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ON THE ASYMPTOTIC BEHAVIOUR OF A MODULUS OF CONTINUITY WITH DISCRETE DESCRIPTION

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ABSTRACT. The paper deals with a characterization of the behaviour of a modulus of continuity $w(t)$ which is described by a discrete countable set of values.

Many authors frequently deal with a so called modulus of continuity with respect to its applications in the theory of approximation and numerical methods (see e.g. [1], [3] and [2], [4] and the long lists of references therein). For charac terization of classes H_p^w and $W^k H_p^w$ (see e.g. [2] and [4]) it is necessary to know the behaviour of the modulus of continuity w. In this paper, we prove that some discrete description of the function w is sufficient for describing the behaviour of this modulus of continuity, and from there it follows the whole characterization of classes H_p^w.

Any function $w(t)$ defined, continuous and nondecreasing on $[0;1]$ which satisfies two conditions:

(i) $w(0) = 0$,

(ii) $w(t_1 + t_2) \le w(t_1) + w(t_2)$ for any $t_1 \ge 0,\ t_2 \ge 0,\ t_1 + t_2 \le 1$

is called a *modulus of continuity*. For $t > 1$ we put $w(t) = w(1)$.

Remark. Let $w(t) \not\equiv 0$ be an arbitrary modulus of continuity. Then we can prove (see e.g. [2; pp. 182 183]) that there exists a concave modulus of continuity $w_1(t)$ such that

$$w(t) \le w_1(t) \le 2 \cdot w(t)$$

for each $t \in [0;1]$. From here we get that $w(t)$ and $w_1(t)$ are asymptotically equivalent functions for $t \to 0+$. Therefore we can consider any modulus of continuity $w(t)$ to be concave.

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LEMMA 1. (See e.g. [4; p. 109]) *Let $f(t)$ be a continuous and nondecreasing function defined on $[0;1]$ such that*

(a) $f(0)=0$,
(b) *$f(t)/t$ is a nonincreasing function on $(0;1]$.*

Then $f(t)$ is the modulus of continuity.

LEMMA 2. *Let $x>0$. Suppose $w_1(t)$ and $w_2(t)$ are moduli of continuity for which $w_1(2x)\le w_2(2x)$. Then*

$$w_1(t)\le 2\cdot w_2(t)$$

for any $t\in[x;2x]$.

Proof. For $t\in[x;2x]$, according to the property (ii), we have

$$w_1(t)\le w_1(2x)\le w_2(2x)\le 2\cdot w_2(x)\le 2\cdot w_2(t).$$

□

THEOREM. *Let w be a concave modulus of continuity. Then there exists a piecewise linear modulus of continuity w^* with the following properties:*

$$w^*(1/n)=w(1/n) \qquad \text{for any natural } n, \tag{1}$$
$$0,5\cdot w(t)\le w^*(t)\le w(t) \qquad \text{for any } t\in(0;1], \tag{2}$$

and $w^*(0)=0$.

Proof. Putting

$$w^*(0)=0,\qquad w^*(t)=k_n\cdot t+q_n,$$
$$k_n=n\cdot(n+1)\cdot\big[w(1/n)-w\big(1/(n+1)\big)\big],$$
$$q_n=(n+1)\cdot w\big(1/(n+1)\big)-n\cdot w(1/n)$$

for $t\in\left(\frac{1}{n+1};\frac{1}{n}\right]$ $(n=1,2,\dots)$, we get the required function w^*. We can easily verify that w^* is a continuous function on $(0;1]$ (for the "critical" points $t_n=1/n$ it holds $w^*(t_n+)=w^*(t_n)$, $n=1,2,\dots$) with $w^*(0+)=w^*(0)=0$.

Using Lemma 1, we prove that $w^*(t)$ is a modulus of continuity. Suppose $n\in\mathbb{N}$. Denote $\big(1/(n+1);1/n\big)$ by I_n and $\big[1/(n+1);1/n\big]$ by J_n. From concavity of the function $w(t)$ with respect to (i), we obtain that for $t\in J_n$

$$q_n\ge 0,$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{w^*(t)}{t}\right]=-q_n\cdot t^{-2}\le 0$$

for $t \in I_n$. Therefore the function $w^*(t)/t$ is nonincreasing on J_n, i.e. the function $w^*(t)$ is a modulus of continuity.

The first inequality in (2) easily follows from Lemma 2, taking $w^*(2x) = w(2x)$ for $x = 1/2n$, and the second inequality in (2) follows from concavity of w.

Moreover, we will show that w^* is a concave function on the set $[0;1]$. We derive this fact from concavity of the given modulus of continuity $w(t)$. Suppose $n = 2, 3, \dots$. From the concavity condition for $w(t)$ we have

$$w\Big(\frac{\alpha}{n+1} + \frac{1-\alpha}{n-1}\Big) \ge \alpha \cdot w\Big(\frac{1}{n+1}\Big) + (1-\alpha)\cdot w\Big(\frac{1}{n-1}\Big), \qquad \alpha \in [0;1],$$

and putting $\alpha = (n+1)/2n$ we obtain

$$w\Big(\frac{1}{n}\Big) \ge \frac{1}{2n}\Big\{(n+1)\cdot w\Big(\frac{1}{n+1}\Big) + (n-1)\cdot w\Big(\frac{1}{n-1}\Big)\Big\}.$$

Since $w^*(1/m) = w(1/m)$ for $m = 1, 2, \dots$, then one has

$$w^*\Big(\frac{1}{n}\Big) \ge \frac{1}{2n}\Big\{(n+1)\cdot w^*\Big(\frac{1}{n+1}\Big) + (n-1)\cdot w^*\Big(\frac{1}{n-1}\Big)\Big\},$$

or

$$w^*\Big(\frac{\alpha}{n+1} + \frac{1-\alpha}{n-1}\Big) \ge \alpha \cdot w^*\Big(\frac{1}{n+1}\Big) + (1-\alpha)\cdot w^*\Big(\frac{1}{n-1}\Big), \qquad \alpha = \frac{n+1}{2n}.$$

Using linearity of $w^*(t)$ on the sets $\big[1/(n+1); 1/n\big]$ and $\big[1/n; 1/(n-1)\big]$ we conclude that $w^*(t)$ is a concave function on $\big[1/(n+1); 1/(n-1)\big]$ for $n = 2, 3, \dots$. Therefore, owing to continuity of w^*, we obtain that the modulus of continuity w^* is concave on $[0;1]$. This completes the proof of Theorem. □

R e m a r k . Let $\{w(t_n)\}$ be a sequence of values of a concave modulus of continuity $w(t)$ for a sequence of positive numbers $t_n \to 0+$ provided $t_1 \le 1$ and $t_{n+1}/t_n \ge b$ with some positive b. Then $\{w(t_n)\}$ is sufficient for describing the asymptotic behaviour of $w(t)$. This fact can be proved analogously as our Theorem.

R e m a r k . The concavity condition for w in Theorem is essential. For instance, we can investigate the function

$$w(t) = \big(45t - |24t-7| + 8\cdot|3t-1| - 3\cdot|8t-5| + 7\cdot|3t-2|\big)/28,$$

which is a modulus of continuity, but the corresponding function w^* is not a modulus of continuity. One can verify this fact with the following inequality:

$$w^*(1/3 + 1/3) = 43/42 > 1 = w^*(1/3) + w^*(1/3).$$

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