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One Singular Multivariate Linear Model with Nuisance Parameters

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Abstract

The multivariate linear model, in which the matrix of the first order parameters is divided into two matrices: to the matrix of the useful parameters and to the matrix of the nuisance parameters, is considered.

Key words: Singular multivariate linear model, useful and nuisance parameters, BLUE.

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1 Introduction

There are two approaches in the problem of nuisance parameters in the linear models of various structures.

The first one respects the structure of the model and seeks to find classes of linear functionals of useful (main) parameters such that their estimators allow the nuisance parameters to be neglected; the estimators computed under disregarding nuisance parameters remain to be unbiased and efficient. The variance of the estimator belonging to the abovementioned class could behave analogously. The determination of the class having such attributes is of a great importance in practice because the number of nuisance parameters in real situations can be greater than the number of useful parameters.

The second approach solves the problem of nuisance parameters by their elimination by a transformation of the observation vector provided this transformation is not allowed to cause a loss of information on the useful parameters (see [7]).

The aim of this paper is to apply the first approach to one of the multivariate models.

2 Notations and auxiliary statements

Let R^n denote the space of all n-dimensional real vectors, let $\mathbf{u}_p$ and $\mathbf{A}_{m,n}$ denote a real column p-dimensional vector and a real $m \times n$ matrix, respectively. The symbols $\mathbf{A}', \mathbf{A}^{(j)}, \mathscr{M}(\mathbf{A}), \mathscr{N}(\mathbf{A}), r(\mathbf{A}), Tr(\mathbf{A})$ will denote transpose, j-th column, range, null space, rank and trace of the matrix $\mathbf{A}$, respectively. Further $vec(\mathbf{A})$ will denote the column vector $((\mathbf{A}^{(1)})', \ldots, (\mathbf{A}^{(n)})')'$ created by the columns of the matrix $\mathbf{A}$. The symbol $\mathbf{A} \otimes \mathbf{B}$ will denote the Kronecker (tensor) product of the matrices $\mathbf{A}, \mathbf{B}$; $\mathbf{A}^-$ will denote an arbitrary generalized inverse of $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$), $\mathbf{A}^+$ will denote a Moore–Penrose generalized inverse of the matrix $\mathbf{A}$ (satisfying $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $(\mathbf{A}\mathbf{A}^+)' = \mathbf{A}\mathbf{A}^+$, $(\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$). Moreover $\mathbf{P}_A$ and $\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A$ will stand for the ortogonal projector onto $\mathscr{M}(\mathbf{A})$ and $\mathscr{M}^\perp(\mathbf{A}) = \mathscr{N}(\mathbf{A}')$, respectively. The symbol $\mathbf{I}$ denotes the identity matrix, $\mathbf{O}_{m,n}$ the $m \times n$ null matrix, $\mathbf{o}$ the null element. We write

$$\mathbf{A} \overset{\leq}{_L} \mathbf{B} \iff \mathbf{B} - \mathbf{A} \text{ is p.s.d.}$$

If $\mathscr{M}(\mathbf{A}) \subset \mathscr{M}(\mathbf{V})$, $\mathbf{V}$ p.s.d., then the symbol $\mathbf{P}_A^V$ denotes the projector on the subspace $\mathscr{M}(\mathbf{A})$ in the $\mathbf{V}$-seminorm given by the matrix $\mathbf{V}$,

$$||\mathbf{x}||_V = \sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}}; \qquad \mathbf{M}_A^V = \mathbf{I} - \mathbf{P}_A^V = \mathbf{I} - \mathbf{A}(\mathbf{A}'\mathbf{V}\mathbf{A})^-\mathbf{A}'\mathbf{V}.$$

Let $\mathbf{N}_{n,n}$ is p.d. (p.s.d.) matrix and $\mathbf{A}_{m,n}$ an arbitrary matrix, then the symbol $\mathbf{A}^-_{m(N)}$ denotes the matrix satisfying

$$\mathbf{A}\mathbf{A}^-_{m(N)}\mathbf{A} = \mathbf{A} \quad \text{and} \quad \mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A} = [\mathbf{N}\mathbf{A}^-_{m(N)}\mathbf{A}]'.$$

($\mathbf{A}^-_{m(N)}\mathbf{y}$ is a solution of the consistent system $\mathbf{A}\mathbf{x} = \mathbf{y}$ whose N-seminorm is minimal, see [4], p.151). $\mathbf{A}^-_{m(N)}$ is called a minimum N-seminorm g-inverse of the matrix $\mathbf{A}$. Let $\mathscr{A}^-_{m(N)}$ be a class of all matrices $\mathbf{A}^-_{m(N)}$.

Assertion 1 (see [1], Lemma 10.1.18)

$$\mathscr{M}(\mathbf{A}') \subset \mathscr{M}(\mathbf{N}) \implies \mathbf{N}^-\mathbf{A}'(\mathbf{A}\mathbf{N}^-\mathbf{A}')^- \in \mathscr{A}^-_{m(N)},$$

otherwise

$$(\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}'[\mathbf{A}(\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}']^- \in \mathscr{A}^-_{m(N)}.$$

Assertion 2 (see [1], Lemma 10.1.35) Let $\mathbf{S}$ be any $n \times k$ matrix and $\mathbf{N}$ an $n \times n$ p.s.d. matrix.

1. If $\mathbf{N}$ is p.d., then $(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = \mathbf{N}^{-1} - \mathbf{N}^{-1}\mathbf{S}(\mathbf{S}'\mathbf{N}^{-1}\mathbf{S})^-\mathbf{S}'\mathbf{N}^{-1}$.
2. If $\mathbf{N}$ is not p.d., however $\mathscr{M}(\mathbf{S}) \subset \mathscr{M}(\mathbf{N})$, then

$$(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = \mathbf{N}^+ - \mathbf{N}^+\mathbf{S}(\mathbf{S}'\mathbf{N}^-\mathbf{S})^-\mathbf{S}'\mathbf{N}^+.$$

3. In general case

$$(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = (\mathbf{N}+\mathbf{S}\mathbf{S}')^+ - (\mathbf{N}+\mathbf{S}\mathbf{S}')^+\mathbf{S}[\mathbf{S}'(\mathbf{N}+\mathbf{S}\mathbf{S}')^-\mathbf{S}]^-\mathbf{S}'(\mathbf{N}+\mathbf{S}\mathbf{S}')^+.$$

4. $$(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = (\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+\mathbf{M}_S = \mathbf{M}_S(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+ = \mathbf{M}_S(\mathbf{M}_S\mathbf{N}\mathbf{M}_S)^+\mathbf{M}_S.$$

Assertion 3 (see [2], Lemma 7, p. 65)

$$\mathscr{M}(\mathbf{B}) \subset \mathscr{M}(\mathbf{A}) \iff \mathbf{A}\mathbf{A}^-\mathbf{B} = \mathbf{B},$$
$$\mathscr{M}(\mathbf{B}') \subset \mathscr{M}(\mathbf{A}') \iff \mathbf{B}\mathbf{A}^-\mathbf{A} = \mathbf{B}.$$

Assertion 4 (see [2], Lemma 8, p. 65)

$$\mathbf{A}\mathbf{B}^-\mathbf{C} \text{ is invariant to the choice of the g-inverse } \mathbf{B}^-$$

$$\iff \mathscr{M}(\mathbf{A}') \subset \mathscr{M}(\mathbf{B}') \text{ and } \mathscr{M}(\mathbf{C}) \subset \mathscr{M}(\mathbf{B}).$$

Assertion 5 If $\mathbf{N}$ is p.s.d. and $\mathbf{A}$ such matrices that $\mathscr{M}(\mathbf{A}) \subset \mathscr{M}(\mathbf{N})$, then

$$\mathscr{M}(\mathbf{A}') = \mathscr{M}(\mathbf{A}'\mathbf{N}^-\mathbf{A}).$$

Proof $\mathbf{A}'\mathbf{N}^-\mathbf{A}$ is invariant to the choice of g-inverse. As $\mathscr{M}(\mathbf{A}'\mathbf{N}^-\mathbf{A}) \subset \mathscr{M}(\mathbf{A}')$, it is sufficient to prove, that $r(\mathbf{A}'\mathbf{N}^+\mathbf{A}) = r(\mathbf{A}')$. Let $\mathbf{N}^+ = \mathbf{J}\mathbf{J}'$, then $r(\mathbf{A}'\mathbf{N}^+\mathbf{A}) = r(\mathbf{A}'\mathbf{J})$. There exists a matrix $\mathbf{F}$ such that $\mathbf{A} = \mathbf{N}\mathbf{F}$. Thus $r(\mathbf{A}') = r(\mathbf{F}'\mathbf{N}) = r(\mathbf{F}'\mathbf{N}\mathbf{N}^+\mathbf{N}) = r(\mathbf{A}'\mathbf{N}^+\mathbf{N}) \le r(\mathbf{A}'\mathbf{N}^+) \le r(\mathbf{A}'\mathbf{J}) \le r(\mathbf{A}')$. □

3 Singular multivariate linear regression model

Let

$$\mathbf{Y} = \mathbf{X}_1\mathbf{B}_1\mathbf{Z}_1 + \mathbf{X}_2\mathbf{B}_2\mathbf{Z}_2 + \boldsymbol{\varepsilon}, \tag{1}$$

be a multivariate linear model under consideration.

Here $\mathbf{Y}$ is an $n \times m$ observation matrix, $\mathbf{X}_1$ of the type $n \times k$, $\mathbf{Z}_1$ of the type $r \times m$, $\mathbf{X}_2$ of the type $n \times l$, $\mathbf{Z}_2$ of the type $s \times m$ are known nonzero matrices.

$\mathbf{B}_1$ of the type $k \times r$ and $\mathbf{B}_2$ of the type $l \times s$ are matrices of unknown nonrandom parameters and $\boldsymbol{\varepsilon}$ of the type $n \times m$ is a random matrix.

Let us consider the situation, where $\mathbf{B}_1$ is a matrix of useful parameters which (or their functions) have to be estimated from the observation matrix and $\mathbf{B}_2$ is a matrix of nuisance parameters.

As it was already said the purpose of this paper is to characterize the class of all linear functions of the useful parameters $vec(\mathbf{B})$ which are unbiasedly estimable under the model with nuisance parameters and under the model, where the nuisance parameters are neglected and estimators of which have the same variance in both models mentioned.

A parametric function $\mathbf{p}'vec(\mathbf{B}_1)$ is said to be unbiasedly estimable under the model (1) if there exists an estimator $\mathbf{f}'vec(\mathbf{Y}), \mathbf{f} \in R^{mn}$, such that $E[\mathbf{f}'vec(\mathbf{Y})] = \mathbf{p}'vec(\mathbf{B}_1)$, $\forall vec(\mathbf{B}_1)$, $\forall vec(\mathbf{B}_2)$.

Lemma 1 *The model (1) can be equivalently written in the form*

$$vec(\mathbf{Y}) = [\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{Z}_2' \otimes \mathbf{X}_2] \begin{pmatrix} vec(\mathbf{B}_1) \\ vec(\mathbf{B}_2) \end{pmatrix} + vec(\boldsymbol{\varepsilon}).$$

Proof The assertion is a consequence of

$$vec(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})vec(\mathbf{B}),$$

valid for all matrices of corresponding types. □

Suppose that the observation vector $vec(\mathbf{Y})$ has the mean value

$$E(vec(\mathbf{Y})) = [\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{Z}_2' \otimes \mathbf{X}_2] \begin{pmatrix} vec(\mathbf{B}_1) \\ vec(\mathbf{B}_2) \end{pmatrix},$$

and that the columns of the observation matrix $\mathbf{Y}$ satisfy

$$cov(\mathbf{Y}^{(i)}, \mathbf{Y}^{(j)}) = \mathbf{O}, \ \forall i \neq j, \quad var[\mathbf{Y}^{(j)}] = \boldsymbol{\Sigma}, \ \forall j = 1, \ldots, m,$$

where $\boldsymbol{\Sigma}$ is at least positive semidefinite known matrix. Thus

$$var[vec(\mathbf{Y})] = \mathbf{I}_{m,m} \otimes \boldsymbol{\Sigma}_{n,n}.$$

We consider the linear model

$$\left[vec(\mathbf{Y}), (\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{Z}_2' \otimes \mathbf{X}_2) \begin{pmatrix} vec(\mathbf{B}_1) \\ vec(\mathbf{B}_2) \end{pmatrix}, \mathbf{I} \otimes \boldsymbol{\Sigma} \right], \tag{2}$$

with nuisance parameters (great model) and the linear model

$$\left[vec(\mathbf{Y}), (\mathbf{Z}_1' \otimes \mathbf{X}_1) vec(\mathbf{B}_1), \mathbf{I} \otimes \boldsymbol{\Sigma} \right], \tag{3}$$

where nuisance parameters are neglected (small model).

The paper [5] deals with following assumption

$$\mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{Z}_2' \otimes \mathbf{X}_2) \subset \mathscr{M}(\mathbf{I} \otimes \boldsymbol{\Sigma}). \tag{4}$$

Here the general situation will be considered.

Notation 2 Let $\mathscr{E}_a$ and $\mathscr{E}$ denote the sets of all linear functions of $vec(\mathbf{B}_1)$ which are unbiasedly estimable under the model (2) and (3), respectively (see [8]). The index a will indicate, that the estimator is considered in the complete model, i.e. in the model with nuisance parameters.

Lemma 2

$$\mathscr{E} = \{\mathbf{p}'vec(\mathbf{B}) : \mathbf{p} \in \mathscr{M}(\mathbf{Z}_1 \otimes \mathbf{X}_1')\}. \quad (5)$$

$$\begin{aligned}\mathscr{E}_a &= \{\mathbf{p}'vec(\mathbf{B}) : \mathbf{p} \in \mathscr{M}[(\mathbf{Z}_1 \otimes \mathbf{X}_1')\mathbf{M}_{Z_2'\otimes X_2}] \\ &= \mathscr{M}[(\mathbf{Z}_1 \otimes \mathbf{X}_1') - (\mathbf{Z}_1\mathbf{P}_{Z_2'} \otimes \mathbf{X}_1'\mathbf{P}_{X_2})]\}. \quad (6)\end{aligned}$$

Proof see [5], Lemma 2.

Comparing (5) and (6) it is obvious that

$$\mathscr{E}_a \subset \mathscr{E}.$$

Moreover,

Lemma 3 *Under the condition* $\mathscr{E}_a \subset \mathscr{E}$

$$\mathscr{E}_a = \mathscr{E} \iff \mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1) \cap \mathscr{M}(\mathbf{Z}_2' \otimes \mathbf{X}_2) = \{\mathbf{o}\} \quad (7)$$

Proof see [5], Lemma 3.

We assume throughout that $\mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1) \not\subset \mathscr{M}(\mathbf{Z}_2' \otimes \mathbf{X}_2)$. If $\mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1) \subset \mathscr{M}(\mathbf{Z}_2' \otimes \mathbf{X}_2)$, then $\mathscr{M}[(\mathbf{Z}_1 \otimes \mathbf{X}_1') - (\mathbf{Z}_1\mathbf{P}_{Z_2'} \otimes \mathbf{X}_1'\mathbf{P}_{X_2})] = \{\mathbf{o}\}$.

Notation 3 Let us denote

$$\mathbf{T} = (\mathbf{I} \otimes \mathbf{\Sigma}) + (\mathbf{Z}_1' \otimes \mathbf{X}_1)(\mathbf{Z}_1 \otimes \mathbf{X}_1') = (\mathbf{I} \otimes \mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1 \otimes \mathbf{X}_1\mathbf{X}_1').$$

Theorem 1 *The BLUE of the vector function* $(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\mathbf{B}_1)$ *under the model (3) is given by*

$$\widehat{(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\mathbf{B}_1)} = \mathbf{P}^{T+}_{Z_1'\otimes X_1} vec(\mathbf{Y}), \quad (8)$$

$$\begin{aligned}&var[\widehat{(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\mathbf{B}_1)}] = \\ &= (\mathbf{Z}_1' \otimes \mathbf{X}_1)\left\{\left[(\mathbf{Z}_1 \otimes \mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1' \otimes \mathbf{X}_1)\right]^- - \mathbf{I}\right\}(\mathbf{Z}_1 \otimes \mathbf{X}_1'). \quad (9)\end{aligned}$$

Proof According to Theorem 3.1.3 in [1]

$$\widehat{(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\mathbf{B}_1)} = (\mathbf{Z}_1' \otimes \mathbf{X}_1)\left\{[(\mathbf{Z}_1' \otimes \mathbf{X}_1)'_{m(I\otimes\Sigma)}]^-\right\}' vec(\mathbf{Y})$$

$$= (\mathbf{Z}_1' \otimes \mathbf{X}_1)[(\mathbf{Z}_1 \otimes \mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1' \otimes \mathbf{X}_1)]^-(\mathbf{Z}_1 \otimes \mathbf{X}_1')\mathbf{T}^+ vec(\mathbf{Y}) = \mathbf{P}^{T+}_{Z_1'\otimes X} vec(\mathbf{Y}),$$

where Assertion 1, the inclusion $\mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1) \subset \mathscr{M}(\mathbf{T})$ and the fact that under the model (3)

$$P[vec(\mathbf{Y}) \in \mathscr{M}(\mathbf{Z}_1' \otimes \mathbf{X}_1, \mathbf{I} \otimes \mathbf{\Sigma})] = 1$$

have been utilized. Further

$$\begin{aligned}
var[\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}] &= (\mathbf{Z}_1'\otimes\mathbf{X}_1)\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+ \\
&\times[\mathbf{T}-(\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')]\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)]^-(\mathbf{Z}_1\otimes\mathbf{X}_1') \\
&= (\mathbf{Z}_1'\otimes\mathbf{X}_1)\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+\mathbf{T}\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1) \\
&\times\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1') \\
&-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1) \\
&\times(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^-(\mathbf{Z}_1\otimes\mathbf{X}_1') \\
&= (\mathbf{Z}_1'\otimes\mathbf{X}_1)\left\{\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right]^- - \mathbf{I}\right\}(\mathbf{Z}_1\otimes\mathbf{X}_1').
\end{aligned}$$

The Assertion 3, the equality $\mathscr{M}[\mathbf{Z}_1'\otimes\mathbf{X}_1] = \mathscr{M}[(\mathbf{Z}_1'\otimes\mathbf{X}_1)\mathbf{T}^+(\mathbf{Z}_1\otimes\mathbf{X}_1')]$ and the fact, that under the model (3) $P[vec(\mathbf{Y})\in\mathscr{M}(\mathbf{Z}_1'\otimes\mathbf{X}_1,\mathbf{I}\otimes\mathbf{\Sigma})]=1$ have been taken into account. □

Theorem 2 *Let us assume that* $\mathscr{M}(\mathbf{Z}_2'\otimes\mathbf{X}_2)\subset\mathscr{M}(\mathbf{M}_{Z_1'\otimes X_1})$, *then the BLUE of the parametric function* $\mathbf{p}'vec(\mathbf{B}_1)$, $\mathbf{p}\in\mathscr{M}[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{M}_{Z_2'\otimes X_2}]$ *in the model (2) is of the form* $\mathbf{g}'vec(\mathbf{Y})$ *where*

$$\begin{aligned}
\mathbf{g} &= \left[\mathbf{M}_{Z_2'\otimes X_2}^{M_{Z_1'\otimes X_1}}(\mathbf{I}\otimes\mathbf{\Sigma})+(\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1) \\
&\times\left\{(\mathbf{Z}_1\otimes\mathbf{X}_1')\left[\mathbf{M}_{Z_2'\otimes X_2}^{M_{Z_1'\otimes X_1}}(\mathbf{I}\otimes\mathbf{\Sigma})+(\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right\}^-\mathbf{p}.
\end{aligned}$$

Proof Let us denote $\mathscr{U}_0$ the class of all unbiased estimators of the null function $\mathbf{p}'vec(\mathbf{B}_1)=0$, i.e.

$$\begin{aligned}
\mathscr{U}_0 = \{\mathbf{g}_0'vec(\mathbf{Y}) : E[\mathbf{g}_0'vec(\mathbf{Y})] &= \mathbf{g}_0'[(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)+(\mathbf{Z}_2'\otimes\mathbf{X}_2)vec(\mathbf{B}_2)] \\
&= \mathbf{p}'vec(\mathbf{B}_1)=0,\ \forall vec(\mathbf{B}_1),\ \forall vec(\mathbf{B}_2)\} \\
&= \left\{\mathbf{u}'\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}vec(\mathbf{Y}) : \mathbf{u}\in R^{rk+sl}\right\}.
\end{aligned}$$

According to the basic lemma on the best estimators (see [3], p. 84) the statistic $\mathbf{g}'vec(\mathbf{Y})$ is the BLUE of the function $\mathbf{p}'vec(\mathbf{B}_1)$ iff

$$\begin{gathered}
cov[\mathbf{u}'\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}vec(\mathbf{Y}), \mathbf{g}'vec(\mathbf{Y})] = \\
= \mathbf{u}'\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g}=0,\quad \forall\mathbf{u}\in R^{rk+sl}, \\
\iff \mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g}=\mathbf{o}.
\end{gathered}$$

Thus we have to find a vector $\mathbf{g}$ such that

$$\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g}=\mathbf{o}\wedge(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g}=\mathbf{p}.$$

Using the relation (see [6], Lemma 1)

$$\mathbf{M}_{(Z_1'\otimes X_1, Z_2'\otimes X_2)} = \mathbf{M}_{Z_1'\otimes X_1}\mathbf{M}_{Z_2'\otimes X_2}^{M_{Z_1'\otimes X_1}},$$

and notation $\mathbf{A} = \mathbf{Z}_1' \otimes \mathbf{X}_1,\ \mathbf{B} = \mathbf{Z}_2' \otimes \mathbf{X}_2$ we get

$$\mathbf{P}_A\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} + \mathbf{M}_A\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} = \mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g},$$

it means we must find the vector $\mathbf{g}$ such that

$$(\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{A}'\mathbf{A})^-\mathbf{A}'\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} = \mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} \wedge (\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g} = \mathbf{p},$$

i.e. vector $\mathbf{g}$ such that

$$(\mathbf{Z}_1'\otimes\mathbf{X}_1)\mathbf{v} = \mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} \wedge (\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g} = \mathbf{p}.$$

We have

$$\begin{aligned}
&\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{g} + (\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g} = (\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{v}+\mathbf{p}),\\
&\Longrightarrow\ \mathbf{g} = \left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{v}+\mathbf{p}).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{p} &= (\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{g}\\
&= (\mathbf{Z}_1\otimes\mathbf{X}_1')\left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)(\mathbf{v}+\mathbf{p}),\\
\Longrightarrow\ \mathbf{v}+\mathbf{p} &= \left\{(\mathbf{Z}_1\otimes\mathbf{X}_1')\left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right\}^-\mathbf{p},\\
\Longrightarrow\ \mathbf{g} &= \left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\\
&\times\left\{(\mathbf{Z}_1\otimes\mathbf{X}_1')\left[\mathbf{M}_A^{M_B}(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')\right]^-(\mathbf{Z}_1'\otimes\mathbf{X}_1)\right\}^-\mathbf{p}.
\end{aligned}$$

□

Theorem 3 *The BLUE of the vector function*

$$(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1) + (\mathbf{Z}_2'\otimes\mathbf{X}_2)vec(\mathbf{B}_2)$$

under the model (2) is given by

$$\begin{aligned}
&[(\mathbf{Z}_1'\otimes\mathbf{X}_1)\widehat{vec(\mathbf{B}_1) + (\mathbf{Z}_2'\otimes}\mathbf{X}_2)vec(\mathbf{B}_2)]_a\\
&= \left[\mathbf{P}_A^{U^+} + \mathbf{M}_A^{U^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\right]vec(\mathbf{Y}),
\end{aligned}$$

where $\mathbf{U} = (\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1') + (\mathbf{Z}_2'\mathbf{Z}_2\otimes\mathbf{X}_2\mathbf{X}_2')$, $\mathbf{A} = \mathbf{Z}_1'\otimes\mathbf{X}_1$, $\mathbf{S} = \mathbf{Z}_2'\otimes\mathbf{X}_2$.

Proof According to the Theorem 3.1.3 in [1] we have in the model (2)

$$(\mathbf{A},\mathbf{S})\widehat{\begin{pmatrix} vec(\mathbf{B}_1)\\ vec(\mathbf{B}_2)\end{pmatrix}_a} = (\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^-_{m(I\otimes\Sigma)}\right]' vec(\mathbf{Y})$$

$$= (\mathbf{A},\mathbf{S})\left\{\left[(\mathbf{I}\otimes\mathbf{\Sigma}) + (\mathbf{A},\mathbf{S})\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}\right]^-(\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}\mathbf{U}^-(\mathbf{A},\mathbf{S})\right]^-\right\}' vec(\mathbf{Y}),$$

where $\mathbf{U} = (\mathbf{I}\otimes\mathbf{\Sigma}) + \mathbf{A}\mathbf{A}' + \mathbf{S}\mathbf{S}'$.

Using the following Rohde's formula for generalized inverse of partitioned p.s.d. matrix (see [2], Lemma 13, p. 68)

$$\begin{pmatrix}\mathbf{A}, & \mathbf{B}\\ \mathbf{B}', & \mathbf{C}\end{pmatrix}^- = \begin{pmatrix}\mathbf{A}^- + \mathbf{A}^-\mathbf{B}(\mathbf{C}-\mathbf{B}'\mathbf{A}^-\mathbf{B})^-\mathbf{B}'\mathbf{A}^-, & -\mathbf{A}^-\mathbf{B}(\mathbf{C}-\mathbf{B}'\mathbf{A}^-\mathbf{B})^-\\ -(\mathbf{C}-\mathbf{B}'\mathbf{A}^-\mathbf{B})^-\mathbf{B}'\mathbf{A}^-, & (\mathbf{C}-\mathbf{B}'\mathbf{A}^-\mathbf{B})^-\end{pmatrix}$$

we get

$$\begin{pmatrix}\mathbf{A}'\mathbf{U}^-\mathbf{A}, & \mathbf{A}'\mathbf{U}^-\mathbf{S}\\ \mathbf{S}'\mathbf{U}^-\mathbf{A}, & \mathbf{S}'\mathbf{U}^-\mathbf{S}\end{pmatrix}^- = \begin{pmatrix}\mathbf{A}_{11}, & \mathbf{A}_{12}\\ \mathbf{A}_{21}, & \mathbf{A}_{22}\end{pmatrix},$$

$$\begin{aligned}
\mathbf{A}_{11} &= (\mathbf{A}'\mathbf{U}^-\mathbf{A})^- + (\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}\\
&\quad\times[\mathbf{S}'\mathbf{U}^+\mathbf{S} - \mathbf{S}'\mathbf{U}^+\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^+\mathbf{S}]^-\mathbf{S}'\mathbf{U}^+\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\\
&= (\mathbf{A}'\mathbf{U}^-\mathbf{A})^- + (\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-,\\
\mathbf{A}_{12} &= -(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}']^- = (\mathbf{A}_{21})',\\
\mathbf{A}_{22} &= [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-.
\end{aligned}$$

After some calculations we get

$$(\mathbf{A},\mathbf{S})\widehat{\begin{pmatrix} vec(\mathbf{B}_1)\\ vec(\mathbf{B}_2)\end{pmatrix}_a} = (\mathbf{A},\mathbf{S})$$

$$\times\begin{pmatrix}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^- - (\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\\ [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\end{pmatrix} vec(\mathbf{Y}).$$

Since $\mathscr{M}(\mathbf{A}) \subset \mathscr{M}(\mathbf{U})$, $\mathscr{M}(\mathbf{S}) \subset \mathscr{M}(\mathbf{U})$, the expressions $\mathbf{A}'\mathbf{U}^-\mathbf{A}$, $\mathbf{A}'\mathbf{U}^-\mathbf{A}$ are invariant to the choice of g-inverse. Thus using the fact that

$$P\{vec(\mathbf{Y}) \in \mathscr{M}[(\mathbf{A},\mathbf{S}),(\mathbf{I}\otimes\mathbf{\Sigma})]\} = 1$$

we can write

$$\widehat{\mathbf{A}vec(\mathbf{B}_1)_a} = \left[\mathbf{P}_A^{U^+} - \mathbf{P}_A^{U^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\right] vec(\mathbf{Y}),$$

$$\widehat{\mathbf{S}vec(\mathbf{B}_2)_a} = \mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+ vec(\mathbf{Y}),$$

i.e.

$$\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)_a + (\mathbf{Z}_2'}\otimes\mathbf{X}_2)vec(\mathbf{B}_2)_a$$

$$= \left[\mathbf{P}_A^{U^+} + \mathbf{M}_A^{U^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\right] vec(\mathbf{Y}). \quad \square$$

Corollary 1 *Let in the Theorem 3 the condition* $\mathcal{M}(\mathbf{S}) \subset \mathcal{M}(\mathbf{T})$, *where* $\mathbf{T} = (\mathbf{I} \otimes \mathbf{\Sigma}) + \mathbf{A}\mathbf{A}'$, $\mathbf{A} = \mathbf{Z}_1' \otimes \mathbf{X}_1$, $\mathbf{S} = \mathbf{Z}_2' \otimes \mathbf{X}_2$, *is valid. Then*

$$[(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\widehat{\mathbf{B}_1) + (\mathbf{Z}_2' \otimes} \mathbf{X}_2)vec(\mathbf{B}_2)]_a$$
$$= \left[\mathbf{P}_A^{T^+} + \mathbf{M}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\right] vec(\mathbf{Y}).$$

Proof Under the assumption $\mathcal{M}(\mathbf{S}) \subset \mathcal{M}(\mathbf{T})$ one of the matrices

$$\left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix}^-_{m(I\otimes\Sigma)}\right]',$$

is the matrix

$$\left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix}^-_{m(T)}\right]' = \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix}^- \begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix} \mathbf{T}^-,$$

since
a) this matrix is g-inverse of the matrix $(\mathbf{A}, \mathbf{S})$,
b) the matrix

$$(\mathbf{A}, \mathbf{S}) \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix}^- \begin{pmatrix} \mathbf{A}'\mathbf{T}^- \\ \mathbf{S}'\mathbf{T}^- \end{pmatrix} (\mathbf{I} \otimes \mathbf{\Sigma})$$

$$= (\mathbf{A}, \mathbf{S}) \left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix}^-_{m(T)}\right]' \mathbf{T} - (\mathbf{A}, \mathbf{S}) \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix}^- \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A} \end{pmatrix} \mathbf{A}'$$

$$= (\mathbf{A}, \mathbf{S}) \left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix}^-_{m(T)}\right]' \mathbf{T} - \mathbf{A}\mathbf{A}',$$

is symmetrical. Here the relation [valid under the assumption $\mathcal{M}(\mathbf{S}) \subset \mathcal{M}(\mathbf{T})$]

$$(\mathbf{A}, \mathbf{S}) \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix}^- \begin{pmatrix} \mathbf{A}'\mathbf{T}^-\mathbf{A}, & \mathbf{A}'\mathbf{T}^-\mathbf{S} \\ \mathbf{S}'\mathbf{T}^-\mathbf{A}, & \mathbf{S}'\mathbf{T}^-\mathbf{S} \end{pmatrix} = (\mathbf{A}, \mathbf{S}),$$

was utilized. Thus enables us to use the matrix $\mathbf{T}$ instead of the matrix $\mathbf{U}$ in the assertion of the Theorem 3. □

Theorem 4 *The variance of the BLUE of the function*

$$\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}(\mathbf{Z}_1' \otimes \mathbf{X}_1)vec(\mathbf{B}_1), \quad \mathbf{g} \in R^{mn},$$

in the model (2) is given by

$$var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}]_a =$$

$$= var\Big[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\Big\{\mathbf{P}_A^{U^+} - \mathbf{P}_A^{U^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\Big\} vec(\mathbf{Y})\Big]_a$$

$$= \mathbf{g}'\mathbf{M}_S\Big[\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}' - \mathbf{A}\mathbf{A}' + \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}$$
$$\times \big\{[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^- - \mathbf{I}\big\}\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'$$
$$+ \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+$$
$$\times \mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\Big]\mathbf{M}_S\mathbf{g}.$$

Proof We get the assertion after some calculations using the facts that

$$[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+$$
$$= [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]\mathbf{P}_{[S'(M_AUM_A)^+S]} = [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}],$$

$$\mathbf{U}\mathbf{U}^+\mathbf{A} = \mathbf{A}, \quad (\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{A} = \mathbf{O},$$

and that the expressions are invariant to the choice of g-inverses (since it is the variance of the BLUE). □

Remark 1 For the variances

$$var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}], \quad \mathbf{g}\in R^{mn}$$

in the model (2) and in the model (3) holds

$$var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}] = \mathbf{g}'\mathbf{M}_S[\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}' - \mathbf{A}\mathbf{A}']\mathbf{M}_S\mathbf{g}$$
$$\le var[\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}\widehat{(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}]_a$$
$$= \mathbf{g}'\mathbf{M}_S\Big[\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}' - \mathbf{A}\mathbf{A}' + \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}$$
$$\times \big\{[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^- - \mathbf{I}\big\}\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}' + \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}$$
$$\times [\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\Big]\mathbf{M}_S\mathbf{g}.$$

The inequality is a consequence of the fact, that

$$\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}' \underset{L}{\leq} \mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'$$

and that the other two matrices are p.s.d. The matrix

$$\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}\left\{[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^- - \mathbf{I}\right\}\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}$$
$$= \mathbf{P}_A^{U^+}\mathbf{S}\{[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^- - \mathbf{I}\}\mathbf{S}'(\mathbf{P}_A^{U^+})'$$
$$= \mathbf{P}_A^{U^+}\mathbf{S}\{[\mathbf{S}'\mathbf{U}^+\mathbf{S} - \mathbf{S}'\mathbf{U}^+\mathbf{A}(\mathbf{A}'\mathbf{U}^+\mathbf{A})^-\mathbf{A}'\mathbf{U}^+\mathbf{S}]^- - \mathbf{I}\}\mathbf{S}'(\mathbf{P}_A^{U^+})'$$
$$= \mathbf{P}_A^{U^+}\mathbf{S}\{(\mathbf{S}'\mathbf{U}^+\mathbf{S})^- + (\mathbf{S}'\mathbf{U}^+\mathbf{S})^-\mathbf{S}'\mathbf{U}^+\mathbf{A}[\mathbf{A}'(\mathbf{M}_S\mathbf{U}\mathbf{M}_S)^+\mathbf{A}]^+$$
$$\times\,\mathbf{A}'\mathbf{U}^+\mathbf{S}(\mathbf{S}'\mathbf{U}^+\mathbf{S})^- - \mathbf{I}\}\mathbf{S}'(\mathbf{P}_A^{U^+})'$$
$$= \mathbf{P}_A^{U^+}\mathbf{S}\{(\mathbf{S}'\mathbf{U}^+\mathbf{S})^+ - \mathbf{I}\}\mathbf{S}'(\mathbf{P}_A^{U^+})'$$
$$+\,\mathbf{P}_A^{U^+}\mathbf{S}(\mathbf{S}'\mathbf{U}^+\mathbf{S})^-\mathbf{S}'\mathbf{U}^+\mathbf{A}[\mathbf{A}'(\mathbf{M}_S\mathbf{U}\mathbf{M}_S)^+\mathbf{A}]^+\mathbf{A}'\mathbf{U}^+\mathbf{S}(\mathbf{S}'\mathbf{U}^+\mathbf{S})^-\mathbf{S}'(\mathbf{P}_A^{U^+})',$$

is positive semidefinite because $\mathbf{S}[(\mathbf{S}'\mathbf{U}^+\mathbf{S})^+ - \mathbf{I}]\mathbf{S}'$ is p.s.d. It can be proved as follows (see considerations next the Corollary 1.11.6 in [4]):

$$\mathbf{U} = (\mathbf{I}\otimes\boldsymbol{\Sigma}) + \mathbf{A}\mathbf{A}' + \mathbf{S}\mathbf{S}' \underset{L}{\gtreqless} \mathbf{S}\mathbf{S}' \iff \mathbf{U}^+ \underset{L}{\lesseqgtr} (\mathbf{S}\mathbf{S}')^+,$$

$$\implies \mathbf{S}'\mathbf{U}^+\mathbf{S} \underset{L}{\lesseqgtr} \mathbf{S}'(\mathbf{S}\mathbf{S}')^+\mathbf{S} \iff (\mathbf{S}'\mathbf{U}^+\mathbf{S})^+ \underset{L}{\gtreqless} [\mathbf{S}'(\mathbf{S}\mathbf{S}')^+\mathbf{S}]^+ = \mathbf{S}'(\mathbf{S}\mathbf{S}')^+\mathbf{S},$$

$$\implies \mathbf{S}(\mathbf{S}'\mathbf{U}^+\mathbf{S})^+\mathbf{S}' \underset{L}{\lesseqgtr} \mathbf{S}\mathbf{S}'(\mathbf{S}\mathbf{S}')^+\mathbf{S}\mathbf{S}' = \mathbf{S}\mathbf{S}' \iff \mathbf{S}[(\mathbf{S}'\mathbf{U}^+\mathbf{S})^+ - \mathbf{I}]\mathbf{S}' \underset{L}{\gtreqless} \mathbf{O}.$$

The matrix

$$\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A}'\mathbf{U}^-\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+$$
$$\times\,\mathbf{S}'\mathbf{U}^-\mathbf{A}(\mathbf{A}'\mathbf{U}^-\mathbf{A})^-\mathbf{A},$$

is also p.s.d. since $[\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}][\mathbf{S}'(\mathbf{M}_A\mathbf{U}\mathbf{M}_A)^+\mathbf{S}]^+$ is a projection matrix.

We need to find a class of such functions of the useful parameters which are unbiasedly estimable in both models (2), (3) and estimators of which have the same variance. Thus we consider the functions from the class $\mathscr{E}_a$ only.

In [5] was proved (see Theorem 1) that under condition (4) the class of functios mentioned above is

$$\{\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1):$$
$$(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{M}_{Z_2'\otimes X_2}\mathbf{g} \in \mathscr{M}[(\mathbf{Z}_1\otimes\mathbf{X}_1')(\mathbf{I}\otimes\boldsymbol{\Sigma})(\mathbf{Z}_1'\otimes\mathbf{X}_1)\mathbf{M}_{(Z_1\otimes X_1')(I\otimes\Sigma)(Z_2'\otimes X_2)}\}.$$

From the Remark it is obvious that in the general case it is impossible to find conditions uder which

$$var[\widehat{\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}] = var[\widehat{\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}]_a.$$

If we confine us to the situation when the condition

$$\mathscr{M}(\mathbf{S}) \subset \mathscr{M}(\mathbf{T}), \tag{10}$$

i.e.

$$\mathscr{M}(\mathbf{Z}_2'\otimes\mathbf{X}_2) \subset \mathscr{M}[(\mathbf{I}\otimes\boldsymbol{\Sigma}) + (\mathbf{Z}_1'\mathbf{Z}_1\otimes\mathbf{X}_1\mathbf{X}_1')],$$

is valid, it is possible to prove following statement (see [4], Theorem 1.11.7).

Theorem 5 *Let in model (2) the condition (10) be true. Then*

$$var[\widehat{\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}] = var[\widehat{\mathbf{g}'\mathbf{M}_{Z_2'\otimes X_2}(\mathbf{Z}_1'\otimes\mathbf{X}_1)vec(\mathbf{B}_1)}]_a,$$

if and only if

$$(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{M}_{Z_2'\otimes X_2}\mathbf{g}\in\mathscr{M}\left[(\mathbf{Z}_1\otimes\mathbf{X}_1')\mathbf{T}^+(\mathbf{Z}_1'\otimes\mathbf{X}_1)\mathbf{M}_{(Z_1\otimes X_1')T^+(Z_2'\otimes X_2)}\right].$$

Proof Using notation $\mathbf{A}=\mathbf{Z}_1'\otimes\mathbf{X}_1$, $\mathbf{S}=\mathbf{Z}_2'\otimes\mathbf{X}_2$ and condition (10), we have in the model (2)

$$\begin{gathered}
var[\widehat{\mathbf{g}'\mathbf{M}_S\mathbf{A}vec(\mathbf{B}_1)}_a] = \\
= var[\mathbf{g}'\mathbf{M}_S\{\mathbf{P}_A^{T^+}-\mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\}vec(\mathbf{Y})] \\
= \mathbf{g}'\mathbf{M}_S\{\mathbf{P}_A^{T^+}-\mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\}(\mathbf{T}-\mathbf{A}\mathbf{A}') \\
\times\{\mathbf{P}_A^{T^+}-\mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\}'\mathbf{M}_S\mathbf{g} \\
= \mathbf{g}'\mathbf{M}_S\{\mathbf{P}_A^{T^+}\mathbf{T}-\mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{T}-\mathbf{A}\mathbf{A}'\} \\
\times\{\mathbf{P}_A^{T^+}-\mathbf{P}_A^{T^+}\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^-\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\}'\mathbf{M}_S\mathbf{g} \\
= \mathbf{g}'\mathbf{M}_S\{\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'-\mathbf{A}\mathbf{A}' \\
+\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{T}^+\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^+\mathbf{S}'\mathbf{T}^+\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\}\mathbf{M}_S\mathbf{g} \\
= var[\widehat{\mathbf{g}'\mathbf{M}_S\mathbf{A}vec(\mathbf{B}_1)}] \\
+\mathbf{g}'\mathbf{M}_S\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{T}^+\mathbf{S}[\mathbf{S}'(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{S}]^+\mathbf{S}'\mathbf{T}^+\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{M}_S\mathbf{g}.
\end{gathered}$$

The second term is zero iff

$$\mathbf{g}'\mathbf{M}_S\mathbf{A}(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{T}^+\mathbf{S}=\mathbf{o}'.$$

It is equivalent to

$$(\mathbf{A}'\mathbf{T}^+\mathbf{A})^-\mathbf{A}'\mathbf{M}_S\mathbf{g}\in\mathscr{M}(\mathbf{M}_{A'T^+S}) \iff \mathbf{A}'\mathbf{M}_S\mathbf{g}\in\mathscr{M}[\mathbf{A}'\mathbf{T}^+\mathbf{A}\mathbf{M}_{A'T^+S}].$$

In the course of the proof the relations $(\mathbf{M}_A\mathbf{T}\mathbf{M}_A)^+\mathbf{A}=\mathbf{O}$, $\mathbf{T}\mathbf{T}^+\mathbf{A}=\mathbf{A}$, $(\mathbf{A}'\mathbf{T}^+\mathbf{A})(\mathbf{A}'\mathbf{T}^+\mathbf{A})^+\mathbf{A}'=\mathbf{A}'$ and the fact, that the expressions are invariant to the choice of the g-inverses have been utilized. □

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