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A Groupoid Characterization of Orthomodular Lattices *

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Abstract

We prove that an orthomodular lattice can be considered as a groupoid with a distinguished element satisfying simple identities.

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A bounded lattice is called an *ortholattice* if there is a unary operation $x \mapsto x^\perp$ called *orthocomplementation* such that

$x \vee x^\perp = 1$ and $x \wedge x^\perp = 0$ (i.e. $x^\perp$ is a *complement* of x)

$x^{\perp\perp} = x$ (it is an *involution*)

$x \leq y$ implies $y^\perp \leq x^\perp$ (it is *antitone*).

An ortholattice is thus considered as an algebra $\mathcal{L} = (L; \vee, \wedge, {}^\perp, 0, 1)$ of type $(2, 2, 1, 0, 0)$. Due to the above mentioned properties of orthocomplementation, it satisfies the De Morgan laws, i.e.

$(x \vee y)^\perp = x^\perp \wedge y^\perp$ and $(x \wedge y)^\perp = x^\perp \vee y^\perp$.

Hence, it can be considered also in the signature $(\vee, {}^\perp, 0)$ of type $(2, 1, 0)$ because $\wedge$ can be expressed by De Morgan laws as a term function in $\vee$ and ${}^\perp$ and $1 = 0^\perp$.

An ortholattice $\mathcal{L} = (L; \vee, \wedge, {}^\perp, 0, 1)$ is called *orthomodular* if it satisfies the implication

$x \leq y \Rightarrow x \vee (x^\perp \wedge y) = y$ (the *orthomodular law*)

which is equivalent to $x \leq y \Rightarrow y \wedge (y^\perp \vee x) = x$.

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The orthomodular law is apparently equivalent to the following identity

$$x \vee (x^{\perp} \wedge (x \vee y)) = x \vee y \tag{OMI}$$

or, equivalently,

$$(x \vee y) \wedge ((x \vee y)^{\perp} \vee x) = x.$$

In what follows we will show that an orthomodular lattice can be discern as an algebra of type $(2,0)$ in the signature $(\circ, 0)$, i.e. as a groupoid with a distingushed element. Let us note that Boolean algebras were characterized in this way already by the author in [4].

Definition 1 An algebra $\mathcal{A} = (A; \circ, 0)$ of type $(2,0)$ is called an *OI-algebra* if it satisfies the following identities

(I0) $0 \circ x = 1$, where 1, denotes $0 \circ 0$

(I1) $(x \circ y) \circ x = x$

(I2) $(x \circ y) \circ y = (y \circ x) \circ x$

The proofs of the following lemmas are taken from [1].

Lemma 1 *Every OI-algebra satisfies the following identities*

(a) $x \circ (x \circ y) = x \circ y$

(b) $x \circ x = (x \circ y) \circ (x \circ y)$

Proof Applying (I1) twice, we obtain $x \circ (x \circ y) = ((x \circ y) \circ x) \circ (x \circ y) = x \circ y$, proving (a). For (b), we apply (I1), (I2) and (a):

$$x \circ x = ((x \circ y) \circ x) \circ x = (x \circ (x \circ y)) \circ (x \circ y) = (x \circ y) \circ (x \circ y). \qquad \square$$

Lemma 2 *Every OI-algebra satisfies the identities*

$$x \circ x = 1, \quad 1 \circ x = x, \quad x \circ 1 = 1.$$

Proof By Lemma 1(b) used twice we conclude $x \circ x = (x \circ y) \circ (x \circ y) = ((x \circ y) \circ y) \circ ((x \circ y) \circ y) = ((y \circ x) \circ x) \circ ((y \circ x) \circ x)(y \circ x) \circ (y \circ x) = y \circ y$. For $y = 0$ we obtain $x \circ x = 0 \circ 0 = 1$.

Now, $1 \circ x = (x \circ x) \circ x = x$ by (I1) and $x \circ 1 = x \circ (x \circ x) = x \circ x = 1$ by Lemma 1 and the firstly proved identity. $\square$

Definition 2 An OI-algebra $\mathcal{A} = (A; \circ, 0)$ is called *antitone* if it satisfies the identity

(I3) $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = 1$ (where $1 = 0 \circ 0$).

Lemma 3 *Let $\mathcal{A} = (A; \circ, 0)$ be an antitone OI-algebra. Define a binary relation $\leq$ on A as follows*

$$x \leq y \quad \text{if and only if} \quad x \circ y = 1.$$

Then $\leq$ is an order on A such that $0 \leq x \leq 1$ for each $x \in A$ and

$$x \leq y \ \text{ implies } \ y \circ z \leq x \circ z \quad \text{for all } x, y, z \in A.$$

Proof Due to Lemma 2, $\leq$ is reflexive.

Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1$ and $y \circ x = 1$ thus, by (I2), $y = 1 \circ y = (x \circ y) \circ y = (y \circ x) \circ x = 1 \circ x = x$, i.e. $\leq$ is antisymmetric. Prove transitivity of $\leq$. Let $x \leq y$ and $y \leq z$. Then $x \circ y = 1$, $y \circ z = 1$ and, by (I3),

$$\begin{aligned} 1 &= (((x \circ y) \circ y) \circ z) \circ (x \circ z) = ((1 \circ y) \circ z) \circ (x \circ z) \\ &= (y \circ z) \circ (x \circ z) = 1 \circ (x \circ z) = x \circ z \end{aligned}$$

thus $x \leq z$. Hence, $\leq$ is an order on A. Due to (I0), $0 \leq x$ and, by Lemma 2, $x \leq 1$ for each $x \in A$.

Suppose $x \leq y$. Then $x \circ y = 1$ and, by (I3),

$$(y \circ z) \circ (x \circ z) = (((x \circ y) \circ y) \circ z) \circ (x \circ z) = 1,$$

whence $y \circ z \leq x \circ z$. □

In spite of Lemma 3, the relation $\leq$ on an antitone OI-algebra $\mathcal{A}$ will be called the *induced order of* $\mathcal{A}$.

Theorem 1 *Let $\mathcal{A} = (A; \circ, 0)$ be an antitone OI-algebra, $\leq$ the induced order on A. Then $(A; \leq)$ is a bounded lattice where $x \vee y = (x \circ y) \circ y$, and the mapping $x \mapsto x \circ 0$, is an antitone involution on $(A; \leq)$.*

Proof Since $y \leq 1$ for each $y \in A$, Lemma 3 yields $x = 1 \circ x \leq y \circ x$, i.e. $\mathcal{A}$ satisfies the identity

$$x \circ (y \circ x) = 1. \tag{B}$$

Suppose now $a, b \in A$. Then, by (B), $b \circ ((a \circ b) \circ b) = 1$ and, by (B) and (I2), $a \circ ((a \circ b) \circ b) = a \circ ((b \circ a) \circ a) = 1$, i.e. $a \leq (a \circ b) \circ b$ and $b \leq (a \circ b) \circ b$.

Suppose further $a \leq c$ and $b \leq c$. Then $b \circ c = 1$ and, by Lemma 3, $c \circ b \leq a \circ b$. Hence

$$(a \circ b) \circ b \leq (c \circ b) \circ b = (b \circ c) \circ c = 1 \circ c = c.$$

We have shown that $(a \circ b) \circ b$ is the least common upper bound of a, b, i.e.

$$a \vee b = (a \circ b) \circ b$$

and $(A; \vee)$ is a $\vee$-semilattice.

Consider the mapping $x \mapsto x \circ 0$. Then $(x \circ 0) \circ 0 = x \vee 0 = x$, i.e. it is an involution on A. By Lemma 3, this involution is antitone. Hence, we can apply De Morgan law to prove $a \wedge b = ((a \circ 0) \vee (b \circ 0)) \circ 0$ for each $a, b \in A$, i.e. $(A; \vee, \wedge)$ is a bounded lattice. □

Definition 3 An antitone OI-algebra is called an *OML-algebra* if it satisfies the identity

(I4) $(x \circ y) \circ y = (((x \circ y) \circ y) \circ 0) \circ x$.

Remark 1 By Theorem 1, (I4) can be read as

$$x \vee y = ((x \vee y) \circ 0) \circ x \tag{C}$$

which being equivalent to

$$x \leq y \Rightarrow y = (y \circ 0) \circ x. \tag{D}$$

Let $\mathcal{A}$ be an antitone OI-algebra, $\leq$ its induced order. By Theorem 1, $(A; \leq)$ is a bounded lattice. Denote this lattice by $\mathcal{L}(\mathcal{A})$ and call it the *assigned lattice of* $\mathcal{A}$.

Theorem 2 *Let* $\mathcal{A} = (A; \circ, 0)$ *be an OML-algebra. Then its assigned lattice* $\mathcal{L}(\mathcal{A})$ *is an orthomodular lattice where the orthocomplement of* $x \in A$ *is*

$$x^{\perp} = x \circ 0.$$

Proof Take $y = 0$ in (I4). We obtain

$$x = (x \circ 0) \circ 0 = (((x \circ 0) \circ 0) \circ 0) \circ x = (x \circ 0) \circ x,$$

thus

$$1 = x \circ x = ((x \circ 0) \circ x) \circ x = (x \circ 0) \vee x.$$

By Theorem 1, $x \mapsto x \circ 0$ is an antitone involution, thus, due to De Morgan laws,

$$0 = (x \circ 0) \wedge x$$

and hence $x^{\perp} = x \circ 0$ is an orthocomplement of $x \in A$.

By Theorem 1, we obtain immediately

$$x \circ y = ((x \circ y) \circ y) \circ y. \tag{E}$$

It remains to prove the orthomodular law. Let $x \leq y$. Then $x \circ y = 1$ and, by (I4), (I2) and (E), we derive

$$\begin{aligned} y = (y \circ 0) \circ x &= (((y \circ 0) \circ x) \circ x) \circ x = ((x \circ (y \circ 0)) \circ (y \circ 0)) \circ x \\ &= ((((x \circ (y \circ 0)) \circ (y \circ 0)) \circ 0) \circ x) \circ x = (((((y \circ 0) \circ x) \circ x) \circ 0) \circ x) \circ x \\ &= (y^{\perp} \vee x)^{\perp} \vee x = (y \wedge x^{\perp}) \vee x. \end{aligned}$$

Thus the assigned lattice $\mathcal{L}(\mathcal{A})$ is an orthomodular lattice. □

Also, conversely, to every orthomodular lattice $\mathcal{L} = (L; \vee, \wedge, {}^{\perp}, 0, 1)$ an OML-algebra can be assigned as follows.

Theorem 3 *Let* $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ *be an orthomodular lattice. Consider the term function*

$$x \circ y = (x \vee y)^{\perp} \vee y.$$

Then $\mathcal{A}(\mathcal{L}) = (L; \circ, 0)$ *is an OML-algebra.*

Proof Of course, $0 \circ 0 = 0^{\perp} \vee 0 = 1 \vee 0 = 1$. Further,

$$0 \circ x = (0 \vee x)^{\perp} \vee x = x^{\perp} \vee x = 1$$

proving (I0). To prove (I2), we use the identity (OMI) equivalent to the orthomodular law:

$$\begin{aligned}(x \circ y) \circ y &= (((x \vee y)^{\perp} \vee y) \vee y)^{\perp} \vee y = ((x \vee y)^{\perp} \vee y)^{\perp} \vee y \\ &= ((x \vee y) \wedge y^{\perp}) \vee y = x \vee y,\end{aligned}$$

i.e. also $(y \circ x) \circ x = y \vee x = x \vee y = (x \circ y) \circ y$. We prove (I1):

$$(x \circ y) \circ x = (((x \vee y)^{\perp} \vee y) \vee x)^{\perp} \vee x = 1^{\perp} \vee x = 0 \vee x = x.$$

For (I3), we firstly prove the following

Claim: $x \leq y$ if and only if $x \circ y = 1$.

Proof: If $x \leq y$ then $x \circ y = (x \vee y)^{\perp} \vee y = y^{\perp} \vee y = 1$. Conversely, suppose $x \circ y = 1$. Then $(x \vee y)^{\perp} \vee y = 1$, hence by the orthomodular law

$$x \vee y = (x \vee y) \wedge ((x \vee y)^{\perp} \vee y) = y,$$

i.e. $x \leq y$. □

Due to the previous part and the Claim, (I3) can be rewritten as

$$(x \vee y) \circ z \leq x \circ z.$$

However,

$$(x \vee y) \circ z = (x \vee y \vee z)^{\perp} \vee z \leq (x \vee z)^{\perp} \vee z = x \circ z$$

thus (I3) is valid in $\mathcal{A}(\mathcal{L})$.

It remains to prove (I4). We have by (OMI)

$$\begin{aligned}(x \circ y) \circ y &= x \vee y = ((x \vee y) \wedge x^{\perp}) \vee x = ((x \vee y)^{\perp} \vee x)^{\perp} \vee x \\ &= ((x \vee y) \circ 0) \circ x = (((x \circ y) \circ y) \circ 0) \circ x.\end{aligned}$$

□

Remark 2 Since $\circ$ is a term function in $\vee$ and $^{\perp}$ and $\vee, \wedge, ^{\perp}$ are term functions in $\circ$ and 0, one can easily verify that the assigning of an OML-algebra to an orthomodular lattice and conversely are mutual inverse correspondences, hence we have

$$\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L} \quad \text{and} \quad \mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}.$$

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