(-\mu_{m+3})}^{\phi(-r_1)} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(-\mu_{m+4})}^{\phi(-\mu_{m+3})} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(M_j(\mu_j))}^{\phi(\mu_{j+1})} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)},$$

$$\int_{\phi(-\mu_{m+4+j})}^{\phi(M^{-1}_{m+1-j}(-\mu_{m+3+j}))} \frac{ds}{\omega(|\phi^{-1}(s)|)} > r + \|k\|_{L(J)}$$

for $j = 1, \dots, m$. We estimate u' from above. Assume that there is $\beta_1 \in (\xi_0, t_1]$ such that

$$\max\{u'(t) : t \in [\xi_0, t_1]\} = u'(\beta_1) = c_1 \ > \ r_1.$$

Then we can find $\alpha_1 \in (\xi_0, \beta_1)$ such that $u'(\alpha_1) = r_1$, $u'(t) > r_1$ for all $t \in (\alpha_1, \beta_1]$. Integrating the inequality

$$\frac{[\phi(u'(t))]'}{\omega(|u'(t)|)} \le (k(t) + |u'(t)|),$$

which holds for a.e. $t \in (\alpha_1, \beta_1)$, we obtain

$$\int_{\alpha_1}^{\beta_1} \frac{[\phi(u'(t))]'dt}{\omega(u'(t))} \le \int_{\alpha_1}^{\beta_1} \big(k(t) + u'(t)\big)dt.$$

Using substitution $s = \phi(u'(t))$ we get that

$$\int_{\alpha_1}^{\beta_1} \frac{[\phi(u'(t))]'dt}{\omega(u'(t))} = \int_{\phi(r_1)}^{\phi(c_1)} \frac{ds}{\omega(\phi^{-1}(s))}.$$

Moreover,

$$\int_{\alpha_1}^{\beta_1} \big(k(t) + u'(t)\big)dt = \int_{\alpha_1}^{\beta_1} k(t)dt + u(\beta_1) - u(\alpha_1) \le \|k\|_{L(J)} + |\sigma_2(\beta_1) - \sigma_1(\alpha_1)|$$
$$\le \|k\|_{L(J)} + (\|\sigma_2\|_{C(J)} + \|\sigma_1\|_{C(J)}) = r + \|k\|_{L(J)}.$$

So we have

$$\int_{\phi(r_1)}^{\phi(c_1)} \frac{ds}{\omega(\phi^{-1}(s))} \le r + \|k\|_{L(J)},$$

which implies that $\phi(c_1) < \phi(\mu_1)$. Since function ϕ is increasing, it means that $c_1 < \mu_1$. Thus $u'(t) < \mu_1$ for all $t \in [\xi_0, t_1]$.

Next assume that there exists $\beta_2 \in (t_1, t_2]$ such that

$$\sup\{u'(t) : t \in (t_1, t_2]\} = u'(\beta_2) = c_2 \ > \ M_1(\mu_1).$$

Then we can find such $\alpha_2 \in (t_1, \beta_2)$ that $u'(\alpha_2) = M_1(\mu_1)$, $u'(t) > M_1(\mu_1)$ for all $t \in (\alpha_2, \beta_2]$. Integrating inequality

$$\frac{[\phi(u'(t))]'}{\omega(|u'(t)|)} \le k(t) + |u'(t)|,$$

which holds for a.e. $t \in (\alpha_2, \beta_2)$, we get

$$\int_{\alpha_2}^{\beta_2} \frac{[\phi(u'(t))]'dt}{\omega(u'(t))} = \int_{\phi(M_1(\mu_1))}^{\phi(c_2)} \frac{ds}{\omega(\phi^{-1}(s))} \le r + \|k\|_{L(J)},$$

so it must be $c_2 < \mu_2$. We have proved that $u'(t) < \mu_2$ for all $t \in [t_1, t_2]$. Continuing inductively over all intervals (t_j, t_{j+1}), we obtain the estimate $u'(t) < \mu_{m+1}$

for all $t \in [t_m, T]$, from this $u'(0) < \mu_{m+1}$ follows. Using the previous procedure we deduce that $u'(t) < \mu_{m+2}$ for all $t \in [0, \xi_0]$.

Similarly we estimate u' from below. Assume that there exists $\beta_{m+3} \in [0, \xi_0)$ such that

$$\min\{u'(t) : t \in [0, \xi_0]\} = u'(\beta_{m+3}) = -c_{m+3} < -r_1.$$

Then we prove that $-c_{m+3} > -\mu_{m+3}$, tj. $u'(t) > -\mu_{m+3}$ on $[0, \xi_0]$, $u'(T) > -\mu_{m+3}$. From the assumption

$$\inf\{u'(t) : t \in (t_m, T]\} = u'(\beta_{m+4}) = -c_{m+4} < -\mu_{m+3}$$

we get $-c_{m+4} > -\mu_{m+4}$, i.e. $-\mu_{m+4} < u'(t)$ for all $t \in [t_m, T]$. Assume that there exists $\beta_{m+5} \in [t_{m-1}, t_m)$ such that

$$\inf\{u'(t) : t \in (t_{m-1}, t_m]\} = u'(\beta_{m+5}) = -c_{m+5} < M_m^{-1}(-\mu_{m+4}).$$

Then we get $-c_{m+5} > -\mu_{m+5}$, i.e. $-\mu_{m+5} < u'(t)$ for all $t \in [t_{m-1}, t_m]$. We can again prove inductively that $-u'(t) > -\mu_{2m+4}$ for every $t \in [\xi_0, t_1]$. If we put $\mu_* = \mu_{2m+4}$, then $\mu_* > \mu_j$ for all $j \in \{1, \ldots, 2m+3\}$ and therefore $|u'(t)| \leq \mu_*$ for all $t \in J$. □

Proof of Theorem 1.2 Define functions

$$\chi(s, r^*) = \begin{cases} 1 & \text{if } \ 0 \leq s \leq r^*, \\ 2 - \frac{s}{r^*} & \text{if } \ r^* < s < 2r^*, \\ 0 & \text{if } \ s \geq 2r^* \end{cases}$$

and

$$g(t, x, y) = \chi(|x| + |y|, r^*) \cdot f(t, x, y),$$

for $t \in J$, $x, y \in R$, where $r^* = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + \max\{\mu_*, \ \|\sigma_1'\|_\infty, \ \|\sigma_2'\|_\infty\}$ for μ_* given by Lemma 3.1. For $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times R$, the function $g(t, x, y)$ is bounded on J by a Lebesgue integrable function. In addition, σ_1, σ_2 are respectively lower and upper functions of the problem

$$\frac{d}{dt}[\phi(x'(t))] = g(t, x(t), x'(t)), \quad (0.2), (0.3). \tag{3.14}$$

According to Theorem 2.1 there exists a solution u of the problem (3.14) fulfiling $\sigma_1 \leq u \leq \sigma_2$ on J. Moreover,

$$g(t, x, y) =$$
$$= \chi(|x| + |y|, r^*) \cdot f(t, x, y) \leq \chi(|x| + |y|, r^*) \cdot \omega(|y|)(k + |y|) \leq \omega(|y|)(k + |y|)$$

for a.e. $t \in J$, for all $x \in [\sigma_1, \sigma_2]$, every $y \in R$. It means that function g satisfies condition (1.12) which implies that

$$[\phi(u'(t))]' = g(t, u(t), u'(t)) \leq \omega(|u'(t)|)(k(t) + |u'(t)|) \quad \text{for a.e } t \in J.$$

Then, according to Lemma 3.1, $|u'(t)| \leq \mu_*$ holds for all $t \in J$. So $\|u\|_\infty + \|u'\|_\infty < \ r^*$ and $g(t, u, u') = f(t, u, u')$ for a.e. $t \in J$. It means that a solution u of the problem (3.14) is a solution of the problem (0.1)–(0.3), too. It concludes the proof of Theorem 1.2. □

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