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Mathematica Slovaca, Vol. 53 (2003), No. 1, 75--89

Persistent URL: http://dml.cz/dmlcz/133303

Math. Slovaca, **53** (2003), No. 1, 75–89

Mathematica Slovaca
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Slovak Academy of Sciences

NECESSARY AND SUFFICIENT CONDITIONS FOR THE NONOSCILLATION OF A FIRST ORDER NEUTRAL EQUATION WITH SEVERAL DELAYS

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(Communicated by Milan Medveď)

ABSTRACT. In this paper, necessary and sufficient conditions have been obtained so that every solution of Neutral Delay Differential Equation (NDDE)

$$\left(y(t) - \sum_{j=1}^{k} p_j y(t-\tau_j)\right)' + Q(t)G\big(y(t-\sigma)\big) = f(t)$$

is oscillatory or tends to zero as $t \to \infty$ for different ranges of $\sum\limits_{j=1}^{k} p_j$. This paper improves and generalizes two recent works [DAS, P.—MISRA, N.: *A necessary and sufficient condition for the solution of a functional differential equation to be oscillatory or tend to zero*, J. Math. Anal. Appl. **204** (1997), 78–87] and [Parhi, N.—Rath, R. N.: *On oscillation criteria for a forced neutral differential equation*, Bull. Inst. Math. Acad. Sinica **28** (2000), 59–70]. The results of this paper hold for linear, sublinear and superlinear equations. Also, it holds for homogeneous equations. The results can be extended to NDDE with variable coefficients with out assumption of any further condition on the coefficient functions.

1. Introduction

In the present work, the author has obtained necessary and sufficient conditions so that every solution of

$$\left(y(t) - \sum_{j=1}^{k} p_j y(t-\tau_j)\right)' + Q(t)G\big(y(t-\sigma)\big) = f(t) \tag{E}$$

2000 Mathematics Subject Classification: Primary 34C10, 34C15, 34K40.
Keywords: oscillation, non-oscillation, neutral equation, asymptotic behaviour.

oscillates or tends to zero as $t \to \infty$ on various ranges of $\sum_{j=1}^{k} p_j$, where each p_j is a scalar, $G \in C(\mathbb{R},\mathbb{R})$, $Q \in C([0,\infty),[0,\infty))$, $f \in C([0,\infty),\mathbb{R})$, $\tau_j \geq 0$, $\sigma \geq 0$. We further assume the following conditions for its use in the sequel.

(H_1) There exists $F \in C^1([0,\infty),\mathbb{R})$ such that $F'(t) = f(t)$ and $\lim_{t\to\infty} F(t) = 0$.

(H_2) G is non-decreasing and $xG(x) > 0$ for $x \neq 0$.

(H_3) $\int_0^\infty Q(t)\, \mathrm{d}t = \infty$.

(H_4) Let G satisfy Lipschitz condition on the intervals of the type $[a,b]$, $0 < a < b$.

The following ranges for p_j $(j = 1,2,\dots,k)$ are considered in this paper.

(A_1) $0 < \sum_{j=1}^{k} p_j < 1$, where each $p_j > 0$,

(A_2) $-1 < \sum_{j=1}^{k} p_j < 0$, where each $p_j < 0$,

(A_3) $\sum_{j=1}^{k} p_j < -1$, where each $p_j < -1$ and $p_i < -1 + \sum_{j\neq i} p_j$ for some $i \in \{1,2,3,\dots,k\}$.

(A_4) $\sum_{j=1}^{k} p_j > 1$, where each $p_j > 1$ and $p_i > 1 + \sum_{j\neq i} p_j$ for some $i \in \{1,2,3,\dots,k\}$.

Our results also hold for the equation

$$\Big(y(t) - \sum_{j=1}^{k} p_j y(t-\tau_j)\Big)' + \sum_{j=1}^{m} Q_j G\big(y(t-\sigma_j)\big) = f(t) \tag{1}$$

under the assumption

$$\int_0^\infty \sum_{j=1}^{m} Q_j(t)\, \mathrm{d}t = \infty \tag{2}$$

in place of (H_3).

In the literature very few results (see [1], [8], [9], [11]) are available regarding the oscillation criteria for solutions of neutral differential equations with several delays. Most of these results are concerned with NDDE's where the several delay terms are not taken under the derivative sign. Virtually these results are related to the equation

$$\big(y(t) - py(t-\tau)\big)' + \sum_{j=1}^{m} Q_j(t) G\big(y(t-\sigma_j)\big) = f(t)\,. \tag{3}$$

Whatever results we find in the literature for (E) are concerned with mostly (A_1) as the range for $\sum_j p_j$. It seems that very little work is done in other ranges of p_j, i.e. for (A_2), (A_3) or (A_4). In a recent paper [13], the author has obtained necessary and sufficient conditions so that every solution of

$$\big(y(t) - py(t-\tau)\big)' + Q(t) G\big(y(t-\sigma)\big) = f(t) \tag{4}$$

oscillates or tends to zero, where p is a scalar not equal to ± 1 and G, Q, f, τ, σ are same as stated earlier. One may take interest and find that the results of [13] are true for the equation (3) under primary assumption (2) in place of (H_3). Hence it seems equations with several delays outside the derivative sign do not pose much of a problem to study. But surprisingly the technique and the method used in [13] fail when one attempts to work out the same problem for (E), and this really motivated the author for the present work. [5; Lemma 1.5.1] was repeatedly used to get the results in [13]. The notes 1.8 given in [5; p. 31] suggests to extend Lemma 1.5.1 for application to neutral equations with several delays. But it seems hard to prove the extended lemma as suggested in [5]. So the author became more interested to study the oscillatory and asymptotic behaviour of solutions of (E).

The results of this paper hold when G is linear, sublinear or super linear, also when $f(t) \equiv 0$. This paper is an improvement and generalization of the work in [2] (see Remark 2) where the results are true only for sublinear equations and non-homogeneous equations. While studying the same problem in [13], f is assumed to be non negative and G is assumed to be Lipschizian. But in the present work there is no such restriction on f, that means f can be ultimately positive, negative or oscillatory. We could relax the Lipschitzian condition on G for the range (A_1), but not for other ranges of p_j, i.e. for (A_2), (A_3) or (A_4). Hence this paper is an improvement and generalization of [13] also.

The authors of the paper [2] have rightly observed that there are very few results concerning necessary and sufficient conditions for oscillation of all solutions of (4) except a few with $f(t) \equiv 0$ and the coefficient functions are real constants (see [6], [7]). The oscillatory behaviour of such equations are usually characterized by the non existence of real roots of the associated characteristic equations. The present work is an attempt in this direction, where all the four theorems provide both necessary and sufficient conditions for the oscillatory and asymptotic behaviour of all solutions of (E).

By a solution of (E) we mean a function $y \in C([T-r,\infty),\mathbb{R})$ such that $\left(y(t) - \sum_{j=1}^{k} p_j y(t-\tau_j)\right)$ is continuously differentiable and (E) is satisfied for $t \geq T$, where $r = \max\{\tau_1,\tau_2,\dots,\tau_k,\sigma\}$ and T is depending on y. Such a solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros, otherwise, it is called *nonoscillatory.*

Special remark. Hence forth it is to be understood that $\sum p_j$ means $\sum_{j=1}^{k} p_j$ and $\sum_{j\neq i} p_j$ means $\left(\sum_{j=1}^{k} p_j\right) - p_i$ for some $i \in \{1,2,\dots,k\}$.

2. Main results

THEOREM 2.1. *Let* (H_1) *and* (H_2) *hold. Suppose that* p_j *is in the range* (A_1). *Then every solution of* (E) *is oscillatory or tends to zero as* $t \to \infty$ *if and only if* (H_3) *holds.*

Proof. Let us first prove the sufficiency part. Let (H_3) hold. Suppose that $y(t)$ is a non-oscillatory solution for $t \geq t_0$. Setting

$$z(t) = y(t) - \sum p_j y(t-\tau_j) \tag{5}$$

and

$$w(t) = z(t) - F(t) \tag{6}$$

for $t \geq t_0 + r$, we obtain

$$w'(t) = -Q(t)G\big(y(t-\sigma)\big)\,. \tag{7}$$

If $y(t) > 0$ for large t, then $w'(t) \leq 0$, which implies that $w(t) > 0$ or $w(t) < 0$ for $t > t_1 \geq t_0 + r$. In both the cases we claim that $y(t)$ is bounded. If not, then there exists a sequence $\langle t_n \rangle$ such that $t_n \to +\infty$, $y(t_n) \to +\infty$ as $n \to \infty$ and $y(t_n) = \max\{y(s) :\ t_1 \leq s \leq t_n\}$. We may choose n sufficiently large such that $t_n - r > t_1$. Then

$$\begin{aligned} w(t_n) &= y(t_n) - \sum p_j y(t_n - \tau_j) - F(t_n) \\ &\geq \left(1 - \sum p_j\right) y(t_n) - F(t_n) \end{aligned}$$

implies that $w(t_n) \to \infty$ as $n \to \infty$, a contradiction whether $w(t) > 0$ or $w(t) < 0$. Hence our claim holds and as a consequence $\liminf_{t\to\infty} y(t)$, $\limsup_{t\to\infty} y(t)$

and $\lim_{t\to\infty} w(t)$ exists. If $\liminf_{t\to\infty} y(t) > 0$, then $y(t) > \beta > 0$ for $t > t_2 > t_1$. Hence

$$\int_{t_3}^{t} Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s > G(\beta)\int_{t_3}^{t} Q(s)\,\mathrm{d}s\,,$$

where $t_3 > t_2$ implies that

$$\int_{t_3}^{\infty} Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s = \infty \tag{8}$$

due to (H_3). However, from (7) one obtains

$$\int_{t_3}^{\infty} Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s < \infty\,, \tag{9}$$

a contradiction. Thus $\liminf_{t\to\infty} y(t) = 0$. Let $\lim_{t\to\infty} w(t) = \ell \in \mathbb{R}$. Then from (H_1) it follows that $\lim_{t\to\infty} z(t) = \ell$. If $\ell > 0$, then

$$\begin{aligned}
0 < \ell = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty}\Big(y(t) - \sum p_j y(t-\tau_j)\Big)\\
&\le \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty}\Big(-\sum p_j y(t-\tau_j)\Big)\\
&\le \sum \limsup_{t\to\infty}(-p_j y(t-\tau_j))\\
&= \sum\Big(-p_j \liminf_{t\to\infty} y(t-\tau_j)\Big) = 0\,,
\end{aligned}$$

a contradiction. If $\ell < 0$, then

$$\begin{aligned}
0 > \ell = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}\Big(y(t) - \sum p_j y(t-\tau_j)\Big)\\
&\ge \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\Big(\sum -p_j y(t-\tau_j)\Big)\\
&\ge \limsup_{t\to\infty} y(t) + \sum \limsup_{t\to\infty}(-p_j y(t-\tau_j))\\
&\ge \Big(\limsup_{t\to\infty} y(t)\Big)\Big(1 - \sum p_j\Big)\,.
\end{aligned}$$

Hence if $\limsup_{t\to\infty} y(t) > 0$, then we obtain $0 > \ell > 0$, a contradiction. If $\limsup_{t\to\infty} y(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$. Hence $\ell = 0$. Again proceeding as above we get

$$0 = \ell = \limsup_{t\to\infty} z(t) \ge \Big(\limsup_{t\to\infty} y(t)\Big)\Big(1 - \sum p_j\Big)\,,$$

which implies $\limsup_{t\to\infty} y(t) \le 0$. Hence $\lim_{t\to\infty} y(t) = 0$. If $y(t) < 0$ for large t, then we proceed as above and prove $\lim_{t\to\infty} y(t) = 0$. Thus the sufficiency part of the theorem is proved.

In order to prove that the condition (H_3) is necessary, we assume that

$$\int_0^\infty Q(t)\,\mathrm{d}t < \infty, \tag{10}$$

and show that (E) admits a positive solution which does not tend to zero as $t \to \infty$, when the limit exists. From (10) and (H_1), it follows that there exists $t_1 > 0$ such that if $t \ge t_1$, then

$$G(1)\int_t^\infty Q(t)\,\mathrm{d}t < \left(1 - \sum p_j\right)/10$$

and

$$|F(t)| < \left(1 - \sum p_j\right)/10 .$$

We set $X = BC\big([t_1,\infty),\mathbb{R}\big)$, the space of real valued bounded continuous functions on $[t_1,\infty)$. Clearly, X is a Banach space with respect to "supremum" norm. Let $K = \{x \in X :\ x(t) \ge 0,\ t \ge t_1\}$. Thus X is a partially ordered Banach space ([5; p. 30]). For $u, \nu \in X$, we define $u \le \nu$ if and only if $\nu - u \in K$. Let

$$S = \left\{u \in X :\ (1-\sum p_j)/10 \le u(t) \le 1 \ \text{ for all } \ t \in [t_1,\infty)\right\}.$$

If $u_0(t) = (1-\sum p_j)/10$, $t \ge t_1$, then $u_0 \in S$ and $u_0 = \inf S$. Let $\varphi \subset S^* \subset S$. If $\nu_0(t) = \sup\{\nu(t) :\ \nu \in S^*\}$, then $\nu_0 = \sup S^*$ and $\nu_0 \in S^*$. For $y \in S$, define

$$Ty(t) = \begin{cases} Ty(t_1+r), & t \in [t_1, t_1+r],\\ \sum p_j y(t-\tau_j) + \int_t^\infty Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s & \\ \qquad + F(t) + \big((1-\sum p_j)/5\big), & t \ge t_1 + r. \end{cases}$$

Thus Ty is a real valued continuous function on $[t_1,\infty)$ for every $y \in S$. Further,

$$\begin{aligned} Ty(t) &< \left(\sum p_j\right) + \left(\left(1-\sum p_j\right)/5\right) + \left(\left(1-\sum p_j\right)/5\right)\\ &= \left(2 + 3\sum p_j\right)/5 < 1 \end{aligned}$$

and

$$\begin{aligned} Ty(t) &> (F(t)) + \left(\left(1-\sum p_j\right)/5\right) > \left(-\left(1-\sum p_j\right)/10\right) + \left(\left(1-\sum p_j\right)/5\right)\\ &= \left(\left(1-\sum p_j\right)/10\right) \qquad \text{for} \quad t \ge t_1 . \end{aligned}$$

Hence $Ty \in S$ for every $y \in S$, that is, $T\colon S \to S$. Let $y_1, y_2, \in S$. If $y_1 \le y_2$, then $Ty_1 \le Ty_2$. Hence T has a fixed point $y_0 \in S$ by Knaster-Tarski fixed point theorem (see [5; Theorem 1.7.3]). Thus y_0 is a positive solution of (E) on $[t_1, \infty)$ such that $\liminf_{t\to\infty} y(t) > 0$. This completes the proof of the theorem. □

Remark 1. $(\mathrm{H}_1) \iff \left| \int_0^\infty f(t)\,\mathrm{d}t \right| < \infty$.

Remark 2. Theorem 2.1 is an improvement and generalization of the work in [2] in view of Remark 1, and it also improves [13; Theorems 2.2, 2.3, Corollary 2.4].

THEOREM 2.2. *Let $\sum p_j$ be in the range* (A_2). *Suppose that* (H_1) *and* (H_2) *hold. Then*

(i) (H_3) *holds implies every solution of* (E) *oscillates or tends to zero as* $t \to \infty$;

(ii) *every solution of* (E) *oscillates or tends to zero as $t \to \infty$ such that* (H_4) *holds implies* (H_3) *holds.*

Proof. Let us prove (i). Suppose (H_3) holds, and $y(t)$ be an ultimately positive solution of (E) for large t. Then setting $z(t)$ and $w(t)$ as in (5) and (6), we obtain (7), which implies $w(t) > 0$ or $w(t) < 0$ for $t > t_1$. Suppose $w(t) > 0$ for $t > t_1$, which implies $\lim_{t\to\infty} w(t) = \ell \in \mathbb{R}$ exists. From (H_1), it follows that $\lim_{t\to\infty} z(t) = \ell$. As $z(t) \ge 0$, so $\ell \ge 0$. We prove $y(t)$ is bounded and $\liminf_{t\to\infty} y(t) = 0$ as in Theorem 2.1. We claim that $\ell = 0$; if not, then $\ell > 0$ which implies

$$\begin{aligned}
\ell = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty}\Big(y(t) - \sum p_j y(t-\tau_j)\Big) \\
&\le \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty}\Big(\sum -p_j y(t-\tau_j)\Big) \\
&\le \sum -p_j \limsup_{t\to\infty} y(t-\tau_j) \\
&\le \Big(-\sum p_j\Big) \limsup_{t\to\infty} y(t) \\
&= \Big(-\sum p_j\Big) m, \qquad \text{where} \quad m = \limsup_{t\to\infty} y(t)\,.
\end{aligned}$$

Hence we get

$$m \ge \Big(\ell / -\sum p_j\Big) > \ell\,. \tag{11}$$

Again

$$\begin{aligned}\ell = \limsup_{t\to\infty} z(t) &= \limsup_{t\to\infty}\Big(y(t) - \sum p_j y(t-\tau_j)\Big)\\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\Big(\sum -p_j y(t-\tau_j)\Big)\\ &\geq m + \sum \liminf_{t\to\infty}(-p_j y(t-\tau_j))\\ &\geq m + \sum -p_j\Big(\liminf_{t\to\infty} y(t-\tau_j)\Big) = m\,,\end{aligned}$$

a contradiction due to inequality (11).

Therefore we conclude $\ell = 0$ and from $z(t) > y(t)$, it follows that $\limsup_{t\to\infty} y(t) \leq 0$. Hence $\lim_{t\to\infty} y(t) = 0$. Further if $w(t) < 0$, then either $\lim_{t\to\infty} w(t) = -\infty$ or $\lim_{t\to\infty} w(t) = \ell < 0$. In both the cases $\lim_{t\to\infty} z(t) = \ell < 0$, which is a contradiction. The case when $y(t) < 0$ for large t can be dealt with similar arguments as above. Hence (i) is proved.

Next let us prove (ii). Suppose to the contrary (H_3) does not hold, that is

$$\int_0^\infty Q(t)\, \mathrm{d}t < \infty\,.$$

From this and (H_1), we can find $t_1 > 0$ such that for $t \geq t_1$

$$K\int_t^\infty Q(s)\, \mathrm{d}s < \Big(1 + \sum p_j\Big)/5 \qquad \text{and} \qquad |F(t)| < \Big(1 + \sum p_j\Big)/10\,,$$

where $K = \max\{G(1), K_1\}$, K_1 is Lipschitz constant of G in $\big[(1+\sum p_j)/10, 1\big]$.

Let

$$X = \Big\{x \in BC\big([t_1,\infty),\mathbb{R}\big) :\ (1+\sum p_j)/10 \leq x(t) \leq 1 \ \text{for all}\ t \in [t_1,\infty)\Big\}\,.$$

For $u, \nu \in X$, we define

$$d(u,\nu) = \sup\big\{|u(t) - \nu(t)| :\ t \geq t_1\big\}\,.$$

Hence (X, d) is a complete metric space. For $y \in X$, define

$$Ty(t) = \begin{cases} Ty(t_1 + r)\,, & t \in [t_1, t_1+r]\,,\\ \sum p_j y(t-\tau_j) + \big((1 - 4\sum p_j)/5\big) & \\ \quad +\int_t^\infty Q(s)G\big(y(s-\sigma)\big)\, \mathrm{d}s + F(t)\,, & t \geq t_1 + r\,. \end{cases}$$

Clearly, $Ty(t)$ is continuous for $t \geq t_1$ and for $t \geq t_1 + r$

$$\begin{aligned} Ty(t) &\leq \left(1 - 4\sum p_j\right)/5 + \left(1 + \sum p_j\right)/5 + \left(1 + \sum p_j\right)/10 \\ &= \left(1 - \sum p_j\right)/2 < 1, \\ Ty(t) &\geq \left(\sum p_j\right) + \left(\left(1 - 4\sum p_j\right)/5\right) - \left(\left(1 + \sum p_j\right)/10\right) \\ &= \left(1 + \sum p_j\right)/10. \end{aligned}$$

Thus $T\colon X \to X$. Further for $y_1, y_2 \in X$,

$$d(Ty_1, Ty_2) \leq \left(\left|\sum p_j\right| + \left(1 + \sum p_j\right)/5\right) d(y_1, y_2).$$

Hence T is a contraction. From the Banach fixed point theorem it follows that T has a unique fixed point $y_0 \in X$ which is the required positive solution such that $\liminf_{t\to\infty} y_0(t) > \left(1 + \sum p_j\right)/10$. Hence the theorem is completely proved. □

THEOREM 2.3. *Let $\sum p_j$ be in the range* (A_3). *Suppose that* (H_1) *and* (H_2) *hold. Then*

(i) (H_3) *holds implies that every solution of* (E) *oscillates or tends to zero as $t \to \infty$;*

(ii) *every solution of* (E) *oscillates or tends to zero as $t \to \infty$ such that* (H_4) *holds implies* (H_3) *holds.*

Proof. First let us prove (i): Suppose that (H_3) holds and $y(t)$ be an ultimately positive solution of (E). Then setting $z(t)$ and $w(t)$ as in (5) and (6), we obtain (7), which implies $w(t) > 0$ or $w(t) < 0$ for $t > t_1$. Suppose $w(t) > 0$ for $t > t_1$, which implies $\lim_{t\to\infty} w(t) = \ell \in \mathbb{R}$. Hence $\lim_{t\to\infty} z(t) = \ell \geq 0$. We prove $y(t)$ is bounded and $\liminf_{t\to\infty} y(t) = 0$ as in Theorem 2.1. Then

$$\begin{aligned} \ell = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty}\left(y(t) + \sum -p_j y(t - \tau_j)\right) \\ &\leq \limsup_{t\to\infty}\left(y(t) + \sum_{j\neq i} -p_j y(t - \tau_j)\right) + \liminf_{t\to\infty} -p_i y(t - \tau_i) \\ &= \left(1 - \sum_{j\neq i} p_j\right) \limsup_{t\to\infty} y(t), \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
\ell = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}\Big(y(t) - \sum p_j y(t-\tau_j)\Big) \\
&\ge \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty}\Big(\sum -p_j y(t-\tau_j)\Big) \\
&\ge \limsup_{t\to\infty}(-p_i y(t-\tau_j)) + \liminf_{t\to\infty}\Big(\sum_{j\ne i} -p_j y(t-\tau_j)\Big) \\
&\ge -p_i \limsup_{t\to\infty} y(t) + \sum_{j\ne i} \liminf_{t\to\infty}(-p_j y(t-\tau_j)) \\
&\ge -p_i \limsup_{t\to\infty} y(t) + \sum_{j\ne i} (-p_j) \liminf_{t\to\infty} y(t-\tau_j) \\
&\ge -p_i \limsup_{t\to\infty} y(t)\,.
\end{aligned}
\tag{13}
$$

From the inequalities (12) and (13) we obtain

$$\Big(1 - \sum_{j\ne i} p_j\Big) \limsup_{t\to\infty} y(t) \ge -p_i \limsup_{t\to\infty} y(t)\,,$$

which implies

$$\bigg(\Big(1 - \sum_{j\ne i} p_j\Big) + p_i\bigg) \limsup_{t\to\infty} y(t) \ge 0\,.$$

Hence by (A_3) we obtain $\limsup_{t\to\infty} y(t) \le 0$. Thus we have $\lim_{t\to\infty} y(t) = 0$. If $w(t) < 0$, then $\lim_{t\to\infty} w(t) = -\infty$ or $\lim_{t\to\infty} w(t) = \ell < 0$ exists. In both the cases we get $z(t) < 0$ for large t, a contradiction. We can use similar arguments for the case $y(t) < 0$ for large t, hence (i) is proved.

Next let us prove (ii). If possible, let

$$\int_0^\infty Q(t)\,\mathrm{d}t < \infty\,.$$

Choose

$$0 < \varepsilon < \Big(\sum_{j\ne i} p_j\Big) - 1 - p_i\,,$$

and

$$0 < \lambda > \varepsilon\Big(1 - \sum p_j\Big) \Big/ \bigg(\Big(\sum_{j\ne i} p_j\Big) - 1 - p_i\bigg)\,.$$

Set

$$H = -(\lambda+\varepsilon)/p_i \qquad \text{and} \qquad h = \bigg(-(\lambda+\varepsilon)\Big(1 - \sum_{j\ne i} p_j\Big) + p_i(\varepsilon - \lambda)\bigg) \Big/ p_i^2$$

clearly $H > h > 0$. Then one may complete the proof by proceeding as in the proof of Theorem 2.2 and with the following changes:

$$K\int_t^\infty Q(s)\,\mathrm{d}s < \frac{\varepsilon}{2} \quad \text{and} \quad |F(t)| < \frac{\varepsilon}{2} \qquad \text{for} \quad t \ge t_1\,,$$

where $K = \max\{K_1, G(H)\}$, K_1 is the Lipschitz constant of G in $[h, H]$,

$$X = \Big\{x \in BC\big([t_1,\infty),\mathbb{R}\big) \,:\, h \le x(t) \le H \text{ for all } t \in [t_1,\infty)\Big\}$$

and for $y \in X$, define

$$Ty(t) = \begin{cases} \frac{1}{p_i}y(t+\tau_i) - \frac{1}{p_i}\sum\limits_{j\neq i} p_j y(t-\tau_j+\tau_i) & \\ -\frac{1}{p_i}\int\limits_{t+\tau_i}^{\infty} Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s - \frac{\lambda}{p_i} - \frac{1}{p_i}F(t+\tau_i)\,, & t \ge t_1 + r\,, \\ Ty(t_1+r)\,, & t \in [t_1, t_1+r]\,, \end{cases}$$

where $r = \max\{\sigma, \tau_1, \tau_2, \dots, \tau_k\}$. Clearly, $T\colon X \to X$ and

$$d(Ty_1, Ty_2) \le \mu d(y_1, y_2) \qquad \text{where} \quad 0 < \mu = \bigg(\Big(\sum_{j\neq i} p_j\Big) - 1 - \frac{\varepsilon}{2}\bigg)\Big/ p_i < 1\,.$$

Hence equation (E) admits a solution $y_0(t)$ on $[t_1+r, \infty)$ with $0 < h \le y_0(t) \le H$ by Banach contraction principle. Thus the theorem is proved. □

EXAMPLE 1. We may note that $y(t) = \mathrm{e}^t$ is an unbounded positive solution of the equation

$$\big(y(t) - 2y(t-\ln 2) - 6y(t-\ln 3)\big)' + 2\mathrm{e}\,y(t-1) = 0\,, \qquad t \ge 2\,.$$

Here $\sum p_j$ is in the range (A_4).

The above example is a source of motivation for the next theorem.

THEOREM 2.4. *Let $\sum p_j$ be in the range (A_4). Suppose that (H_1) and (H_2) hold. Then*

(i) *(H_3) holds implies that every bounded solution of (E) oscillates or tends to zero as $t \to \infty$;*
(ii) *every bounded solution of (E) oscillates or tends to zero as $t \to \infty$ such that (H_4) holds implies (H_3) holds.*

Proof. First let us prove (i). Let $y(t) > 0$ be any bounded nonoscillatory solution of (E). Setting $z(t)$ and $w(t)$ as in (5) and (6), we obtain (7), which implies $w(t) > 0$ or $w(t) < 0$ for $t > t_1$. In any case $\lim_{t\to\infty} w(t) = \ell \in \mathbb{R}$ exists and by (H_1), we get $\lim_{t\to\infty} z(t) = \ell$. We prove $\liminf_{t\to\infty} y(t) = 0$ as in Theorem 2.1. Let $\limsup_{t\to\infty} y(t) = m$. If $\ell \geq 0$, then

$$\begin{aligned}
0 \leq \ell &= \liminf_{t\to\infty}\Big(y(t) - \sum p_j y(t-\tau_j)\Big) \\
&\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\Big(\sum -p_j y(t-\tau_j)\Big) \\
&\leq m + \liminf_{t\to\infty}\big(-p_i y(t-\tau_i)\big) + \limsup_{t\to\infty}\Big(\sum_{j\neq i} -p_j y(t-\tau_j)\Big) \\
&\leq (1-p_i)m + \sum_{j\neq i} \limsup_{t\to\infty}\big(-p_j y(t-\tau_j)\big) \\
&\leq (1-p_i)m .
\end{aligned}$$

Hence $m = 0$ as $p_i > 1$, which implies $\lim_{t\to\infty} y(t) = 0$.

If $\ell < 0$, then we get

$$\begin{aligned}
\ell &= \lim_{t\to\infty} z(t) = \liminf_{t\to\infty}\Big(y(t) - \sum p_j y(t-\tau_j)\Big) \\
&\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\Big(\sum -p_j y(t-\tau_j)\Big) \\
&\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\big(-p_i y(t-\tau_i)\big) + \limsup_{t\to\infty}\Big(\sum_{j\neq i} -p_j y(t-\tau_j)\Big) \qquad (14) \\
&\leq (1-p_i)m + \sum_{j\neq i} -p_j \liminf_{t\to\infty} y(t-\tau_j) \\
&= (1-p_i)m ,
\end{aligned}$$

and

$$\begin{aligned}
\ell &= \limsup_{t\to\infty} z(t) = \limsup_{t\to\infty}\Big(y(t) - \sum p_j y(t-\tau_j)\Big) \\
&\geq \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty} \sum -p_j y(t-\tau_j) \\
&\geq \limsup_{t\to\infty}\big(-p_i y(t-\tau_i)\big) + \liminf_{t\to\infty} \sum_{j\neq i} -p_j y(t-\tau_j) \qquad (15) \\
&\geq -p_i \liminf_{t\to\infty} y(t-\tau_i) + \Big(-\sum_{j\neq i} p_j \limsup_{t\to\infty} y(t-\tau_j)\Big) \\
&= \Big(-\sum_{j\neq i} p_j\Big) \limsup_{t\to\infty} y(t) = \Big(-\sum_{j\neq i} p_j\Big) m .
\end{aligned}$$

From inequalities (14) and (15) we obtain

$$(1-p_i)m \geq \ell \geq \Big(-\sum_{j\neq i} p_j\Big)m\,,$$

which implies

$$\Big(1-p_i+\sum_{j\neq i} p_j\Big)m \geq 0\,.$$

Hence $\limsup_{t\to\infty} y(t) \leq 0$ since $p_i > 1+\sum_{j\neq i} p_j$, which implies $\lim_{t\to\infty} y(t) = 0$. Hence (i) is proved. Proof of (ii) is similar to the proof of Theorem 2.2(ii), hence it is omitted. □

Remark 3. [13; Theorems 2.5, 2.6, Corollary 2.7] are particular cases of Theorem 2.4 of this paper.

Remark 4. We may note that the conditions $p_i < -1+\sum_{j\neq i} p_j$ in (A_3) and $p_i > 1+\sum_{j\neq i} p_j$ in (A_4) are essential for both the necessary and sufficient part of the proofs of the Theorems 2.3 and 2.4.

EXAMPLE 2. Consider

$$\Big(y(t)-\frac{1}{2}y(t-\ln 2)-\frac{1}{3}y(t-\ln 3)\Big)' + \mathrm{e}^{2t-3}\,y^3(t-1) = 2\,\mathrm{e}^{-t}\,, \qquad t \geq 2\,.$$

From the sufficiency part of Theorem 2.1 it follows that every solution is oscillatory or tends to zero as $t\to\infty$. In particular, $y(t)=\mathrm{e}^{-t}$ is a solution of the equation, which tends to zero as $t\to\infty$. Here $0<\sum p_j<1$.

EXAMPLE 3. Clearly $y(t)=1+(1/t)$ is a bounded positive solution of

$$\Big(y(t)-\frac{1}{2}y(t-1)-\frac{1}{3}y(t-2)\Big)' + t^{-2}\big(1-t^{-1}\big)^3 y^3(t-1)$$
$$= \big((t-1)^{-2}/2\big)+\big((t-2)^{-2}/3\big)\,, \qquad t\geq 3\,.$$

This illustrates the necessary part of Theorem 2.1. Here $Q(t)=t^{-2}(1-t^{-1})^3$ and it does not satisfy (H_3).

Note. Similar examples as above can be found out to illustrate the Theorems 2.2, 2.3 and 2.4.

Remark 5. One may easily find that our results hold for the solutions of the equation with variable coefficients, i.e. for the equation:

$$\Big(y(t)-\sum p_j(t)y(t-\tau_j)\Big)' + \sum_{j=1}^{m} Q_j(t)G\big(y(t-\sigma_j)\big) = f(t)\,. \tag{F}$$

Here the primary assumption is (2), where $p_j \in C([0,\infty),\mathbb{R})$ and $0 \le \sum \limsup_{t\to\infty} p_j(t) \le p < 1$, p is a scalar. This result improves [11; Theorem 2.2] for $n = 1$. Similarly, one may study the same problem for (F) in other ranges of $p_j(t)$.

Remark 6. The author is pained for not being able to find answer to the problem: Can we obtain necessary and sufficient conditions for all solutions of (F) to oscillate or tend to zero under assumption (2), and with no extra condition on G when $\sum p_j(t) = \pm 1$.

Acknowledgements

The author is thankful and indebted to Prof. N. Parhi of Berhampur University (Orissa), India, for his guidance and inspiration in completion of this paper.

The author is also thankful to the referee for his helpful comments, which helped to improve the presentation of the paper.

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Received August 23, 2001
Revised February 15, 202

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