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ON SOME PROPERTIES OF DISPERSION OF BLOCK SEQUENCES OF POSITIVE INTEGERS

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ABSTRACT. Properties of distribution functions of block sequences were investigated in [STRAUCH, O.—TÓTH, J. T.: *Distribution functions of ratio sequences*, Publ. Math. Debrecen **58** (2001), 751–778]. The present paper is a continuation of the study of relations between the density of the block sequence and so called dispersion of the block sequence.

Preliminaries

In this part we recall some basic definitions. Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. For $X \subset \mathbb{N}$ let $X(n) = \operatorname{card}\{x \in X : x \le n\}$. In the whole paper we will assume that X is infinite. Denote by $R(X) = \left\{\frac{x}{y} : x \in X,\ y \in X\right\}$ the *ratio set of* X and say that a set X is (R)*-dense* if $R(X)$ is (topologically) dense in the set $\mathbb{R}^+$. Let us notice that the concept of (R)-density was defined and first studied in papers [Š1] and [Š2].

Now let $X = \{x_1, x_2, \dots\}$ where $x_n < x_{n+1}$ are positive integers. The following sequence of finite sequences derived from X

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

is called the *block sequence* of the sequence X. Thus the block sequence is formed by blocks $X_1, X_2, \dots, X_n, \dots$ where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \qquad n = 1, 2, \dots,$$

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is called the *nth block*. This kind of block sequences were studied in the paper [S–T2]. For every $n \in \mathbb{N}$ let

$$D(X_n) = \max\left\{\frac{x_1}{x_n}, \frac{x_2 - x_1}{x_n}, \dots, \frac{x_{i+1} - x_i}{x_n}, \dots, \frac{x_n - x_{n-1}}{x_n}\right\},$$

the maximum distance between two consecutive terms in the nth block. In this paper we will consider the following characteristics, called the *dispersion* of the sequence X

$$\underline{D}(X) = \liminf_{n\to\infty} D(X_n),$$

and its relations to the previously mentioned (R)-density. Notice that the (R)-density of the set X is equivalent to the density of its block sequence in the interval $(0,1)$.

At the end of this section, let us mention the concept of a dispersion of a general sequence of numbers in the interval $[0,1]$. Let $(x_n)_{n=0}^{\infty}$ be a sequence in $[0,1]$. For every $N \in \mathbb{N}$ let $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_N}$ be reordering of its first N terms into a nondecreasing sequence and denote

$$d_N = \frac{1}{2}\max\{\max\{x_{i_{j+1}} - x_{i_j} : j = 1,2,\dots N-1\},\ x_{i_1},\ 1 - x_{i_N}\}$$

the dispersion of the finite sequence $x_0, x_1, x_2, \dots x_N$. Properties of this concept can be found for example in [N], where it is also proven that

$$\limsup_{N\to\infty} N d_N \ge \frac{1}{\log 4}$$

holds for every one-to-one infinite sequence $x_n \in [0,1)$. Notice that the density of the whole sequence $(x_n)_{n=0}^{\infty}$ is equivalent to $\lim_{N\to\infty} d_N = 0$. Also notice that the analogy of this property for the concept of dispersion of block sequences defined in the present paper does not hold.

Results

When calculating the value $\underline{D}(X)$ the following theorem is often useful.

THEOREM 1. *Let*

$$X = \{x_1, x_2, \dots\} = \bigcup_{n=1}^{\infty} \langle c_n, d_n \rangle \cap \mathbb{N}$$

where $x_n < x_{n+1}$ and let $c_n \le d_n < c_{n+1} - 1$, for $n \in \mathbb{N}$, be positive integers. Then

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : i = 1,2,\dots,n\}}{d_{n+1}}. \tag{2}$$

Proof. Let n be a fixed positive integer and let $k \in \mathbb{N}$ be such that $c_{k+1} \le x_n \le d_{k+1}$. Then

$$\begin{aligned} D(X_n) &= \max\left\{\frac{x_1}{x_n}, \frac{x_{i+1}-x_i}{x_n} : i = 1,2,\dots n-1\right\} \\ &= \max\left\{\frac{x_1}{x_n}, \frac{c_2-d_1}{x_n}, \frac{c_3-d_2}{x_n}, \dots, \frac{c_{k+1}-d_k}{x_n}\right\} \\ &= \frac{\max\{x_1,\, c_{i+1}-d_i : i = 1,2,\dots,k\}}{x_n}. \end{aligned}$$

For $x_n \in \langle c_{k+1}, d_{k+1}\rangle$ the minimal value of $D(X_n)$ will be obtained w en $x_n = d_{k+1}$. Thus

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{x_1,\, c_{i+1}-d_i : i = 1,2,\dots,n\}}{d_{n+1}}.$$

Now notice that the set of all $k \in \mathbb{N}$ for which $x_1 = \max\{x_1, c_{i+1}-d_i : i = 1,2,\dots,k\}$ is either empty or finite or equals to $\mathbb{N}$. Thus in the first two cases the term x_1 in the nominator of the fraction on the right side in the last equation can be omitted. In the third case $\underline{D}(X) = 0$ and, consequently, in all cases the relation (2) holds. □

The following corollary is a straightforward consequence of the previous heorem.

Corollary 1. *Let X be of the same form as in Theorem 1. Suppose that there exists such a positive integer n_0 that for all integers $n > n_0$*

$$c_{n+1} - d_n \le c_{n+2} - d_{n+1}.$$

Then

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{c_{n+1}-d_n}{d_{n+1}}. \tag{3}$$

Theorem 2. *If $\underline{D}(X) = 0$, then the block sequence* (1) *is dense in the interval* $(0,1)$.

Proof. Let $\underline{D}(X) = 0$ and let $0 < a < b < 1$ be given numbers. Then there exists a positive integer n such that $D(X_n) < b - a$. Then, by definition of $D(X_n)$, we have

$$\frac{x_{i+1}}{x_n} - \frac{x_i}{x_n} < b - a \quad \text{for } i = 1,2,\dots,n-1 \qquad \text{and} \qquad \frac{x_1}{x_n} < b - a.$$

Consequently at least one of the numbers $\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_{n-1}}{x_n}, \frac{x_n}{x_n}$ belongs to the interval (a,b), i.e. the block sequence (1) is dense in $(0,1)$. □

THEOREM 3. *If the block sequence* (1) *is dense in the interval* $(0,1)$, *then* $\underline{D}(X) \le \frac{1}{2}$.

P r o o f . Suppose the contrary. Then there exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for all $n > n_0$ there is $D(X_n) \ge \frac{1}{2} + \varepsilon$. By definition of $D(X_n)$, for every $n > n_0$ there exists an interval $I_n \subset (0,1)$ with length $|I_n| = D(X_n)$ such that $X_n \cap I_n = \emptyset$. Obviously $\left(\frac{1}{2} - \varepsilon,\, \frac{1}{2} + \varepsilon\right) \subset \bigcap_{n=n_0}^{\infty} I_n$ and therefore only finite number of terms of the block sequence (1) belong to the interval $\left(\frac{1}{2} - \varepsilon,\, \frac{1}{2} + \varepsilon\right)$. Consequently, the block sequence (1) is not dense in $\left(\frac{1}{2} - \varepsilon,\, \frac{1}{2} + \varepsilon\right)$, which is a contradiction. □

THEOREM 4. *For every* $\alpha \in \left\langle 0, \frac{1}{2}\right\rangle$ *there exists a set* $X \subset \mathbb{N}$ *such that* $\underline{D}(X) = \alpha$ *and the block sequence* (1) *is dense in the interval* $(0,1)$.

P r o o f . First let us notice that for $\alpha = 0$ the set $X = \mathbb{N}$ fulfils the statement of the theorem. So, let $\alpha \in \left(0, \frac{1}{2}\right\rangle$ be given. Let $(r_n)_{n=1}^{\infty}$ be a sequence dense in the interval $\left(\frac{1}{\alpha}, \infty\right)$. Let us consider the set $X = \bigcup_{n=1}^{\infty} \langle c_n, d_n\rangle \cap \mathbb{N}$ defined as follows.

$$c_1 = 1\,, \quad d_1 = 2 \qquad \text{and}$$
$$c_{n+1} = [r_n d_n]\,, \quad d_{n+1} = \left[\frac{r_n - 1}{\alpha} d_n\right] + 1 \qquad \text{for} \quad n = 1, 2, 3, \dots$$

where $[x]$ means the integer part of x. For $\alpha \in \left(0, \frac{1}{2}\right\rangle$ we have $\frac{1}{1-\alpha} \le \frac{1}{\alpha}$ and, as $r_n \ge \frac{1}{\alpha}$, also $r_n \ge \frac{1}{1-\alpha}$, which is equivalent to the inequality $r_n \le \frac{r_n - 1}{\alpha}$. Consequently for every positive integer n we have $c_{n+1} \le d_{n+1} < c_{n+2} - 1$, which proofs that the set X is defined correctly.

Now we are going to prove that $\underline{D}(X) = \alpha$. As $\alpha \le \frac{1}{2}$ and $r_{n+1} > \frac{1}{\alpha}$, we have $2 < \frac{r_{n+1} - 1}{\alpha}$ and, consequently, $d_n(r_n - 1) < \frac{d_n(r_n - 1)}{\alpha}(r_{n+1} - 1)$. Thus we have for all $n \in \mathbb{N}$

$$\begin{aligned}
c_{n+1} - d_n &= [r_n d_n] - d_n \le d_n(r_n - 1) \le \frac{d_n(r_n - 1)}{\alpha}(r_{n+1} - 1) - 1 \\
&< \left(\left[\frac{r_n - 1}{\alpha} d_n\right] + 1\right)(r_{n+1} - 1) - 1 \\
&= d_{n+1}(r_{n+1} - 1) - 1 = d_{n+1} r_{n+1} - 1 - d_{n+1} < [r_{n+1} d_{n+1}] - d_{n+1} \\
&= c_{n+2} - d_{n+1}\,.
\end{aligned}$$

Thus the assumptions of Corollary 1 are fulfilled and so

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{[r_n d_n] - d_n}{\left[\frac{r_n - 1}{\alpha} d_n\right] + 1} = \liminf_{n\to\infty} \frac{d_n(r_n - 1)}{\frac{r_n - 1}{\alpha} d_n + 1} = \alpha\,.$$

Now we are going to prove that the block sequence (1) is dense in $(0,1)$. Clearly. this is equivalent to density of $R(X)$ in $(1,\infty)$. Density of $R(X)$ in the interval $\left(\frac{1}{\alpha},\infty\right)$ follows easily from the definition of c_{n+1}:

$$\frac{c_{n+1}}{d_n} = r_n - \frac{\{r_n d_n\}}{d_n}\,, \qquad \frac{c_{n+1}}{d_n} \in R(X)\,, \qquad \lim_{n\to\infty} \frac{\{r_n d_n\}}{d_n} = 0$$

and the sequence $(r_n)_{n=1}^{\infty}$ is dense in $\left(\frac{1}{\alpha},\infty\right)$. In the above, $\{r_n d_n\}$ means the fractional part of $r_n d_n$. Thus to finish the proof it suffices to prove that $R(X)$ is dense in $\left(1,\frac{1}{\alpha}\right)$. Let $1 < a < b < \frac{1}{\alpha}$. Then there is $n \in \mathbb{N}$ such that

$$\frac{1}{c_{n+1}} < b - a \qquad \text{and} \qquad r_n > \frac{1}{\alpha(b-a)}\,. \tag{4}$$

The difference of succeeding terms in the finite sequence

$$A_n = \left\{\frac{c_{n+1}}{c_{n+1}}, \frac{c_{n+1}+1}{c_{n+1}}, \frac{c_{n+1}+2}{c_{n+1}}, \dots, \frac{d_{n+1}}{c_{n+1}}\right\} \subset R(X)$$

is $\frac{1}{c_{n+1}} < b - a$. From the definition of numbers c_{n+1}, d_{n+1} and (4) it follows that

$$\frac{d_{n+1}}{c_{n+1}} \ge \frac{\frac{r_n-1}{\alpha} d_n}{r_n d_n} = \left(1 - \frac{1}{r_n}\right)\frac{1}{\alpha} > \big(1 - \alpha(b-a)\big)\frac{1}{\alpha} = \frac{1}{\alpha} - (b-a)\,.$$

Thus

$$1 < \frac{c_{n+1}+1}{c_{n+1}} < 1 + b - a \qquad \text{and} \qquad \frac{1}{\alpha} - (b-a) < \frac{d_{n+1}}{c_{n+1}}$$

and, consequently, $A_n \cap (a,b) \neq \emptyset$ and also $R(X) \cap (a,b) \neq \emptyset$ which completes the proof. □

THEOREM 5. *For every $c \in (0,1\rangle$ there exists a set $X \subset \mathbb{N}$ such that $\underline{D}(X) = c$ and the block sequence (1) is not dense in the interval $(0,1)$.*

P r o o f. For $c = 1$ the set $X = \left\{2^{2^n} : n \in \mathbb{N}\right\}$ fulfils the statement of the theorem. So let $c \in (0,1)$. Let $1 < a < b$. Let us consider the set $X = \bigcup\limits_{n-1}^{\infty} \langle c_n, d_n\rangle \cap \mathbb{N}$ where

$$c_n = [a^n b^n] + 1\,, \quad d_n = [a^{n+1} b^n] \qquad \text{for} \quad n = 1, 2, \dots\,.$$

In the paper [M–T] it is proved that $R(X) \cap (a,b) = \emptyset$, so the block sequence (1) is not dense in $(0,1)$. Notice that for every $c \in (0,1)$ there are $1 < a < b$ such that $c = \frac{b-1}{ab}$. Thus to prove the theorem it suffices to show that $\underline{D}(X) = \frac{b-1}{ab}$. Let $n_0 \in \mathbb{N}$ be such that $2 \le a^{n+1} b^n (b-1)(ab-1)$ or, equivalently,

$$a^{n+1} b^n (b-1) + 1 \le a^{n+2} b^{n+1} (b-1) - 1\,.$$

So for every $n > n_0$ we have

$$\begin{aligned} c_{n+1} - d_n &= \left[a^{n+1}b^{n+1}\right] - \left[a^{n+1}b^n\right] \le a^{n+1}b^{n+1} - \left(a^{n+1}b^n - 1\right) \\ &= a^{n+1}b^n(b-1) + 1 \le a^{n+2}b^{n+1}(b-1) - 1 \\ &= a^{n+2}b^{n+2} - 1 - a^{n+2}b^{n+1} < \left[a^{n+2}b^{n+2}\right] - \left[a^{n+2}b^{n+1}\right] \\ &= c_{n+2} - d_{n+1} \,. \end{aligned}$$

This shows that the assumptions of Corollary 1 are fulfilled and we can calculate

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\left[a^{n+1}b^{n+1}\right] - \left[a^{n+1}b^n\right]}{\left[a^{n+2}b^{n+1}\right]} = \frac{b-1}{ab} \,.$$

□

In the rest of the paper we will suppose that

$$X = \{x_1 < x_2 < \dots\} = \bigcup_{n=1}^{\infty} \langle c_n, d_n \rangle \cap \mathbb{N}, \qquad \text{where} \quad c_n \le d_n < c_{n+1} \ \text{ for } \ n \in \mathbb{N} \tag{5}$$

and we denote

$$M(X) = \left\{n \in \mathbb{N} :\ c_{n+1} - d_n = \max\{c_{i+1} - d_i :\ i = 1, 2, \dots, n\}\right\}.$$

In the sequel we will consider only sets $X \subset \mathbb{N}$ such that $M(X)$ is infinite. Notice that otherwise $\underline{D}(X) = 0$ and the results in the sequel are trivial in this case. Also define

$$q(X) = \liminf_{m\to\infty,\, m\in M(X)} \frac{c_{m+1}^2}{d_m d_{m+1}} \,.$$

Remark 1. Notice that as the sequence $\left(\frac{c_{m+1}-d_m}{d_{m+1}}\right)_{m\in M(X)}$ is a subsequence of $\left(\frac{\max\{c_{i+1}-d_i:\ i=1,2,\dots,n\}}{d_{n+1}}\right)_{n\in\mathbb{N}}$, an immediate consequence of Theorem 1 is the following

$$\underline{D}(X) \le \liminf_{m\to\infty,\, m\in M(X)} \frac{c_{m+1} - d_m}{d_{m+1}} \,.$$

The following theorem provides an upper bound of $\underline{D}(X)$ by means of $q(X)$ and can be useful when $q(X)$ is not very large.

THEOREM 6. *For every $X \subset \mathbb{N}$ we have*

$$\underline{D}(X) \le \frac{q(X)}{4} \,.$$

Proof. Let us denote $q_m = \frac{c_{m+1}^2}{d_m d_{m+1}}$ and $x_m = \frac{d_{m+1}}{c_{m+1}}$ for all $m \in M(X)$. Then we have

$$\frac{c_{m+1} - d_m}{d_{m+1}} = \frac{1 - \frac{d_m}{c_{m+1}}}{\frac{d_{m+1}}{c_{m+1}}} = \frac{\frac{d_{m+1}}{c_{m+1}} - \frac{d_m d_{m+1}}{c_{m+1}^2}}{\left(\frac{d_{m+1}}{c_{m+1}}\right)^2} = \frac{x_m - \frac{1}{q_m}}{x_m^2}\,.$$

Using methods of elementary analysis one can easily verify that for fixed q_m the last fraction takes its maximal value $\frac{q_m}{4}$ when $x_m = \frac{2}{q_m}$ and thus we have

$$\frac{c_{m+1} - d_m}{d_{m+1}} \le \frac{q_m}{4}\,.$$

Now, by Remark 1, we have

$$\underline{D}(X) \le \liminf_{m\to\infty,\, m\in M(X)} \frac{c_{m+1} - d_m}{d_{m+1}} \le \liminf_{m\to\infty,\, m\in M(X)} \frac{q_m}{4} = \frac{q(X)}{4}\,.$$

□

The following theorem is of the similar kind as the previous one.

Theorem 7. *Let* $X = \bigcup_{n=1}^{\infty} \langle c_n, d_n \rangle \cap \mathbb{N}$. *Suppose that there exists an increasing sequence* $(k_n)_{n=1}^{\infty}$ *of positive integers such that* $k_n \in M(X)$ *holds for all sufficiently large* $n \in \mathbb{N}$ *and*

$$a = \lim_{n\to\infty} \frac{d_{k_n+1}}{c_{k_n+1}} \ge \lim_{n\to\infty} \frac{c_{k_n+1}}{d_{k_n}} = b\,. \tag{6}$$

Then

$$\underline{D}(X) \le \frac{b-1}{ab} \le \frac{1}{4}\,.$$

Proof. Again, we will use Remark 1 to calculate

$$\begin{aligned}\underline{D}(X) &\le \liminf_{m\to\infty,\, m\in M(X)} \frac{c_{m+1} - d_m}{d_{m+1}} \le \liminf_{n\to\infty} \frac{c_{k_n+1} - d_{k_n}}{d_{k_n+1}} \\ &\le \liminf_{n\to\infty} \frac{\frac{c_{k_n+1}}{d_{k_n}} - 1}{\frac{d_{k_n+1}}{c_{k_n+1}} \frac{c_{k_n+1}}{d_{k_n}}} = \frac{b-1}{ab} \le \frac{b-1}{b^2} \le \frac{1}{4}\,.\end{aligned}$$

□

The next theorem completes Theorem 7.

Theorem 8. *Let $\alpha \in \langle 0, \frac{1}{4} \rangle$. Then there exists a set $X \subset \mathbb{N}$ fulfilling conditions of Theorem 7 such that $\underline{D}(X) = \alpha$.*

Proof. First, let us consider the case $\alpha = 0$. Then put $X = \bigcup_{n=1}^{\infty} \langle 2^{n-1}+n-1, 2^n \rangle \cap \mathbb{N}$. One can easily see that in this case $M(X) = \mathbb{N}$ and the sequence $k_n = n$ for every $n \in \mathbb{N}$ fulfils (6). An easy calculation using Corollary 1 shows that

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}} = \liminf_{n\to\infty} \frac{n}{2^{n+1}} = 0\,.$$

Now, let $\alpha \in \left(0, \frac{1}{4}\right\rangle$. Denote $t = \frac{1}{1-2\alpha}$. Then $t \in (1, 2\rangle$ and also $\alpha = \frac{1}{2} - \frac{1}{2t}$. Set

$$X = \bigcup_{n=1}^{\infty} \langle [2^{n-1}t^n], [2^n t^n] \rangle \cap \mathbb{N}.$$

As $t > 1$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$2 \le 2^n t^n (t-1)(2t-1)$$

and consequently

$$\begin{aligned} c_{n+1} - d_n &= \left[2^n t^{n+1}\right] - \left[2^n t^n\right] \le 2^n t^{n+1} - \left(2^n t^n - 1\right) \\ &= 2^n t^n (t-1) + 1 \le 2^{n+1} t^{n+1} (t-1) - 1 \\ &= 2^{n+1} t^{n+2} - 1 - 2^{n+1} t^{n+1} < \left[2^{n+1} t^{n+2}\right] - \left[2^{n+1} t^{n+1}\right] \\ &= c_{n+2} - d_{n+1}\,. \end{aligned}$$

Thus $M(X)$ is a cofinite set and so we can set $k_n = n$ for every $n \in \mathbb{N}$. For such a choice the condition (6) is equivalent to $2 \ge t$, which holds, and the set X fulfils the condition of Theorem 7. To finish the proof, it suffices to show that $\underline{D}(X) = \alpha$.

Again, we can use Corollary 1 and (3)

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}} = \liminf_{n\to\infty} \frac{2^n t^{n+1} - 2^n t^n}{2^{n+1} t^{n+1}} = \frac{1}{2} - \frac{1}{2t} = \alpha\,.$$

□

Theorem 3 states the upper bound for dispersions of (R)-dense sets and Theorem 4 shows that dispersions of (R)-dense sets can take any positive values less than or equal to this upper bound. The sets constructed in Theorem 4 can have very irregular structure. A natural question arises whether the upper bound in Theorem 3 can be improved when knowing that structure of the (R)-dense set is regular in some sense. The next two theorems give some answers to this question. In their proofs we will use the following lemma.

Lemma 1. *Let $X \subset \mathbb{N}$ be (R)-dense set. Then*

$$\limsup_{n\to\infty} \frac{d_n}{c_n} \ge \liminf_{n\to\infty} \frac{c_{n+1}}{d_n} .$$

P r o o f . Suppose the contrary, i.e.

$$a = \limsup_{n\to\infty} \frac{d_n}{c_n} < \liminf_{n\to\infty} \frac{c_{n+1}}{d_n} = b .$$

First assume that $b < \infty$. Then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $a+\varepsilon < b-\varepsilon$, and for all $n > n_0$ it is $\frac{d_n}{c_n} < a+\varepsilon$ and $\frac{c_n}{d_{n-1}} > b-\varepsilon$. For the proof that X is not (R)-dense it is sufficient to show that there are only finitely many fractions $\frac{p}{q}$: $p, q \in X$ in some open subinterval of $(1,\infty)$. So suppose that $\frac{p}{q} \in (a+\varepsilon, b-\varepsilon)$ for some $p, q \in X$. If both p and q belong to the same interval. say $\langle c_n, d_n\rangle \bigcap \mathbb{N}$, then

$$\frac{d_n}{c_n} \ge \frac{p}{q} > a+\varepsilon$$

and so $n \le n_0$. Now let $p \in \langle c_n, d_n\rangle \bigcap \mathbb{N}$ and $q \in \langle c_m, d_m\rangle \bigcap \mathbb{N}$ for some $n > m$. Then

$$\frac{c_n}{d_{n-1}} \le \frac{c_n}{d_m} \le \frac{p}{q} < b-\varepsilon$$

and so $m < n \le n_0$. In the case $b = \infty$ the idea of proof is similar to the previous one and we omit it. □

The following theorem is a consequence of Theorem 7.

Theorem 9. *Let $X \subset \mathbb{N}$ be (R)-dense set and let there exist a proper limit $\lim\limits_{n\to\infty} \frac{c_{n+1}}{d_n} = b$. Then*

$$\underline{D}(X) \le \frac{b-1}{b^2} \le \frac{1}{4} .$$

P r o o f . First, let us consider the case $b = 1$. Then a simple use of Theorem 1 yields $\underline{D}(X) = 0$ and the statement of Theorem 9 holds in this case. Now let $b > 1$. Then it can be easily seen that $M(X)$ contains almost all positive integers and, by Lemma 1, the (R)-density of X implies that the assumptions of Theorem 7 are fulfilled. □

Theorem 10. *Let $X \subset \mathbb{N}$ be (R)-dense and let there exist a proper limit $\lim\limits_{n\to\infty} \frac{d_n}{c_n} = a$. Then*

$$\underline{D}(X) \le \min\left\{\frac{1}{a+1}, \max\left\{\frac{a-1}{a^2}, \frac{1}{a^2}\right\}\right\} ,$$

i.e.

$$\underline{D}(X) \le \begin{cases} \frac{1}{a+1} & \textit{if } a \in \left\langle 1, \frac{1+\sqrt{5}}{2} \right), \\ \frac{1}{a^2} & \textit{if } a \in \left\langle \frac{1+\sqrt{5}}{2}, 2 \right), \\ \frac{a-1}{a^2} & \textit{if } a \in \langle 2, \infty) . \end{cases}$$

Proof. First we will prove that $\underline{D}(X) \le \max\{\frac{a-1}{a^2}, \frac{1}{a^2}\}$.

By Lemma 1 there is a sequence $(k_n)_{n=1}^{\infty}$ of positive integers such that

$$a = \lim_{m\to\infty} \frac{d_m}{c_m} \ge \lim_{n\to\infty} \frac{c_{k_n+1}}{d_{k_n}} = b .$$

If there are infinitely many k_n's in $M(X)$, Theorem 7 can be applied to get

$$\underline{D}(X) \le \frac{b-1}{ab} \le \frac{a-1}{a^2} \le \max\left\{ \frac{a-1}{a^2}, \frac{1}{a^2} \right\} .$$

Now suppose that there are only finitely many k_n's in $M(X)$. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ be such that $\frac{d_{k_n}}{c_{k_n}} > a - \varepsilon$ and k_n does not belong to $M(X)$ for all $n > n_0$. Then $j \le k_n - 1$ holds for such a j that $c_{j+1} - d_j = \max\{c_{i+1} - d_i : i = 1, 2, \dots, k_n\} \le c_{k_n}$, for all $n > n_0$. Let us calculate, using Theorem 1,

$$\begin{aligned} \underline{D}(X) &= \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : \ i = 1, 2, \dots, n\}}{d_{n+1}} \\ &\le \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : \ i = 1, 2, \dots, k_n\}}{d_{k_n+1}} \\ &\le \liminf_{n\to\infty} \frac{c_{k_n}}{d_{k_n+1}} = \liminf_{n\to\infty} \frac{c_{k_n}}{d_{k_n}} \frac{d_{k_n}}{c_{k_n+1}} \frac{c_{k_n+1}}{d_{k_n+1}} \le \frac{1}{(a-\varepsilon)^2} , \end{aligned}$$

which proves $\underline{D}(X) \le \max\{\frac{a-1}{a^2}, \frac{1}{a^2}\}$, as $\varepsilon > 0$ was arbitrary.

Now we are going to prove that $\underline{D}(X) \le \frac{1}{a+1}$. So, let $X = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ be an (R)-dense set. Notice that the statement is trivial in the case $a = 1$. Thus, let $a > 1$. We will prove the statement by contradiction. So, suppose the contrary, i.e. $\underline{D}(X) > \frac{1}{a+1}$. Let us define the set $Y \subset \mathbb{N}$ as follows.

$$Y = X \cup \left(\bigcup_{k\in K} (d_k, c_{k+1}\rangle \cap \mathbb{N} \right) ,$$

where $k \in K$ if and only if there exists a positive integer $l < k$ such that

$$c_{l+1} - d_l \ge c_{k+1} - d_k .$$

Let us write the set Y in the form $Y = \bigcup_{n=1}^{\infty} \langle c'_n, d'_n \rangle \cap \mathbb{N}$. Then the following statements hold.

(i) $\underline{D}(X) = \underline{D}(Y) = \liminf_{x\to\infty} \frac{c'_{n+1}-d'_n}{d'_{n+1}}$.

(ii) $\liminf_{x\to\infty} \frac{d'_n}{c'_n} \ge a$.

(iii) $\limsup_{x\to\infty} \frac{d'_n}{c'_n} \le \frac{1}{\underline{D}(Y)}$.

(iv) The set Y is (R)-dense.

The statement (i) implies that $\underline{D}(Y) > \frac{1}{a+1}$. Let $\delta > 0$ be any number such that $\underline{D}(Y) > \frac{1}{a+1} + \delta$.

Now, choose an arbitrary $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ the inequalities

$$\frac{1}{a+1} + \delta - \varepsilon < \frac{c'_{n+1} - d'_n}{d'_{n+1}} \qquad \text{and} \qquad a - \varepsilon < \frac{d'_{n+1}}{c'_{n+1}}$$

hold. This implies

$$\begin{aligned} \frac{d'_n}{c'_{n+1}} &< 1 - \left(\frac{1}{a+1} + \delta - \varepsilon\right)(a-\varepsilon) \\ &= \frac{1}{a+1} + \frac{\varepsilon}{a+1} - a\delta + \varepsilon(a - \varepsilon + \delta) \\ &< \frac{1}{a+1} - a\delta + \varepsilon\left(\frac{1}{a+1} + a + \delta\right). \end{aligned}$$

As both $\delta > 0$ and $\varepsilon > 0$ were arbitrary small, the above inequalities imply that

$$\limsup_{x\to\infty} \frac{d'_n}{c'_{n+1}} < \frac{1}{a+1}.$$

The last inequality, together with (iii), gives

$$\limsup_{x\to\infty} \frac{d'_n}{c'_n} < \liminf_{x\to\infty} \frac{c'_{n+1}}{d'_n},$$

and an application of Lemma 1 yields that the set Y is not (R)-dense, which is a contradiction. □

Remark 2. *Notice that the previous theorem implies that if $X \subset \mathbb{N}$ is (R)-dense set and if there exists a proper limit $\lim_{n\to\infty} \frac{d_n}{c_n} = a \ge 2$, then*

$$\underline{D}(X) \le \frac{1}{4}.$$

REMARK 3. *Sometimes it is useful to express subsets of* $\mathbb{N}$ *as composed of blocks in a slightly different form as it is done in* (5), *for example*

$$X = \{x_1 < x_2 < \dots\} = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}, \qquad \textit{where} \quad c_n < d_n < c_{n+1} \quad \textit{for} \quad n \in \mathbb{N},$$

or

$$X = \{x_1 < x_2 < \dots\} = \bigcup_{n=1}^{\infty} \langle c_n, d_n) \cap \mathbb{N}, \qquad \textit{where} \quad c_n < d_n < c_{n+1} \quad \textit{for} \quad n \in \mathbb{N}.$$

Notice that also using any of this notations all theorems in the paper hold without any change.

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