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GENERALIZED STRONGLY (V, λ)-SUMMABLE SEQUENCES DEFINED BY ORLICZ FUNCTIONS

Mikail Et — Çiğdem A. Bektaş

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ABSTRACT. The idea of difference sequence spaces was introduced in [KIZMAZ, H.: *On certain sequence spaces*, Canad. Math. Bull. **24** (1981), 169–176] and this concept was generalized in [ET, M.—ÇOLAK, R.: *On some generalized difference sequence spaces*, Soochow J. Math. **21** (1995), 377–386]. In this paper we introduce concepts of λ^m-statistical convergence and strongly $(V, \lambda)(\Delta^m)$-summable sequence with respect to an Orlicz function and give some relations related to these sequence spaces.

1. Introduction

Let ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$$

where $\mathbb{N} = \{1, 2, \dots\}$, the set of positive integers.

Throughout the paper ω denotes the set of all sequences of complex numbers and m an arbitrary positive integer.

Kızmaz [9] defined the sequence spaces

$$X(\Delta) = \{x \in \omega : \ \Delta x \in X\}$$

for $X = \ell_\infty$, c or c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

The operators Δ^m, $\Sigma^m : \omega \to \omega$ are defined by

$$(\Delta^1 x)_k = \Delta^1 x_k = x_k - x_{k+1}, \quad (\Sigma^1 x)_k = \sum_{j=1}^{k-1} x_j \qquad (k = 1, 2, \dots),$$

$$\Delta^m = \Delta^1 \circ \Delta^{m-1}, \qquad \Sigma^m = \Sigma^1 \circ \Sigma^{m-1} \qquad (m \geq 2),$$

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where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so that

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v} .$$

Then Et and Çolak [3] generalized the above sequence spaces

$$X(\Delta^m) = \{x \in \omega : \ \Delta^m x \in X\}$$

for $X = \ell_\infty$, c and c_0.

The *generalized de la Vallee-Pousin mean* is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k ,$$

where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers such that

$$\lambda_{n+1} \le \lambda_n + 1 , \qquad \lambda_1 = 1 ,$$

$\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)*-summable to a number* L ([11]) if

$$t_n(x) \to L \qquad \text{as} \quad n \to \infty .$$

(V, λ)-summability reduces to $(C, 1)$-summability when $\lambda_n = n$ for all n. We write

$$[C, 1] = \Big\{x = (x_n) \in \omega : \ \lim_{n\to\infty} \tfrac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \ \text{for some } L\Big\}$$

and

$$[V, \lambda] = \Big\{x = (x_n) \in \omega : \ \lim_{n\to\infty} \tfrac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \ \text{for some } L\Big\}$$

for the sets of the sequences $x = (x_k)$ which are strongly Cesaro summable and strongly (V, λ)-summable to L, i.e., $x_k \to L[C, 1]$ and $x_k \to L[V, \lambda]$ respectively.

The idea of statistical convergence was introduced by Fast [5] and studied by various authors ([2], [4], [7], [8], [10], [13], [15]).

Recently, λ-statistical convergence were introduced by Mursaleen [13] as below:

A sequence $x = (x_n)$ is said to be λ*-statistical convergent* or S_λ*-convergent* to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \big|\{k \in I_n : \ |x_k - L| \ge \varepsilon\}\big| = 0 .$$

In this case we write $S_\lambda\text{-}\lim x = L$ or $x_k \to L(S_\lambda)$, and

$$S_\lambda = \{x \in \omega : \ S_\lambda\text{-}\lim x = L \ \text{for some } L\} .$$

An *Orlicz function* is a function $M\colon [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function and they defined the sequence space l_M as follows:

$$l_M = \Big\{x \in \omega : \sum_{k=1}^{\infty} M\big(\tfrac{|x_k|}{\rho}\big) < \infty \text{ for some } \rho > 0\Big\}.$$

The space l_M is a Banach space with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\big(\tfrac{|x_k|}{\rho}\big) \le 1\Big\}$$

and this space is called an *Orlicz sequence space.* They proved that every Orlicz sequence space l_M contains a subspace isomorphic to l_p for some $p \ge 1$. For $M(x) = x^p$, $1 \le p < \infty$, the space l_M coincides with the classical sequence space l_p.

An Orlicz function M is said to *satisfy* Δ_2*-condition* for all values of u, if there exists a constant $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$. The Δ_2-condition is equivalent to the inequality $M(lu) \le KlM(u)$ for all values of u and for $l > 1$ being satisfied.

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Let $x \in \omega$ and $X, Y \subset \omega$. Then we shall write

$$M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : \ ax \in Y \text{ for all } x \in X\}.$$

The set $X^\alpha = M(X, l_1)$ is called *Köthe-Toeplitz dual space* or α*-dual* of X.

Let X be a sequence space. Then X is called:

i) *Solid* (or *normal*) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$, for all sequences (α_k), scalars with $|\alpha_k| \le 1$.
ii) *Monotone* provided X contains the canonical preimages of all its stepspaces.
iii) *Perfect* if $X = X^{\alpha\alpha}$.

It is well known that X is perfect $\implies$ X is normal $\implies$ X is monotone.

In the present paper we introduce the concepts of λ^m-statistical convergence and strongly $(V,\lambda)(\Delta^m)$-summability with respect to an Orlicz function and examine some properties of these sequence spaces.

2. λ^m-statistical convergence

Before giving some inclusion relations we will give a new definition.

DEFINITION 2.1. A sequence $x = (x_n)$ is said to be λ^m*-statistically convergent* or S_{λ^m}*-convergent* to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : \ |\Delta^m x_k - L| \geq \varepsilon\right\}\right| = 0 .$$

In this case we write $S_{\lambda^m}\text{-}\lim x = L$ or $x_k \to L\big(S_\lambda(\Delta^m)\big)$, and

$$S_\lambda(\Delta^m) = \left\{x \in \omega : \ S_{\lambda^m}\text{-}\lim x = L \ \text{ for some } L\right\}.$$

Now we will find the relationship of $S_\lambda(\Delta^m)$ with $[V,\lambda](\Delta^m)$ and $(C,1)(\Delta^m)$, which are the generalizations of well-known sequence spaces of $[V,\lambda]$-summable and $(C,1)$-summable sequences, respectively. We define the sequence spaces $(C,1)(\Delta^m)$, $[C,1](\Delta^m)$, $(V,\lambda)(\Delta^m)$ and $[V,\lambda](\Delta^m)$ as below:

$$(C,1)(\Delta^m) = \left\{x \in \omega : \ \lim_{n\to\infty} \tfrac{1}{n} \sum_{k=1}^{n} (\Delta^m x_k - L) = 0 \ \text{ for some } L\right\},$$

$$[C,1](\Delta^m) = \left\{x \in \omega : \ \lim_{n\to\infty} \tfrac{1}{n} \sum_{k=1}^{n} |\Delta^m x_k - L| = 0 \ \text{ for some } L\right\},$$

$$(V,\lambda)(\Delta^m) = \left\{x \in \omega : \ \lim_{n\to\infty} \tfrac{1}{\lambda_n} \sum_{k\in I_n} (\Delta^m x_k - L) = 0 \ \text{ for some } L\right\},$$

$$[V,\lambda](\Delta^m) = \left\{x \in \omega : \ \lim_{n\to\infty} \tfrac{1}{\lambda_n} \sum_{k\in I_n} |\Delta^m x_k - L| = 0 \ \text{ for some } L\right\}.$$

It is trivial that $[C,1](\Delta^m) \subset (C,1)(\Delta^m)$, $[V,\lambda](\Delta^m) \subset (V,\lambda)(\Delta^m)$ and $X(\Delta^{m-1}) \subset X(\Delta^m)$ for $X = (C,1), [C,1], (V,\lambda)$ or $[V,\lambda]$.

THEOREM 2.2. *The space $(C,1)(\Delta^m)$ is a BK-space with the norm*

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \sup_{n\in\mathbb{N}} \left| n^{-1} \sum_{k=1}^{n} \Delta^m x_k \right|,$$

and the space $[C,1](\Delta^m)$ is a BK-space with the norm

$$\|x\|_{\Delta'} = \sum_{i=1}^{m} |x_i| + \sup_{n\in\mathbb{N}} \left(n^{-1} \sum_{k=1}^{n} |\Delta^m x_k| \right).$$

P r o o f . Proof follows from [4; Theorem 2.2]. □

THEOREM 2.3. *Let $\lambda = (\lambda_n)$ be the same as above, then*

(i) $x_k \to L[V,\lambda](\Delta^m) \implies x_k \to LS_\lambda(\Delta^m)$ *and the inclusion* $[V,\lambda](\Delta^m) \subset S_\lambda(\Delta^m)$ *is proper.*

(ii) *If* $x \in \ell_\infty(\Delta^m)$ *and* $x_k \to LS_\lambda(\Delta^m)$, *then* $x_k \to L[V,\lambda](\Delta^m)$ *and hence* $x_k \to L(C,1)(\Delta^m)$ *provided* $x = (x_k)$ *is not eventually constant.*

(iii) $S_\lambda(\Delta^m) \cap \ell_\infty(\Delta^m) = [V,\lambda](\Delta^m) \cap \ell_\infty(\Delta^m)$.

Proof.

(i) Let $\varepsilon > 0$ and $x_k \to L[V,\lambda](\Delta^m)$. We have

$$\sum_{k\in I_n} |\Delta^m x_k - L| \ge \sum_{\substack{k\in I_n\\ |\Delta^m x_k - L|\ge\varepsilon}} |\Delta^m x_k - L| \ge \varepsilon \big|\{k \in I_n : \ |\Delta^m x_k - L| \ge \varepsilon\}\big| .$$

Therefore $x_k \to L[V,\lambda](\Delta^m) \implies x_k \to LS_\lambda(\Delta^m)$.

To show that the inclusion is strict, define $x = (x_k)$ such that

$$\Delta^m x_k = \begin{cases} k & \text{for } k = n^2 , \ n = 1,2,\dots , \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \notin \ell_\infty(\Delta^m)$, $x_k \to 0S_\lambda(\Delta^m)$ and $x \notin [V,\lambda](\Delta^m)$.

(ii) Suppose that $x_k \to LS_\lambda(\Delta^m)$ and $x \in \ell_\infty(\Delta^m)$ and set $|\Delta^m x_k - L| \le K$ for all k. Given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n} |\Delta^m x_k - L| &= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |\Delta^m x_k - L|\ge\varepsilon}} |\Delta^m x_k - L| + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |\Delta^m x_k - L|<\varepsilon}} |\Delta^m x_k - L| \\ &\le \frac{M}{\lambda_n}\big|\{k \in I_n : \ |\Delta^m x_k - L| \ge \varepsilon\}\big| + \varepsilon .\end{aligned}$$

Hence $x_k \to L[V,\lambda](\Delta^m)$.

Since

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}(\Delta^m x_k - L) &= \frac{1}{n}\sum_{k=1}^{n-\lambda_n}(\Delta^m x_k - L) + \frac{1}{n}\sum_{k\in I_n}(\Delta^m x_k - L) \\ &\le \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n} |\Delta^m x_k - L| + \frac{1}{\lambda_n}\sum_{k\in I_n} |\Delta^m x_k - L| \\ &\le \frac{2}{\lambda_n}\sum_{k\in I_n} |\Delta^m x_k - L|\end{aligned}$$

we obtain $x_k \to L(C,1)(\Delta^m)$. □

DEFINITION 2.4. ([4]) The sequence x is said to be Δ^m*-statistically convergent* if there is a complex number L such that

$$\lim_{n\to\infty} n^{-1}\big|\{k \le n:\ |\Delta^m x_k - L| \ge \varepsilon\}\big| = 0$$

for every $\varepsilon > 0$. In this case we write $x_k \to LS(\Delta^m)$. The set of Δ^m-statistically convergent sequences will be denoted by $S(\Delta^m)$.

It is easy to see that $S_\lambda(\Delta^m) \subset S(\Delta^m)$ for all λ, since $\frac{\lambda_n}{n}$ is bounded.

THEOREM 2.5. $S(\Delta^m) \subset S_\lambda(\Delta^m)$ *if and only if*

$$\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0. \tag{1}$$

Proof. For given $\varepsilon > 0$ we have

$$\{k \le n:\ |\Delta^m x_k - L| \ge \varepsilon\} \supset \{k \in I_n:\ |\Delta^m x_k - L| \ge \varepsilon\}.$$

Therefore

$$\begin{aligned}\frac{1}{n}\big|\{k \le n:\ |\Delta^m x_k - L| \ge \varepsilon\}\big| &\ge \frac{1}{n}\big|\{k \in I_n:\ |\Delta^m x_k - L| \ge \varepsilon\}\big| \\ &\ge \frac{\lambda_n}{n}\frac{1}{\lambda_n}\big|\{k \in I_n:\ |\Delta^m x_k - L| \ge \varepsilon\}\big|.\end{aligned}$$

Taking the limit as $n \to \infty$ and using (1), we get

$$x_k \to LS(\Delta^m) \implies x_k \to LS_\lambda(\Delta^m).$$

Conversely suppose that $\liminf\limits_{n\to\infty} \frac{\lambda_n}{n} = 0$. As in [6; p. 510] we can choose a subsequence $(n(j))$ such that $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$. Define $x = (x_i)$ such that

$$\Delta^m x_i = \begin{cases} 1 & \text{if } i \in I_{n(j)},\ j = 1,2,\dots, \\ 0 & \text{otherwise.}\end{cases}$$

Then $x \in [C,1](\Delta^m)$, and by [4; Theorem 4.2], $x \in S(\Delta^m)$. But on the other hand, $x \notin [V,\lambda](\Delta^m)$ and Theorem 2.3(ii) implies that $x \notin S_\lambda(\Delta^m)$. Hence (1) is necessary. □

3. Some sequence spaces defined by Orlicz functions

In this section we introduce and examine some topological properties of three sequence spaces defined by using an Orlicz function M. It is also shown that if a sequence is strongly $(V,\lambda)(\Delta^m)$-summable with respect to an Orlicz function, then it is S_{λ^m}-statistically convergent.

DEFINITION 3.1. Let M be an Orlicz function, m be a positive integer and $p=(p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets.

$$[V,\lambda,M,p](\Delta^m)=\Big\{x=(x_k):\ \lim_{n\to\infty}\tfrac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\tfrac{|\Delta^m x_k-L|}{\rho}\Big)\Big]^{p_k}=0 \text{ for some } L, \text{ and } \rho>0\Big\},$$

$$[V,\lambda,M,p]_0(\Delta^m)=\Big\{x=(x_k):\ \lim_{n\to\infty}\tfrac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\tfrac{|\Delta^m x_k|}{\rho}\Big)\Big]^{p_k}=0 \text{ for some } \rho>0\Big\},$$

$$[V,\lambda,M,p]_\infty(\Delta^m)=\Big\{x=(x_k):\ \sup_{n\in\mathbb{N}}\tfrac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\tfrac{|\Delta^m x_k|}{\rho}\Big)\Big]^{p_k}<\infty \text{ for some } \rho>0\Big\}.$$

We denote $[V,\lambda,M,p](\Delta^m)$, $[V,\lambda,M,p]_0(\Delta^m)$ and $[V,\lambda,M,p]_\infty(\Delta^m)$ as $[V,\lambda,M](\Delta^m)$, $[V,\lambda,M]_0(\Delta^m)$ and $[V,\lambda,M]_\infty(\Delta^m)$ when $p_k=1$ for all k, respectively.

If $x\in[V,\lambda,M](\Delta^m)$, we say that x is *strongly* $(V,\lambda)(\Delta^m)$*-summable with respect to the Orlicz function* M.

THEOREM 3.2. *Let m be a positive integer. For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $[V,\lambda,M,p](\Delta^m)$, $[V,\lambda,M,p]_0(\Delta^m)$ and $[V,\lambda,M,p]_\infty(\Delta^m)$ are linear spaces over the field $\mathbb{C}$ of complex numbers.*

Proof. Let $x,y\in[V,\lambda,M,p]_0(\Delta^m)$ and $\alpha,\beta\in\mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m x_k|}{\rho_1}\right)\right]^{p_k}=0$$

and

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m y_k|}{\rho_2}\right)\right]^{p_k}=0.$$

Define $\rho_3=\max\{2|\alpha|\rho_1,2|\beta|\rho_2\}$. Since Δ^m is linear and M is non-decreasing and convex,

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m(\alpha x_k+\beta y_k)|}{\rho_3}\right)\right]^{p_k}\\
&=\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\alpha\Delta^m x_k+\beta\Delta^m y_k|}{\rho_3}\right)\right]^{p_k}\\
&\le\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\alpha\Delta^m x_k|}{\rho_3}+\frac{|\beta\Delta^m y_k|}{\rho_3}\right)\right]^{p_k}\\
&\le\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta^m x_k|}{\rho_1}\right)+M\left(\frac{|\Delta^m y_k|}{\rho_2}\right)\right]^{p_k}\\
&\le\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m x_k|}{\rho_1}\right)+M\left(\frac{|\Delta^m y_k|}{\rho_2}\right)\right]^{p_k}\\
&\le C\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m x_k|}{\rho_1}\right)\right]^{p_k}+C\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m y_k|}{\rho_2}\right)\right]^{p_k}\to 0
\end{aligned}$$

where $C=\max\{1,2^{H-1}\}$, $H=\sup_{k\in\mathbb{N}} p_k$; so that $\alpha x+\beta y\in[V,\lambda,M,p]_0(\Delta^m)$. This proves that $[V,\lambda,M,p]_0(\Delta^m)$ is a linear space. The rest can be proved by the same way as above. □

THEOREM 3.3. *Let m be a positive integer. For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $[V,\lambda,M,p]_0(\Delta^m)$ is a paranormed space (not necessarily totally paranormed) with*

$$g(x)=\inf\left\{\rho^{p_n/H}:\ \left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1,\ n=1,2,3,\dots\right\}$$

where $H=\max\left\{1,\sup_{k\in\mathbb{N}} p_k\right\}$.

Proof. Clearly $g(x)=g(-x)$. The subadditivity of g follows from the proof of Theorem 3.2, taking $\alpha=1$, $\beta=1$. It is trivial that $\Delta^m x=0$ for $x=0$. Since $M(0)=0$, we get $\inf\{\rho^{p_n/H}\}=0$ for $x=0$.

Finally, we prove that scalar multiplication is continuous. Let r be any complex number. From the linearity of Δ^m

$$\begin{aligned}
g(rx)&=\inf\left\{\rho^{p_n/H}:\ \left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m(rx_k)|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1,\ n=1,2,3,\dots\right\}\\
&=\inf\left\{\rho^{p_n/H}:\ \left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|r\Delta^m x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1,\ n=1,2,3,\dots\right\}.
\end{aligned}$$

Then

$$g(rx) = \inf\Big\{ \big(|r|s\big)^{p_n/H} : \Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[M\Big(\frac{|\Delta^m x_k|}{s}\Big)\Big]^{p_k}\Big)^{1/H} \le 1,\ n=1,2,3,\ldots\Big\}$$

where $s=\rho/|r|$. Since $|r|^{p_n}\le \max\{1,|r|^{\sup p_n}\}$, we have

$$g(rx) \le \big(\max\{1,|r|^{\sup p_n}\}\big)^{1/H}\cdot \inf\Big\{ s^{p_n/H} : \Big(\lambda_n^{-1}\sum_{k\in I_n}\Big[M\Big(\frac{|\Delta^m x_k|}{s}\Big)\Big]^{p_k}\Big)^{1/H} \le 1,\ n=1,2,3,\ldots\Big\},$$

which converges to zero as $g(x)$ converges to zero in $[V,\lambda,M,p]_0(\Delta^m)$.

Now suppose that $r_i\to 0$ as $i\to\infty$. Let x be a fixed sequence in $[V,\lambda,M,p]_0(\Delta^m)$. For arbitrary $\varepsilon>0$, let N be a positive integer such that

$$\lambda_n^{-1}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m x_k|}{\rho}\right)\right]^{p_k} \le (\varepsilon/2)^H$$

for some $\rho>0$ and all $n>N$. This implies that

$$\left(\lambda_n^{-1}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} \le \varepsilon/2$$

for some $\rho>0$ and all $n>N$.

Let $0<|r|<1$, using convexity of M, for $n>N$, we get

$$\lambda_n^{-1}\sum_{k\in I_n}\left[M\left(\frac{|r\Delta^m x_k|}{\rho}\right)\right]^{p_k} < \lambda_n^{-1}\sum_{k\in I_n}\left[|r|M\left(\frac{|\Delta^m x_k|}{\rho}\right)\right]^{p_k} < (\varepsilon/2)^H.$$

Since M is continuous everywhere in $[0,\infty)$, then for $n\le N$,

$$f(t) = \lambda_n^{-1}\sum_{k\in I_n}\left[M\left(\frac{|t\Delta^m x_k|}{\rho}\right)\right]^{p_k}$$

is continuous at 0. So there is $1>\delta>0$ such that $|f(t)|<\left(\frac{\varepsilon}{2}\right)^H$ for $0<t<\delta$. Let K be such that $|r_i|<\delta$ for $i>K$, then for $i>K$ and $n\le N$,

$$\left(\lambda_n^{-1}\sum_{k\in I_n}\left[M\left(\frac{|r_i\Delta^m x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} < \varepsilon/2.$$

Thus

$$\left(\lambda_n^{-1}\sum_{k\in I_n}\left[M\left(\frac{|r_i\Delta^m x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} < \varepsilon$$

for $i>K$ and all n, so that $g(rx)\to 0$ $(r\to 0)$. □

THEOREM 3.4. *Let X stand for $[V,\lambda,M]$, $[V,\lambda,M]_0$ or $[V,\lambda,M]_\infty$ and $m \geq 1$. Then the inclusion $X(\Delta^{m-1}) \subset X(\Delta^m)$ is strict. In general $X(\Delta^i) \subset X(\Delta^m)$ for all $i = 1,2,\dots,m-1$, and the inclusion is strict.*

Proof. We give the proof for $X = [V,\lambda,M]_\infty$ only. It can be proved in a similar way for $X = [V,\lambda,M]$ or $[V,\lambda,M]_0$. Let $x \in [V,\lambda,M]_\infty(\Delta^{m-1})$. Then we have

$$\sup_{n\in\mathbb{N}} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\frac{|\Delta^{m-1}x_k|}{\rho} \right) \right] < \infty \,, \tag{2}$$

for some $\rho > 0$. Since M is non-decreasing and convex function, we have

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\frac{|\Delta^{m}x_k|}{2\rho} \right) \right] \\
&= \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\frac{|\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}|}{2\rho} \right) \right] \\
&\leq \frac{1}{\lambda_n} \sum_{k\in I_n} \left[\frac{1}{2} M\left(\frac{|\Delta^{m-1}x_k|}{\rho} \right) \right] + \frac{1}{\lambda_n} \sum_{k\in I_n} \left[\frac{1}{2} M\left(\frac{|\Delta^{m-1}x_{k+1}|}{\rho} \right) \right] < \infty \quad \text{by (2).}
\end{aligned}$$

Thus $[V,\lambda,M]_\infty(\Delta^{m-1}) \subset [V,\lambda,M]_\infty(\Delta^m)$. Proceeding in this way one will have $[V,\lambda,M]_\infty(\Delta^i) \subset [V,\lambda,M]_\infty(\Delta^m)$ for $i = 1,2,\dots,m-1$. The inclusion is strict; the sequence $x = (k^m)$, for example, belongs to $[V,\lambda,M]_\infty(\Delta^m)$, but does not belong to $[V,\lambda,M]_\infty(\Delta^{m-1})$ for $M(x) = x$, $p_k = 1$ for all $k \in \mathbb{N}$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. (If $x = (k^m)$, then $\Delta^m x_k = (-1)^m m!$ and $\Delta^{m-1}x_k = (-1)^{m+1} m!(k + (m-1)/2)$ for all $k \in \mathbb{N}$.) □

THEOREM 3.5. *The sequence spaces $[V,\lambda,M,p]_0$ and $[V,\lambda,M,p]_\infty$ are solid.*

Proof. We give the proof for $[V,\lambda,M,p]_0$. Let $(x_k) \in [V,\lambda,M,p]_0$ and α_k be any sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\lambda_n^{-1} \sum_{k\in I_n} \left[M\left(\frac{|\alpha_k x_k|}{\rho} \right) \right]^{p_k} < \lambda_n^{-1} \sum_{k\in I_n} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} \to 0 \qquad (n \to \infty)\,.$$

Hence $(\alpha_k x_k) \in [V,\lambda,M,p]_0$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in [V,\lambda,M,p]_0$. □

Remark. In general it is difficult to predict about the solidity of $[V,\lambda,M,p]_0(\Delta^m)$ and $[V,\lambda,M,p]_\infty(\Delta^m)$ when $m > 0$. For this, consider the following example.

EXAMPLE. Let $m = 1$, $p_k = 1$ for all k and $M(x) = x$. Then $(x_k) = (k) \in [V,\lambda,M,p]_0(\Delta^2)$ but $(\alpha_k x_k) \notin [V,\lambda,M,p]_0(\Delta^2)$ when $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $[V,\lambda,M,p]_0(\Delta^2)$ is not solid.

From Theorem 3.5 we may give the following results:

COROLLARY 3.6.

(i) *The sequence spaces* $[V,\lambda,M,p]_0$ *and* $[V,\lambda,M,p]_\infty$ *are monotone.*

(ii) *The sequence spaces* $[V,\lambda,M,p]_0(\Delta^m)$ *and* $[V,\lambda,M,p]_\infty(\Delta^m)$ *are not perfect.*

LEMMA 3.7. ([1]) *Let* M *be an Orlicz function which satisfies* Δ_2*-condition and let* $0<\delta<1$. *Then for each* $x\ge\delta$ *we have* $M(x)<Kx\delta^{-1}M(2)$ *for some constant* $K>0$.

THEOREM 3.8. *For any Orlicz function* M *which satisfies* Δ_2*-condition, we have* $[V,\lambda](\Delta^m)\subset[V,\lambda,M](\Delta^m)$.

Proof. Let $x\in[V,\lambda](\Delta^m)$ so that

$$A_n\equiv\frac{1}{\lambda_n}\sum_{k\in I_n}|\Delta^m x_k-L|\to 0\,,\quad n\to\infty\,,\qquad\text{for some}\quad L\,.$$

Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $M(t)<\varepsilon$ for $0\le t\le\delta$. We can write

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}M\big(|\Delta^m x_k-L|\big)&=\lambda_n^{-1}\sum_{\substack{k\in I_n\\|\Delta^m x_k-L|<\delta}}M\big(|\Delta^m x_k-L|\big)+\sum_{\substack{k\in I_n\\|\Delta^m x_k-L|\ge\delta}}M\big(|\Delta^m x_k-L|\big)\\&<\lambda_n^{-1}(\lambda_n\varepsilon)+K\delta^{-1}M(2)A_n\end{aligned}$$

by Lemma 3.7, letting $n\to\infty$, it follows that $x\in[V,\lambda,M](\Delta^m)$. □

THEOREM 3.9. *Let* m *be a positive integer. For any Orlicz function* M, $[V,\lambda,M](\Delta^m)\subset S_\lambda(\Delta^m)$.

Proof. Let $x\in[V,\lambda,M](\Delta^m)$ and $\varepsilon>0$ be given. Then

$$\begin{aligned}\lambda_n^{-1}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^m x_k-L|}{\rho}\right)\right]&\ge\lambda_n^{-1}\sum_{\substack{k\in I_n\\|\Delta^m x_k-L|\ge\varepsilon}}\left[M\left(\frac{|\Delta^m x_k-L|}{\rho}\right)\right]\\&>\lambda_n^{-1}M(\varepsilon/\rho)\big|\{k\in I_n:\ |\Delta^m x_k-L|\ge\varepsilon\}\big|\,.\end{aligned}$$

Hence $x\in S_\lambda(\Delta^m)$. □

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Department of Mathematics
Faculty of Sciences and Arts
Firat University
23119-Elazig
TURKEY

E-mail: mikailet@yahoo.com