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INTEGRAL CONDITIONS OF OSCILLATION OF A LINEAR DIFFERENTIAL EQUATION

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Recently some papers have been devoted to the study of the oscillatory behaviour of second and higher order ordinary or retarded linear differential equations. We shall continue in this course. To be more precise, we consider the linear differential equation

$$y^{(n)}(t) + p(t)y(t) = 0, \qquad n \geqslant 2, \tag{1}$$

where $p(t) > 0$ is a continuous function on $[a, \infty)$, $a > 0$.

Our aim is to give new sufficient conditions that guarantee the oscillatory character of equation (1).

We shall use the following definitions.

A solution $y(t)$ of equation (1) is called oscillatory if it has arbitrarily large zeros, and it is called nonoscillatory otherwise.

Equation (1) is said to have the property A if every solution of this equation is oscillatory if n is even, and every solution is either oscillatory or $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, \dots, n-1$ if n is odd.

We introduce the notation:

M_n is a maximum of the function $P_n(x) = x(1-x)\dots(n-1-x)$ on $(0, 1)$.

α_1, α_2 are the fixed points of the function

$$f_n(x) = \frac{\alpha}{(1-x)\dots(n-1-x)} \qquad \text{on } (0, 1), \quad 0 < \alpha \leqslant M_n.$$

In the present paper an attempt is made to establish new conditions under which equation (1) has property A if $0 < \alpha \leqslant M_n$. The case when $\alpha > M_n$ is solved in [1] for the equation (1), and the problem when $\alpha > M_2$ is solved in [6] for the second order retarded differential equation. We can find the other conditions related to the one in Theorem 2 in [4, 5].

We will need the following lemma.

Lemma ([1]). *Equation* (1) *has the property A if and only if the equation*

$$y^{(n)}(t) + (-1)^n p(t)y(t) = 0 \tag{2}$$

has not the solution $y(t)$ *such that*

$$y(t) > 0, \quad t \geqslant t_0 \geqslant a,$$
$$(-1)^{i+1} y^{(i)}(t) > 0, \quad t \geqslant t_0 \geqslant a, \qquad i = 1, \ldots, n-1.$$

We can now establish the main results in the next theorems.

Theorem 1. *Let the following conditions be fulfilled*

$$\liminf_{t \to \infty} t^{n-1} \int_t^{\infty} p(s)\,\mathrm{d}s = \frac{\alpha}{n-1}, \qquad 0 < \alpha \leqslant M_n, \tag{3}$$

$$\liminf_{t \to \infty} t^{n-1-\alpha_1+\varepsilon} \int_t^{\infty} s^{\alpha_1 - \varepsilon} p(s)\,\mathrm{d}s > \alpha_2 (n-2)!, \qquad 0 < \varepsilon < \alpha_1. \tag{4}$$

Then equation (1) *has the property* A.

Proof. Assume the contrary, i.e. equation (1) has not the property A. By the Lemma equation (2) has a solution $v(t)$ such that

$$\begin{aligned} &v(t) > 0, \quad t \geqslant t_0 \geqslant a, \\ &(-1)^{i+1} v^{(i)}(t) > 0, \quad t \geqslant t_0 \geqslant a, \qquad i = 1, \ldots, n-1. \end{aligned} \tag{5}$$

We remind that

$$\int^{\infty} p(s)\,\mathrm{d}s < \infty,$$

which follows from the condition (3).

Integrating equation (2) and according to the fact that $v(t)$ is increasing we obtain

$$(-1)^n v^{(n-1)}(t) \geqslant \int_t^{\infty} p(s) v(s)\,\mathrm{d}s \geqslant v(t) \int_t^{\infty} p(s)\,\mathrm{d}s, \qquad t \geqslant t_0.$$

The condition (3) implies

$$t^{n-1} \int_t^{\infty} p(s)\,\mathrm{d}s > \frac{\nu}{n-1}, \tag{6}$$

where $0 < \nu < \alpha$, $t \geqslant t_1 \geqslant t_0$.

Hence it is obvious that

$$(-1)^n v^{(n-1)}(t) > \frac{\nu}{n-1} \cdot \frac{v(t)}{t^{n-1}}, \qquad t \geqslant t_1.$$

Integrating the above inequality we get

$$(-1)^{n-1} v^{(n-2)}(t) > \frac{\nu}{n-1}\, v(t) \int_t^{\infty} \frac{\mathrm{d}s}{s^{n-1}} = \frac{\nu}{(n-1)(n-2)} \cdot \frac{v(t)}{t^{n-2}},$$

$t \geqslant t_1$. Repeating this argument we find that

$$v'(t) > \frac{\nu}{(n-1)!} \cdot \frac{v(t)}{t}, \qquad t \geqslant t_1,$$

which means that the function

$$\frac{v(t)}{t^r}, \quad \text{where} \quad r = \frac{\nu}{(n-1)!},$$

is increasing on $[t_1, \infty)$.

Then from the equation (2) it follows that

$$(-1)^n v^{(n-1)}(t) \geqslant \int_t^\infty p(s)\, v(s)\, \mathrm{d}s \geqslant \frac{v(t)}{t^r} \int_t^\infty s^r p(s)\, \mathrm{d}s, \qquad t \geqslant t_1.$$

Using (6) in the next integral we have

$$\int_t^\infty s^r p(s)\, \mathrm{d}s > \left(\frac{\nu}{n-1}\right)^{\frac{r}{n-1}} \int_t^\infty p(s) \left(\int_s^\infty p(u)\, \mathrm{d}u\right)^{-\frac{r}{n-1}} \mathrm{d}s =$$

$$= \left(\frac{\nu}{n-1}\right)^{\frac{r}{n-1}} \cdot \frac{n-1}{n-1-r} \left(\int_t^\infty p(s)\, \mathrm{d}s\right)^{\frac{n-1-r}{n-1}},$$

since

$$\int_t^\infty p(s)\, \mathrm{d}s < \infty.$$

By virtue of (6) the last inequality implies

$$\int_t^\infty s^r p(s)\, \mathrm{d}s > \frac{\nu}{n-1-r} \cdot \frac{1}{t^{n-1-r}},$$

and so we obtain

$$(-1)^n v^{(n-1)}(t) > \frac{\nu}{n-1-r} \cdot \frac{v(t)}{t^{n-1}}, \qquad t \geqslant t_1.$$

Integrating this inequality and because of the fact that the function

$$\frac{v(t)}{t^r}$$

is increasing we get

$$(-1)^{n-1} v^{(n-2)}(t) > \frac{\nu}{n-1-r} \cdot \frac{v(t)}{t^r} \int_t^\infty \frac{\mathrm{d}s}{s^{n-1-r}} =$$

$$= \frac{\nu}{(n-1-r)(n-2-r)} \cdot \frac{v(t)}{t^{n-2}}, \qquad t \geqslant t_1.$$

Repeating this argument we find that

$$v'(t) > \frac{\nu}{(n-1-r)\dots(1-r)} \cdot \frac{v(t)}{t}, \qquad t \geqslant t_1.$$

Thus the function

$$\frac{v(t)}{t^k}, \quad \text{where} \quad k = \frac{\nu}{(n-1-r)\dots(1-r)},$$

is increasing on $[t_1, \infty)$. Repeating this procedure we find that

$$v'(t) > x_i \cdot \frac{v(t)}{t}, \qquad t \geqslant t_1,$$

where $x_i = g_n(x_{i-1})$, $i = 1, 2, \dots, x_0 = 0$, and

$$g_n(x) = \frac{\nu}{(n-1-x)\dots(1-x)}.$$

So the functions

$$\frac{v(t)}{t^{x_i}}, \qquad i = 1, 2, \dots$$

are increasing on $[t_1, \infty)$.

According to the properties of the function $g_n(x)$, where $0 < \nu < M_n$, this function has two fixed points ν_1, ν_2 $(\nu_1 < \nu_2)$ on $(0, 1)$ and the sequence $\{x_i\}$ converges to ν_1.

The condition (4) implies

$$t^{n-1-\alpha_1+\varepsilon} \int_t^\infty s^{\alpha_1-\varepsilon} p(s)\,\mathrm{d}s > (\alpha_2 + \varepsilon_1)(n-2)! = \beta(n-2)!, \tag{7}$$

where $\varepsilon_1 > 0$, $t \geqslant t_2 \geqslant t_1$.

We can always choose ν such that $\alpha_1 - \varepsilon < \nu_1$ and $\nu_2 < \beta = \alpha_2 + \varepsilon_1$. Then the function

$$\frac{v(t)}{t^{\alpha_1-\varepsilon}}$$

is increasing on $[t_2, \infty)$ and we obtain from (2) and (7) that

$$(-1)^n v^{(n-1)}(t) \geqslant \int_t^\infty p(s)\,v(s)\,\mathrm{d}s \geqslant \frac{v(t)}{t^{\alpha_1-\varepsilon}} \int_t^\infty s^{\alpha_1-\varepsilon} p(s)\,\mathrm{d}s,$$

$$(-1)^n v^{(n-1)}(t) > \beta (n-2)! \frac{v(t)}{t^{n-1}}, \qquad t \geqslant t_2 .$$

Now integrating as above we get that

$$v'(t) > \beta \cdot \frac{v(t)}{t},$$

i.e. the function

$$\frac{v(t)}{t^{\beta}}$$

is increasing on $[t_2, \infty)$. Repeating the procedure as above we find

$$v'(t) > x_k \cdot \frac{v(t)}{t},$$

where $x_k = g_n(x_{k-1})$, $k = 1, 2, \ldots, p$, $x_0 = \beta$ and $x_p > 1$, which follows from the properties of the function $g_n(x)$. Thus

$$v'(t) > x_p \cdot \frac{v(t)}{t}, \qquad t \geqslant t_2 . \tag{8}$$

On the other hand with regard to (5) we obtain

$$v(t) \geqslant (t - t_0)\, v'(t), \qquad t \geqslant t_0 ,$$

so

$$x_p v(t) - (x_p - 1)\, v(t) \geqslant t v'(t) - t_0 v'(t),$$

$$x_p v(t) + t_0 v'(t) - (x_p - 1)\, v(t) \geqslant t v'(t). \tag{9}$$

Since (3) implies (see e.g. [5])

$$\int^{\infty} t^{n-1} p(t)\, \mathrm{d}t = \infty,$$

and this condition and (5) imply that

$$\lim_{t \to \infty} v(t) = \infty ;$$

then for sufficiently large t we have

$$t_0 (v'(t) - (x_p - 1)\, v(t) < 0.$$

Thus from (9) for sufficiently large t we get

$$x_p v(t) > t v'(t),$$

which is a contradiction with (8). This completes the proof.

Theorem 2. *Assume that condition* (3) *and*

$$\limsup_{t\to\infty}\left(\frac{1}{t}\int_T^t s^n p(s)\,\mathrm{d}s + t^{1-\varepsilon}\int_t^\infty s^{n+\varepsilon-2}p(s)\,\mathrm{d}s\right) > (n-1)!, \tag{10}$$

$\varepsilon\in(0, \alpha_1)$, *are fulfilled.*

Then equation (1) *has the property* A.

Proof. Suppose that equation (1) has not the property A. Then equation (2) has a solution $v(t)$ which satisfies (5). Now using the same argument as in the proof of Theorem 1 we find that the function

$$\frac{v(t)}{t^\varepsilon}, \qquad \varepsilon\in(0, \alpha_1), \tag{11}$$

is increasing on $[t_1, \infty)$, $t_1 \geqslant t_0 \geqslant a$, since $(v(t)\,t^{-\varepsilon})' > 0$.

On the other hand from the identity

$$v^{(j)}(t) = \sum_{i=j}^{k-1}(-1)^{i-j}\frac{(s-t)^{i-j}}{(i-j)!}v^{(i)}(s) + \frac{(-1)^{k-j}}{(k-j-1)!}\int_t^s (u-t)^{k-j-1}v^{(k)}(u)\,\mathrm{d}u,$$

$s \geqslant t \geqslant t_1$, where $1 \leqslant k \leqslant n$, $0 \leqslant j \leqslant n-1$, for $k = n, j = 1$, we obtain

$$v'(t) = \sum_{i=1}^{n-1}(-1)^{i-1}\frac{(s-t)^{i-1}}{(i-1)!}v^{(i)}(s) + \frac{(-1)^{n-1}}{(n-2)!}\int_t^s (u-t)^{n-2}v^{(n)}(u)\,\mathrm{d}u. \tag{12}$$

According to (5) and (12) we get

$$v'(t) \geqslant \frac{(-1)^{n-1}}{(n-2)!}\int_t^\infty (u-t)^{n-2}v^{(n)}(u)\,\mathrm{d}u.$$

Since $v(t)$ satisfies (2) from the last inequality we have

$$v'(t) \geqslant \frac{1}{(n-2)!}\int_t^\infty (u-t)^{n-2}p(u)\,v(u)\,\mathrm{d}u.$$

Integrating this inequality from T to t, $t > T \geqslant t_1$, we get the following one.

$$v(t) \geqslant \frac{1}{(n-1)!}\int_T^t (u-T)^{n-1}p(u)\,v(u)\,\mathrm{d}u + \\ + \frac{1}{(n-2)!}\int_t^\infty p(u)\,v(u)\int_T^t (u-s)^{n-2}\,\mathrm{d}s\,\mathrm{d}u,$$

and using the inequality $a^n - b^n \geqslant (a-b)\,a^{n-1}$, $a \geqslant b \geqslant 0$, we obtain

$$v(t) \geqslant \frac{1}{(n-1)!}\left[\int_T^t (u-T)^{n-1}p(u)\,v(u)\,\mathrm{d}u + (t-T)\int_t^\infty (u-T)^{n-2}p(u)\,v(u)\,\mathrm{d}u\right]. \tag{13}$$

According to (5) we have

$$v(t) \geqslant (t - T)v'(t) + v(T) \geqslant (t - T)v'(t)$$

and the inequality $v(t) \geqslant (t - T)v'(t)$ implies that the function

$$\frac{v(t)}{t - T}, \qquad t > T,$$

is nonincreasing and with regard to (11) from (13) we have

$$(n - 1)! \geqslant \frac{1}{t - T}\int_T^t (u - T)^n p(u)\,\mathrm{d}u + \frac{t - T}{t^\varepsilon}\int_t^\infty u^\varepsilon (u - T)^{n-2} p(u)\,\mathrm{d}u,$$

which contradicts condition (10). This completes the proof.

REFERENCES

[1] CHANTURIA, T. A.: Integral conditions of oscillation of solutions of higher-order linear differential equations (Russian). Differencialnye Uravnenija, 16, 1980, 470—482.

[2] KIGURADZE, I. T.: On the oscillatory and monotone solutions of ordinary differential equations. Arch. Math., 14, 1978, 21—44.

[3] IVANOV, A. F.—KUSANO, T.—ŠEVELO, V. N.: Comparison theorems for functional differential equations (Russian). Inst. Math. Ukrain. Academy Sciences, 1984.

[4] OHRISKA, J.: Oscillation of second-order delay and ordinary differential equation, Czech. Math. J., 34, 1984, 107—112.

[5] OLÁH, R.: Oscillation of linear retarded differential equation. Czech. Math. J., 34, 1984, 371—377.

[6] PHILOS, Ch. G.—SFICAS, Y. G.: Oscillatory and asymptotic behaviour of second and third order retarded differential equations. Czech. Math. J., 32, 1982, 169—182.

[7] WERBOWSKI, J.: Oscillations of first-order differential inequalities with deviating arguments. Ann. Matem. purae et applic., CXL, 1985, 383—392.

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ИНТЕГРАЛЬНЫЕ ПРИЗНАКИ КОЛЕБАНИЯ ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

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Резюме

В работе приведены достаточные условия для того, чтобы каждое решение уравнения (1) при четном n являлось колеблющимся, а при нечетном n, либо колеблющимся, либо удовлетворяло условию $\lim_{t \to \infty} y^{(i)}(t) = 0$, $i = 0, \ldots, n - 1$.