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ON THE CONVERGENCE OF ω-PRIMITIVES

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ABSTRACT. We continue the series of research on oscillations of functions [DUSZYŃSKI, Z.—GRANDE, Z.—PONOMAREV, S. P.: *On the ω-primitive*, Math. Slovaca **51** (2001), 469–476], [EWERT, J.—PONOMAREV, S. P.: *Oscillation and ω-primitives*, Real Anal. Exchange **26** (2000/2001), 687–702]. In the first part we consider first-countable topological spaces X satisfying some neighborhood conditions (weak enough to imply the metrizability of X) and show that given a sequence of upper semicontinuous functions $f_n\colon X \to [0,\infty)$ converging to an upper semicontinuous function $f\colon X \to [0,\infty)$, there exist functions $F_n, F\colon X \to [0,\infty)$, $n \in \mathbb{N}$, such that $\omega(F_n,\cdot) = f_n$, $n \in \mathbb{N}$, $\omega(F,\cdot) = f$ and $F_n \to F$ in the same sense as $f_n \to f$. By $\omega(g,x)$ we denote the oscillation of $g\colon X \to \mathbb{R}$ at x. Quite different technique had to be employed in the second part of the paper where the analogous result is proved for $X = \mathbb{R}^n$ equipped with the usual density topology τ_d.

Throughout the paper we consider topological spaces X without isolated points. The *oscillation* of $F\colon X \to \mathbb{R}$ is the function $\omega(F,\cdot)\colon X \to [0,\infty]$ defined by

$$\omega(F,x) = M(F,x) - m(F,x)\,, \qquad x \in X\,. \tag{1}$$

$M(F,\cdot)$, $m(F,\cdot)$ are respectively *upper* and *lower Baire functions* for F:

$$M(F,x) = \inf_W \sup_{z\in W} F(z)\,, \qquad m(F,x) = \sup_W \inf_{z\in W} F(z)\,, \tag{2}$$

where W runs over all open neighborhoods of x from a neighborhood base at x.

It is well known that $M(F,\cdot)$, $m(F,\cdot)$ are respectively upper and lower semicontinuous functions from X into $\overline{\mathbb{R}}$, therefore $\omega(F,\cdot)$ is nonnegative and upper semicontinuous.

We let USC abbreviate upper semicontinuous.

We say that $F\colon X \to \mathbb{R}$ is an *ω-primitive* for a USC function $f\colon X \to [0,\infty)$ if $\omega(F,x) = f(x)$ for each $x \in X$.

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It is known that the uniform (quasi-uniform) convergence of a sequence $\{F_n : n \in \mathbb{N}\}$ to a function F implies the same type of convergence of oscillations $\{\omega(F_n, \cdot) : n \in \mathbb{N}\}$ to $\omega(F, \cdot)$ ([1], [4]).

We investigate the converse problem. Given a sequence $\{f_n : n \in \mathbb{N}\}$ of USC functions $f_n\colon X \to [0, \infty)$ convergent in some sense to a USC function $f\colon X \to [0, \infty)$, we ask whether there exist ω-primitives $F_n, F\colon X \to \mathbb{R}$ for f_n and f respectively, such that F_n will be convergent to F in the same sense. We show that under some assumptions on X the answer is YES for various types of convergence.

We will use the following definitions and notations.

Let X be a topological space and $A \subset X$; by A^d we denote the *set of all accumulation points* of A.

A topological space X is said to be ([2], [4]):

- *σ-discrete at $x \in X$*, if there is an open neighborhood W of x which is a σ-discrete set, i.e.
$$W = \bigcup_{n=1}^{\infty} A_n ,$$
where A_n are discrete subsets of X (empty set is considered as discrete),
- *massive*, if it is not σ-discrete at any $x \in X$.

Note that each first countable T_1 dense in itself Baire space is massive ([4; Corollary 1.2]), but a massive space need not be Baire ([4; Examples 1.2, 1.3]).

The proof of our first result (Theorem 1) is based on Teichmüller-Tukey's lemma. For convenience of the reader we recall its formulation.

Suppose we are given a set X and a property P pertaining to subsets of X. We say that P is a *property of finite character* (on X) if the following holds:

> A set $A \subset X$ has the property P if and only if each finite subset of A has the property P.

LEMMA 1 (TEICHMÜLLER-TUKEY). (cf. e.g., [3; p. 22]) *Let* P *be a property of finite character related to subsets of a set X. Then each subset $A \subset X$ with the property* P *is contained in a maximal* (*with respect to the inclusion relation*) *set $B \subset X$, which also has the property* P.

A maximal set will be termed as P*-maximal.*

In our first result we deal with first-countable topological spaces. Let X be such space, and τ its topology. For each $x \in X$ denote by
$$\mathcal{N}(x) = \{U_n(x) : n \in \mathbb{N}\}$$
an *open countable base of τ at x*. If $\mathcal{N}(x)$ is chosen for each $x \in X$, then the family
$$\mathcal{N} = \{\mathcal{N}(x) : x \in X\} \tag{3}$$

will be called a *neighborhood system on* X.

The following properties of a neighborhood system will be important for us. Namely we say that:

- $\mathcal{N}$ *satisfies condition* (N1) if
$$(\forall\, n \in \mathbb{N})(\forall\, x \in X)\big(U_{n+1}(x) \subset U_n(x)\big)\,.$$
- $\mathcal{N}$ *satisfies condition* (N2) if
there exists a function $s\colon \mathbb{N} \to \mathbb{N}$, $s(n) \geq n$, such that
$$(\forall\, n \in \mathbb{N})(\forall\, x, y \in X)\big(x \in U_{s(n)}(y) \implies y \in U_n(x)\big)\,.$$

Remark 1. Each metrizable space X has a neighborhood system satisfying (N1) and (N2). Indeed, define for each $x \in X$
$$\mathcal{N}(x) = \{B_n(x) :\ n \in \mathbb{N}\}$$
where $B_n(x)$ is an open ball centered at x and of radius 2^{-n}. Then it is immediate that the corresponding neighborhood system $\mathcal{N}$ satisfies (N1), (N2). More generally, each Moore space (for definitions see [3; p. 414] or [5; p. 426]) also has that property, e.g. such is the Niemytzki plane.

Let X be a first-countable T_1-space with a neighborhood system $\mathcal{N}$. For each nonempty set $A \subset X$ we put
$$N(A) = \big\{n \in \mathbb{N} :\ U_n(x) \cap A = \{x\} \text{ for each } x \in A\big\}\,,$$
and then we define
$$\Delta A = \begin{cases} \sup\{\frac{1}{n} :\ n \in N(A)\} & \text{if } N(A) \neq \emptyset\,, \\ 0 & \text{if } N(A) = \emptyset\,. \end{cases}$$
We let by definition $\Delta\emptyset = \infty$.

Now we define for each $n \in \mathbb{N}$ the property P_n related to subsets of X as follows ([4]):

a set $A \subset X$ *has the property* $\mathrm{P}_n \iff \Delta A \geq \frac{1}{n}$.

It is easy to see that P_n is a property of finite character and is hereditary. We will also need some results from [4] which are stated here as lemmas.

LEMMA 2. ([4; Proposition 1.2]) *A topological space is massive if and only if each* σ*-discrete subset of* X *is a boundary set.*

LEMMA 3. ([4; Theorem 1.2]) *Let Z be a first-countable T_1-space with a neighborhood system $\mathcal{N}$ satisfying* (N1) *and* (N2). *Then the following holds.*

(a) *If $A \subset Z$ and $\Delta A > 0$, then $A^d = \emptyset$; so A is closed and discrete.*

(b) *There exists a sequence $\{E_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of Z such that E_1 is P_1-maximal in Z, E_n is P_n-maximal in $Z \setminus (E_1 \cup \dots \cup E_{n-1})$ for $n > 1$, and the set $\bigcup_{n=1}^{\infty} E_n$ is $\mathcal{F}_\sigma$, σ-discrete and dense in Z.*

(c) *Each σ-discrete set $A \subset Z$ can be written in the form $A = \bigcup_{n=1}^{\infty} A_n$, where A_n are pairwise disjoint and $\Delta A_n > 0$ for $n \in \mathbb{N}$.*

We restrict ourselves to considering only two types of convergence for sequences of functions, although it will be clear from proofs that Theorems 1, 2 remain valid for some other types of convergence.

A sequence $\{f_n : n \in \mathbb{N}\}$ of functions $f_n \colon X \to \mathbb{R}$ is said to be convergent to a function $f \colon X \to \mathbb{R}$

- *quasi-uniformly* ([8]) if
$$(\forall \varepsilon > 0)(\forall x \in X)(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists U(x))$$
$$(\forall z \in U(x))(|f(z) - f_n(z)| < \varepsilon),$$
(where $U(x)$ stands for an open neighborhood of x);
- *in the sense of Arzelà* if $f_n \to f$ pointwise and
$$(\forall \varepsilon > 0)(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(\forall x \in X)\Big(\inf_{k \le m} |f(x) - f_{n+k}(x)| < \varepsilon\Big).$$

THEOREM 1. *Let X be a massive first countable T_1-space with a neighborhood system $\mathcal{N}$ satisfying* (N1), (N2) *and let $f_n, f \colon X \to [0,\infty)$, $n \in \mathbb{N}$, be USC functions. Then there exist ω-primitives F_n, F for f_n, f with the properties:*

(a) *if the sequence $\{f_n : n \in \mathbb{N}\}$ converges to f pointwise (uniformly, quasi-uniformly, in the sense of Arzelà), then $F_n \to F$ in the same sense;*

(b) *if $f = \sum_{n=1}^{\infty} f_n$, then $F = \sum_{n=1}^{\infty} F_n$ and*
$$\omega\left(\sum_{n=1}^{\infty} F_n, \cdot\right) = \sum_{n=1}^{\infty} \omega(F_n, \cdot).$$

Proof. We adopt the method used in the proof of [4; Theorem 2.1]. To unify our notations, let us put $f_0 = f$. For each $j = 0, 1, 2, \dots$, the graph of

f_j will be denoted by $G(f_j)$ and considered as a subspace of the product space $X \times \mathbb{R}$. Next we put

$$W_{j,n}(z) = \left[U_n(x) \times \left(f_j(x)-\tfrac{1}{n}, f_j(x)+\tfrac{1}{n}\right)\right] \cap G(f_j) \qquad \text{for} \quad z = \left(x, f_j(x)\right);$$
$$\mathcal{N}_j(z) = \left\{W_{j,n}(z) :\ n \in \mathbb{N}\right\};$$
$$\mathcal{N}_j = \left\{\mathcal{N}_j(z) :\ z \in G(f_j)\right\} \qquad \text{for} \quad j = 0, 1, 2, \ldots.$$

It is easily checked that each subspace $G(f_j)$ is first countable and that the family $\mathcal{N}_j$ is its neighborhood system satisfying (N1) and (N2). Using arguments of Lemma 3(b), we pick, in each $G(f_j)$, a sequence $\left\{E_{j,n} :\ n \in \mathbb{N}\right\}$ of pairwise disjoint subsets such that

$E_{j,1}$ is P_1-maximal in $G(f_j)$;

$E_{j,n}$ is P_n-maximal in $G(f_j) \setminus \bigcup_{k=1}^{n-1} E_{j,k}$, $n > 1$;

$\bigcup_{n=1}^{\infty} E_{j,n}$ is σ-discrete and dense in $G(f_j)$.

Let $\pi \colon X \times \mathbb{R} \to X$ be the natural projection. We claim that each $\pi(E_{j,n})$ is discrete. Assume, otherwise, that for some $j \in \mathbb{N} \cup \{0\}$ and some $n \in \mathbb{N}$ there is a point $x_0 \in \left(\pi(E_{j,n})\right)^d \cap \pi(E_{j,n})$. Then there exists a sequence $\{x_k :\ k \in \mathbb{N}\}$, $x_k \in \pi(E_{j,n})$, $x_k \neq x_m$ for $k \neq m$, and $x_k \to x_0$. Let s be the function from (N2). There is $k_0 \geq 1$ such that $x_k \in U_{s(n)}(x_0)$ for $k \geq k_0$. Then by (N2) we have that $x_0 \in U_n(x_k)$ for $k \geq k_0$. It follows that we can choose a subsequence $\{x_{k_i} :\ i \in \mathbb{N}\}$ with $x_{k_{i+1}} \in \bigcap_{m=1}^{i} U_n(x_{k_m})$. Since the points $z_i = \left(x_{k_i}, f_j(x_{k_i})\right)$ belong to the P_n-maximal set $E_{j,n}$, we have $z_i \notin W_{j,n}(z_m)$ for $i \neq m$. Therefore $|f_j(x_{k_i}) - f_j(x_{k_m})| > \frac{1}{n}$ for all $i, m \in \mathbb{N}$, $i \neq m$, which implies immediately that f_j is not locally bounded at x_0. But f_j is USC and nonnegative, a contradiction. This proves that $\pi(E_{j,n})$ is discrete for $j = 0, 1, 2, \ldots$, and all $n \in \mathbb{N}$, as stated above. Now we let

$$E = \bigcup_{j=0}^{\infty} \bigcup_{n=1}^{\infty} \pi(E_{j,n}).$$

Clearly the set E is σ-discrete and dense in X. Moreover, by Lemma 2, E is a boundary set. Let φ be the characteristic function of E. Consider the functions $F_j \colon X \to [0, \infty)$, $j = 0, 1, 2, \ldots$, defined by $F_j(x) = f_j(x)\varphi(x)$, $x \in X$, $j = 0, 1, 2, \ldots$. Then we have $m(F_j, x) = 0$ for each $x \in X$, $j = 0, 1, 2, \ldots$, and $\omega(F_j, x) = M(F_j, x) \leq M(f_j, x) = f_j(x)$ for $x \in X$, $j = 0, 1, 2, \ldots$. If $x \in E$, then $f_j(x) = F_j(x) \leq M(F_j, x) = \omega(F_j, x)$. Thus we have $\omega(F_j, x) = f_j(x)$ for $x \in E$, $j = 0, 1, 2, \ldots$.

Now let $x \in X \setminus E$. Since $(x, f_j(x)) \in G(f_j)$ and $\bigcup_{n=1}^{\infty} E_{j,n}$ is dense in $G(f_j)$, there exists a sequence $\{x_k : k \in \mathbb{N}\}$ with $(x_k, f_j(x_k)) \in \bigcup_{n=1}^{\infty} E_{j,n}$ and $(x_k, f_j(x_k)) \to (x, f_j(x))$. So

$$f_j(x) = \lim_{k\to\infty} f_j(x_k) = \lim_{k\to\infty} F_j(x_k) \le M(F_j, x) = \omega(F, x).$$

Thus we have shown that $\omega(F_j, \cdot) = f_j$ for each $j = 0, 1, 2, \ldots$. Denoting $F = F_0$, we see that claim (a) is immediate from the definition of F, F_j, $j \in \mathbb{N}$.

Finally, if $f = \sum_{n=1}^{\infty} f_n$, then the series $\sum_{n=1}^{\infty} f_n\varphi$ is convergent and we have

$$F = f\varphi = \sum_{n=1}^{\infty} f_n\varphi = \sum_{n=1}^{\infty} F_n$$

and

$$\omega\left(\sum_{n=1}^{\infty} F_n, \cdot\right) = \omega(F, \cdot) = f = \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \omega(F_n, \cdot),$$

which completes the proof. □

Remark 2. Theorem 1 obviously holds for each massive metric space X (cf. Remark 1).

The next theorem shows that the first countability of the space X is not necessary in getting results like those obtained in Theorem 1. But then, of course, the technique of proofs should be generally different.

In what follows, we deal with the ordinary density topology τ_d in $\mathbb{R}^n$, see [7; p. 167]. The space $(\mathbb{R}^n, \tau_d)$ is dense in itself and no point of $\mathbb{R}^n$ has a countable τ_d-base ([7; Theorem 2.1]); furthermore, it is a Baire space ([7; Proposition 3.16]).

For a measurable set $A \subset \mathbb{R}^n$, the symbol $|A|$ will denote the Lebesgue measure of A.

Given a function $f \colon \mathbb{R}^n \to \mathbb{R}$, we let $C(f, \tau_d)$ be the set of all points at which f is τ_d-continuous.

THEOREM 2. *Let $f_k, f \colon (\mathbb{R}^n, \tau_d) \to [0, \infty)$, $k \in \mathbb{N}$, be USC functions. Then there exist ω-primitives F_k, F for f_k, f respectively, with the properties:*

(a) *if the sequence $\{f_k : k \in \mathbb{N}\}$ converges to f pointwise (uniformly, quasi-uniformly or in the sense of Arzelà), then $\{F_k : k \in \mathbb{N}\}$ converges to F in the same sense;*

(b) *if $f = \sum_{k=1}^{\infty} f_k$, then $F = \sum_{k=1}^{\infty} F_k$ and $\omega\left(\sum_{k=1}^{\infty} F_k, \cdot\right) = \sum_{k=1}^{\infty} \omega(F_k, \cdot)$.*

Proof. Recall first that the set of discontinuity points of each USC function defined on a topological space, is of first category [3; p. 87]. So the set $(\mathbb{R}^n \setminus C(f,\tau_d)) \cup \bigcup_{k=1}^{\infty} \mathbb{R}^n \setminus C(f_k,\tau_d)$ is of τ_d-first category, hence of Lebesgue measure zero, therefore τ_d-closed. It follows that the set $E := C(f,\tau_d) \cap \bigcap_{k=1}^{\infty} C(f_k,\tau_d)$ is τ_d-open and τ_d-dense. Let B be a Bernstein set in $\mathbb{R}^n$; then $\mathbb{R}^n \setminus B$ is also a Bernstein set. We claim that the sets $E \cap B$, $E \setminus B$ are τ_d-dense in $\mathbb{R}^n$. Let $\emptyset \neq W \in \tau_d$. Then $W \cap E$ is nonempty and τ_d-open, whence $|W \cap E| > 0$. Since each measurable subset of any Bernstein set is of measure zero [6; pp. 58–61], we conclude that $E \cap B \cap W \neq \emptyset$ and $(E \setminus B) \cap W \neq \emptyset$, which shows that $E \cap B$, $E \setminus B$ are τ_d-dense. Put $A = E \cap B$ and let φ be the characteristic function of $\mathbb{R}^n \setminus A$. Consider the functions $F = f\varphi$, $F_k = f_k\varphi$, $k = 1,2,\dots$. We will show that $\omega(F_k,\cdot) = f_k$ on $\mathbb{R}^n$ for all $k = 0,1,2,\dots$, where $f_0 = f$ and $F_0 = F$. Since A is τ_d-dense, we get $m(F_k,x) = 0$ for all $k = 0,1,2,\dots$, and all $x \in \mathbb{R}^n$, whence $\omega(F_k,x) = M(F_k,x) \leq M(f_k,x) = f_k(x)$. Thus we have $\omega(F_k,x) \leq f_k(x)$ for $x \in \mathbb{R}^n$, $k = 0,1,2,\dots$. If $x \in \mathbb{R}^n \setminus A$, then $f_k(x) = F_k(x) \leq M(F_k,x) = \omega(F_k,x)$, so we have $\omega(F_k,x) = f_k(x)$ for $x \in \mathbb{R}^n \setminus A$, $k = 0,1,2,\dots$.

Now let $x \in A$. Fix $k \in \{0,1,2,3,\dots\}$ and let $\varepsilon > 0$ be given. Since $A \subset C(f_k,\tau_d)$, there is a τ_d-neighborhood W of x such that $f_k(x) - \varepsilon < f_k(z)$ for $z \in W$. Taking any $z_1 \in W \setminus A$, $z_2 \in W \cap A$ we obtain $|F_k(z_1) - F_k(z_2)| = f_k(z_1) > f_k(x) - \varepsilon$, which yields $\omega(F_k,x) \geq f_k(x)$ for $x \in A$. Thus we have shown that $\omega(F_k,\cdot) = f_k$ for $k = 0,1,2,\dots$. Now claim (a) is immediate from the definition of F, F_k, $k \in \mathbb{N}$. The proof of (b) is much the same as in Theorem 1. □

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