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ON INFINITELY DISTRIBUTIVE ORDERED SETS

JOSEF NIEDERLE

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ABSTRACT. The notion of infinite distributivity is modified in order to obtain a tractable and consistent property of ordered sets, in particular lattices.

If M is a set, then $\wp(M)$ and $\mathrm{Fin}(M)$ denote the set of all subsets and of all finite subsets of M respectively. If P is a subset of an ordered set A, then we denote the sets of all upper bounds and lower bounds of P in A by $\mathrm{U}_A(P)$ and $\mathrm{L}_A(P)$ respectively, or merely $\mathrm{U}(P)$ and $\mathrm{L}(P)$. We also write $\mathrm{L}(a,b)$ instead of $\mathrm{L}(\{a,b\})$ and $\mathrm{L}(P,Q)$ instead of $\mathrm{L}(P \cup Q)$. In accordance with [1] we denote $\downarrow M := \bigcup_{m\in M} \mathrm{L}(m)$ and $\uparrow M := \bigcup_{m\in M} \mathrm{U}(m)$. $\mathrm{Id}(B)$ is the Frink ideal generated by B. Recall that a subset I of an ordered set A is a *Frink ideal* in A if $\mathrm{L}\,\mathrm{U}(F) \subseteq I$ for each finite subset $F \subseteq I$, see [3] and [5].

The following observation is well known.

OBSERVATION 1. *Let L be a complete lattice. The following conditions are equivalent:*

(i) $a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$ *for each subset* $B \subseteq L$*;*
(ii) $a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$ *for each down-set* B *in* L*;*
(iii) L *is distributive and* $a \wedge \bigvee I = \bigvee\{a \wedge b : b \in I\}$ *for each ideal* I *in* L.

Recall that a complete lattice is said to be *infinitely distributive* if it satisfies conditions (i)–(iii). It is a commonplace that:

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Observation 2. *A complete lattice is infinitely distributive if and only if it is Brouwerian, i.e. relatively pseudocomplemented. In particular, every infinitely distributive complete lattice is pseudocomplemented.*

We will generalize these results from complete lattices to ordered sets, in particular, to all lattices.

Observation 3. $\downarrow B \cap \downarrow C = \bigcup \{\mathrm{L}(b,c) : \ b \in B \ \ \& \ \ c \in C\}$ *for each subsets* B, C *of an ordered set.*

Observation 4. *In every ordered set,* $a = \bigvee M \iff \mathrm{U}(a) = \mathrm{U}(M)$.

Recall that Larmerová and Rachůnek defined an ordered set A to be *distributive* if $\mathrm{L}(a, \mathrm{U}(b,c)) = \mathrm{L}\,\mathrm{U}(\mathrm{L}(a,b), \mathrm{L}(a,c))$ for each $a, b, c \in A$. This is equivalent with $\mathrm{L}(a, \mathrm{U}(B)) = \mathrm{L}\,\mathrm{U}(\downarrow\{a\} \cap \downarrow B)$ for each $a \in A$, $B \in \mathrm{Fin}(A)$, see [5]. An ordered set $(A, \leq)$ is distributive if and only if $(A, A, \leq)$ is a distributive context as defined in [2], see [5] for the proof.

Definition. We say that an ordered set A is *strictly infinitely distributive* if $\mathrm{L}(a, \mathrm{U}(B)) = \mathrm{L}\,\mathrm{U}(\downarrow\{a\} \cap \downarrow B)$ for each $a \in A$, $B \in \wp(A)$, and *ideal-continuous* if $\mathrm{L}(a, \mathrm{U}(I)) = \mathrm{L}\,\mathrm{U}(\downarrow\{a\} \cap I)$ for each Frink ideal I.

Let A be an ordered set and let $a \in A$, $B \in \wp(A)$. We put $a \circ B := \{c \in A : \mathrm{L}(a,c) \subseteq \mathrm{L}(B)\}$. Recall that $B \in \wp(A)$ is a *cut* in A if $B = \mathrm{L}\,\mathrm{U}(B)$.

Lemma 5. *Let* a *be an element of an ordered set* A. *The following conditions are equivalent:*

(i) $a \circ B$ *is a cut for each* $B \in \wp(A)$;
(ii) $a \circ \{b\}$ *is a cut for each* $b \in A$.

Proof.
(i) $\implies$ (ii): follows a fortiori.
(ii) $\implies$ (i):

$$\begin{aligned} c \in a \circ B &\iff \mathrm{L}(a,c) \subseteq \mathrm{L}(B) = \bigcap_{b \in B} \mathrm{L}(b) \\ &\iff (\forall b \in B)(\mathrm{L}(a,c) \subseteq \mathrm{L}(b)) \iff (\forall b \in B)(c \in a \circ \{b\}) \\ &\iff c \in \bigcap_{b \in B} a \circ \{b\}. \end{aligned}$$

Since all $a \circ \{b\}$ are cuts, their intersection is also a cut. □

Proposition 6. *Let A be an ordered set. The following conditions are equivalent:*

(i) *A is strictly infinitely distributive;*

(ii) $\mathrm{L}(a, \mathrm{U}(P)) = \mathrm{L}\,\mathrm{U}(\downarrow\{a\} \cap \downarrow P)$ *for each $a \in A$ and each down-set P in A;*

(iii) $a \in \mathrm{L}\,\mathrm{U}(G) \implies a = \bigvee(\downarrow\{a\} \cap \downarrow G)$ *for each $a \in A$ and each subset $G \subseteq A$;*

(iv) $a \in \mathrm{L}\,\mathrm{U}(G) \implies a = \bigvee(\downarrow\{a\} \cap \downarrow G)$ *for each $a \in A$ and each down-set G in A;*

(v) $\mathrm{L}(\mathrm{U}(P), \mathrm{U}(Q)) = \mathrm{L}\,\mathrm{U}(\downarrow P \cap \downarrow Q)$ *for each subsets $P, Q \subseteq A$;*

(vi) $\mathrm{L}(\mathrm{U}(P), \mathrm{U}(Q)) = \mathrm{L}\,\mathrm{U}(\downarrow P \cap \downarrow Q)$ *for each down-sets P, Q in A;*

(vii) *$a \circ B$ is a cut for each $a \in A$ and $B \in \wp(A)$;*

(viii) *$a \circ \{b\}$ is a cut for each $a \in A$ and $b \in A$;*

Proof. Implications (i) $\implies$ (ii), (iii) $\implies$ (iv), (v) $\implies$ (vi) and (v) $\implies$ (i) follow a fortiori.

(ii) $\implies$ (i): $\mathrm{L}(a, \mathrm{U}(P)) = \mathrm{L}(a, \mathrm{U}(\downarrow P)) = \mathrm{L}\,\mathrm{U}(\downarrow\{a\} \cap \downarrow\downarrow P) = \mathrm{L}\,\mathrm{U}(\downarrow\{a\} \cap \downarrow P)$.

(iv) $\implies$ (iii): $a \in \mathrm{L}\,\mathrm{U}(G) \implies a \in \mathrm{L}\,\mathrm{U}(\downarrow G) \implies a = \bigvee(\downarrow\{a\} \cap \downarrow\downarrow G) = \bigvee(\downarrow\{a\} \cap \downarrow G)$.

(vi) $\implies$ (v): $\mathrm{L}(\mathrm{U}(P), \mathrm{U}(Q)) = \mathrm{L}(\mathrm{U}(\downarrow P), \mathrm{U}(\downarrow Q)) = \mathrm{L}\,\mathrm{U}(\downarrow\downarrow P \cap \downarrow\downarrow Q) = \mathrm{L}\,\mathrm{U}(\downarrow P \cap \downarrow Q)$.

(i) $\implies$ (iii): Suppose $a \in \mathrm{L}\,\mathrm{U}(G)$. Then $\mathrm{U}(a) = \mathrm{U}\,\mathrm{L}(a) = \mathrm{U}\,\mathrm{L}(a, \mathrm{U}(G)) = \mathrm{U}(\downarrow\{a\} \cap \downarrow G)$.

(iii) $\implies$ (v): Clearly $\mathrm{L}\,\mathrm{U}(\downarrow P \cap \downarrow Q) \subseteq \mathrm{L}(\mathrm{U}(P), \mathrm{U}(Q))$. Let $a \in \mathrm{L}(\mathrm{U}(P), \mathrm{U}(Q)) = \mathrm{L}\,\mathrm{U}(P) \cap \mathrm{L}\,\mathrm{U}(Q)$. By assumption $a = \bigvee(\downarrow\{a\} \cap \downarrow P)$ and $a = \bigvee(\downarrow\{a\} \cap \downarrow Q)$. Denote $G := \downarrow\{a\} \cap \downarrow P$ and $H := \downarrow\{a\} \cap \downarrow Q$. Then $h \in H$ implies that $h \in \mathrm{L}(a) = \mathrm{L}\,\mathrm{U}(G)$, which in turn yields $h = \bigvee(\downarrow G \cap \downarrow\{h\})$. Therefore $a = \bigvee_{h \in H} h = \bigvee_{h \in H} \bigvee(\downarrow G \cap \downarrow\{h\}) = \bigvee(\downarrow G \cap \downarrow H)$, and hence $a = \bigvee(\downarrow\{a\} \cap \downarrow P \cap \downarrow Q) \in \mathrm{L}\,\mathrm{U}(\downarrow\{a\} \cap \downarrow P \cap \downarrow Q) \subseteq \mathrm{L}\,\mathrm{U}(\downarrow P \cap \downarrow Q)$.

(vii) $\iff$ (viii): by Lemma 5.

(i) $\implies$ (viii): $\mathrm{L}(a) \cap \mathrm{L}\,\mathrm{U}(a \circ \{b\}) = \mathrm{L}\,\mathrm{U}\Big(\bigcup_{c \in a \circ \{b\}} (\mathrm{L}(a) \cap \mathrm{L}(c))\Big) \subseteq \mathrm{L}\,\mathrm{U}\,\mathrm{L}(b) = \mathrm{L}(b)$.

Thus $\mathrm{L}\,\mathrm{U}(a \circ \{b\}) \subseteq a \circ \{b\}$, and consequently $a \circ \{b\}$ is a cut.

(vii) $\implies$ (i): Since $B \subseteq a \circ \mathrm{L}\,\mathrm{U}\Big(\bigcup_{b \in B} (\mathrm{L}(a) \cap \mathrm{L}(b))\Big)$, and the latter is a cut,

$$\mathrm{L}\,\mathrm{U}(B) \subseteq a \circ \mathrm{L}\,\mathrm{U}\Big(\bigcup_{b \in B} (\mathrm{L}(a) \cap \mathrm{L}(b))\Big).$$

Hence $\mathrm{L}(a) \cap \mathrm{L}\,\mathrm{U}(B) \subseteq \mathrm{L}\,\mathrm{U}\Big(\bigcup_{b \in B} (\mathrm{L}(a) \cap \mathrm{L}(b))\Big)$. □

We can generalize the second part of Observation 2. Recall that an ordered set A is said to be *weakly pseudocomplemented* if $a \circ A$ is a cut for each $a \in A$, see [6].

OBSERVATION 7. *Every strictly infinitely distributive ordered set is weakly pseudocomplemented.*

It might seem surprising that weakly Brouwerian ordered sets were not defined. But such a definition would be superfluous in virtue of Proposition 6. Indeed, in [7] the *relative pseudocomplement* of a with respect to B was defined as the greatest element of $a \circ B$, and a *Brouwerian ordered set* as having all relative pseudocomplements of a with respect to $\{b\}$. Natural generalizations would really be (vii) or (viii), and hence the expected weakly Brouwerian ordered sets coincide with strictly infinitely distributive ones. This generalizes the first part of Observation 2.

PROPOSITION 8. *An ordered set A is Brouwerian if and only if it is strictly infinitely distributive and each $a \circ \{b\}$ has a supremum in A.*

Proof. Let A be Brouwerian. Then every $a \circ \{b\}$ has a greatest element, which is its least upper bound, and therefore $a \circ \{b\}$ is a cut. Let conversely A be strictly infinitely distributive, that is every $a \circ \{b\}$ be a cut, and let every $a \circ \{b\}$ have a least upper bound. Then the least upper bound is the required greatest element. □

PROPOSITION 9. *The following conditions are equivalent for every ordered set A:*

(i) *A is strictly infinitely distributive;*
(ii) *A is distributive and ideal-continuous.*

Proof.

(i) $\Longrightarrow$ (ii): follows a fortiori.

(ii) $\Longrightarrow$ (i): Notice that $\mathrm{L}(a, \mathrm{U}(C)) = \mathrm{L}\,\mathrm{U}\Big(\bigcup\{\mathrm{L}(a,c) :\ c \in \mathrm{L}\,\mathrm{U}(C)\}\Big)$ for each $a \in A$ and $C \subseteq A$. Indeed, for $p \in \mathrm{L}(a, \mathrm{U}(C))$ we have $p \in \mathrm{L}(a) \cap \mathrm{L}\,\mathrm{U}(C)$, and therefore $\mathrm{L}(p) = \mathrm{L}(a,p) \subseteq \bigcup\{\mathrm{L}(a,c) :\ c \in \mathrm{L}\,\mathrm{U}(C)\}$. Hence $\mathrm{L}(p) = \mathrm{L}\,\mathrm{U}\,\mathrm{L}(p) = \mathrm{L}\,\mathrm{U}\Big(\bigcup\{\mathrm{L}(a,c) :\ c \in \mathrm{L}\,\mathrm{U}(C)\}\Big)$. The converse inclusion is obvious because $\mathrm{U}(C) = \mathrm{U}\,\mathrm{L}\,\mathrm{U}(C)$.

Let $a \in A$ and $B \subseteq A$. Then

$$\begin{aligned}
\mathrm{U}\Big(\bigcup\{\mathrm{L}(a,b):\ b \in B\}\Big) &\subseteq \bigcap_{F\in \mathrm{Fin}(B)} \mathrm{U}\Big(\bigcup\{\mathrm{L}(a,b):\ b\in F\}\Big) \\
&= \bigcap_{F\in \mathrm{Fin}(B)} \mathrm{U}\,\mathrm{L}\big(a, \mathrm{U}(F)\big) \qquad \text{as } A \text{ is distributive} \\
&= \bigcap_{F\in \mathrm{Fin}(B)} \mathrm{U}\left(\bigcup\{\mathrm{L}(a,c):\ c\in \mathrm{L}\,\mathrm{U}(F)\}\right) \\
&= \mathrm{U}\Big(\bigcup_{F\in\mathrm{Fin}(B)} \bigcup\{\mathrm{L}(a,c):\ c\in \mathrm{L}\,\mathrm{U}(F)\}\Big) \\
&= \mathrm{U}\Big(\bigcup\Big\{\mathrm{L}(a,c):\ \big(\exists F\in \mathrm{Fin}(B)\big)\big(c\in \mathrm{L}\,\mathrm{U}(F)\big)\Big\}\Big) \\
&= \mathrm{U}\left(\bigcup\{\mathrm{L}(a,c):\ c\in \mathrm{Id}(B)\}\right) \\
&= \mathrm{U}\,\mathrm{L}\left(a, \mathrm{U}\big(\mathrm{Id}(B)\big)\right) \qquad \text{as } A \text{ is ideal-continuous} \\
&= \mathrm{U}\,\mathrm{L}\big(a, \mathrm{U}(B)\big)\,.
\end{aligned}$$

The converse inclusion is obvious. □

We denote by $\mathrm{G}(A)$ the sublattice of the Dedekind-Mac Neille completion $\mathrm{DM}(A)$ of the ordered set A generated by A. We say that $\mathrm{G}(A)$ is the *characteristic lattice* of A. We say that a subset $A \subseteq B$ is *dense* in an ordered set $(B, \leq)$ if $\mathrm{L}_A(A) = \mathrm{L}_B(B) \cap A$ and $b = \bigvee\big(\mathrm{L}_B(b) \cap A\big)$ for each $b \in B$. It is *doubly dense* if it is both dense and dually dense. Recall that an ordered set A is doubly dense both in $\mathrm{DM}(A)$ and in $\mathrm{G}(A)$. Moreover, a complete lattice L is isomorphic to $\mathrm{DM}(A)$ whenever A is a doubly dense subset of L, and a lattice L is isomorphic to $\mathrm{G}(A)$ whenever A is a doubly dense generating subset of L.

PROPOSITION 10. *The Dedekind-Mac Neille completion of a strictly infinitely distributive ordered set is infinitely distributive.*

Proof. Let C, B_i be cuts in a strictly infinitely distributive ordered set A. Then

$$\begin{aligned}
&C \wedge_{\mathrm{DM}(A)} \bigvee\nolimits_{\mathrm{DM}(A)}\{B_i:\ i\in I\} \\
&\quad = C \cap \mathrm{L}\,\mathrm{U}\left(\bigcup\{B_i:\ i\in I\}\right) = \mathrm{L}\Big(\mathrm{U}(C), \mathrm{U}\left(\bigcup\{B_i:\ i\in I\}\right)\Big) \\
&\quad = \mathrm{L}\,\mathrm{U}\left({\downarrow}C \cap {\downarrow}\bigcup\{B_i:\ i\in I\}\right) = \mathrm{L}\,\mathrm{U}\left(\bigcup\{{\downarrow}C\cap{\downarrow}B_i:\ i\in I\}\right) \\
&\quad = \mathrm{L}\,\mathrm{U}\left(\bigcup\{C\cap B_i:\ i\in I\}\right) = \bigvee\nolimits_{\mathrm{DM}(A)}\left\{C\wedge_{\mathrm{DM}(A)} B_i:\ i\in I\right\}.
\end{aligned}$$

□

DEFINITION. A *tree* is a topped ordered set T such that $\mathrm{U}_T(t)$ is a chain for each $t \in T$.

Let T be a tree of finite length the minimal elements of which are labelled with subsets of A, the element v being labelled with $S(v)$. Let $T(v) := \mathrm{L}_T(v)$ with the order and labelling inherited from T. We put

$$\mathrm{U}_A^\hbar(T(v)) = \mathrm{U}_A(S(v)) \qquad \text{and} \qquad \mathrm{L}_A^\hbar(T(v)) = \mathrm{L}_A(S(v))$$

if v is minimal in T, and

$$\mathrm{U}_A^\hbar(T(v)) = \mathrm{U}_A\Big(\bigcup_{u \prec v} \mathrm{L}_A^\hbar(T(u))\Big) \qquad \text{and} \qquad \mathrm{L}_A^\hbar(T(v)) = \mathrm{L}_A\Big(\bigcup_{u \prec v} \mathrm{U}_A^\hbar(T(u))\Big)$$

otherwise, where $\prec$ denotes the covering relation in T.

PROPOSITION 11. *Let A be a doubly dense subset of an ordered set B. Let T be a tree of finite length the minimal elements of which are labelled with subsets of A. Then $\mathrm{U}_A^\hbar(T) = A \cap \mathrm{U}_B^\hbar(T)$ and $\mathrm{L}_A^\hbar(T) = A \cap \mathrm{L}_B^\hbar(T)$.*

Proof. By well-founded induction on the covering relation in T. Let $u \in T$ and assume that $\mathrm{U}_A^\hbar(T(v)) = A \cap \mathrm{U}_B^\hbar(T(v))$ and $\mathrm{L}_A^\hbar(T(v)) = A \cap \mathrm{L}_B^\hbar(T(v))$ whenever $b \prec u$. If u is minimal in T, then

$$\mathrm{U}_A^\hbar(T(u)) = \mathrm{U}_A(S(u)) = A \cap \mathrm{U}_B(S(u)) = A \cap \mathrm{U}_B^\hbar(T(u)) .$$

If not, then

$$\begin{aligned}\mathrm{U}_A^\hbar(T(u)) &= \mathrm{U}_A\Big(\bigcup_{w \prec u} \mathrm{L}_A^\hbar(T(w))\Big) = A \cap \mathrm{U}_B\Big(\bigcup_{w \prec u} \big(A \cap \mathrm{L}_B^\hbar(T(w))\big)\Big) \\ &= A \cap \mathrm{U}_B\Big(A \cap \bigcup_{w \prec u} \mathrm{L}_B^\hbar(T(w))\Big) = A \cap \mathrm{U}_B\Big(\bigcup_{w \prec u} \mathrm{L}_B^\hbar(T(w))\Big) \\ &= A \cap \mathrm{U}_B^\hbar(T(u))\end{aligned}$$

since A is doubly dense in B. □

PROPOSITION 12. *Each doubly dense subset of a strictly infinitely distributive lattice is strictly infinitely distributive.*

Proof. Let A be doubly dense in a strictly infinitely distributive lattice L, and let $a \in A$, $B \subseteq A$. Consider labelled trees

$$T_1 := (\{2\} \dot\cup \{3\}) \oplus \{1\} , \qquad S(2) := \{a\} , \qquad S(3) := B$$

and

$$T_2 := B \oplus \{2\} \oplus \{1\} , \qquad S(b) := \{a, b\} \quad \text{for each} \quad b \in B$$

with B considered as an antichain, see [1] for details. Then by Proposition 11

$$\begin{aligned}
\mathrm{L}_A\big(a, \mathrm{U}_A(B)\big) &= \mathrm{L}_A^{\hbar}(T_1) = A \cap \mathrm{L}_L^{\hbar}(T_1) \\
&= A \cap \mathrm{L}_L\big(a, \mathrm{U}_L(B)\big) = A \cap \mathrm{L}_L\,\mathrm{U}_L\Big(\bigcup_{b\in B} \mathrm{L}_L(a,b)\Big) \\
&= A \cap \mathrm{L}_L^{\hbar}(T_2) = \mathrm{L}_A^{\hbar}(T_2) = \mathrm{L}_A\,\mathrm{U}_A\Big(\bigcup_{b\in B} \mathrm{L}_A\big(a,b\big)\big)\Big)\,.
\end{aligned}$$

□

THEOREM 13. *Let A be an ordered set. The following conditions are equivalent:*

(i) *A is strictly infinitely distributive;*
(ii) *$\mathrm{DM}(A)$ is infinitely distributive;*
(iii) *$\mathrm{G}(A)$ is strictly infinitely distributive;*
(iv) *A is a doubly dense subset of an infinitely distributive complete lattice;*
(v) *A is a doubly dense generating subset of a strictly infinitely distributive lattice.*

Proof.

(i) $\Longrightarrow$ (ii) in virtue of Proposition 10.

(ii) $\Longrightarrow$ (iii) follows from Proposition 12 since $\mathrm{G}(A)$ is doubly dense in $\mathrm{DM}(A)$.

(ii) $\Longrightarrow$ (iv) since A is isomorphic with a doubly dense subset of $\mathrm{DM}(A)$.

(iii) $\Longrightarrow$ (v) since A is isomorphic with a doubly dense generating subset of $\mathrm{G}(A)$.

(iv) $\Longrightarrow$ (i) follows from Proposition 12.

(v) $\Longrightarrow$ (i) follows from Proposition 12. □

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Katedra algebry a geometrie
Přírodovědecká fakulta
Masarykova universita
Janáčkovo náměstí 2a
CZ 662 95 Brno
CZECH REPUBLIC
E-mail: niederle@math.muni.cz