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MATCHING WITTS LOCALLY AND GLOBALLY

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ABSTRACT. Reciprocity equivalence of global fields is a set of local conditions implying the Witt equivalence of global fields, that is, the existence of an isomorphism of their Witt rings of symmetric bilinear forms. This paper proves the conjecture that Witt equivalence of global fields actually coincides with reciprocity equivalence.

Introduction

The *Witt ring* of non-degenerate symmetric bilinear forms over a field F is an important invariant of the field F. Thus it is natural to ask if two fields F and E are *Witt equivalent*, i.e., if their Witt rings are isomorphic. We will be interested in the classical situation when F and E are global fields (algebraic number and function fields). Although the theory of quadratic forms over global fields is well developed, with its central pillar the celebrated *Hasse-Minkowski Principle*, the question of Witt equivalence of global fields has been studied only for a few years. At the Chlébské 1983 conference we asked if there are Witt equivalent but non-isomorphic number fields ([12], p. 99). This question was answered in 1985 by A. Czogała, and independently by R. Perlis and the author. It turned out that there are examples of infinitely many distinct quadratic number fields with isomorphic Witt rings. An analysis of those examples led to the notion of *reciprocity equivalence* of global fields, which is a set of local conditions on two global fields implying their Witt equivalence. Reciprocity equivalence is a property of an independent interest and its ramifications ("tame" and "wild") attracted the attention of several authors (cf. [3], [4], [5], [10]).

The present paper gives the proof that reciprocity equivalence and Witt equivalence actually coincide. We mention that, in the meantime, another proof has been found and will appear in [11]. It is based on an analysis of the 2-torsion in the *Brauer group* of a global field.

Section 1 presents reciprocity equivalence and explains its relation to matching Witts locally, that is, to matching Witt rings of completions of the two global fields. Sections 2 and 3 contain some preliminary material on valuations and rigid

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and basic elements in general fields. We apply these results to construct in section 4 an almost reciprocity equivalence of two Witt equivalent global fields. The construction in section 4 uses a valuation-theoretic result of R. W a r e [14] (see [1] for a more general version) relating some rigidity properties of the field — well preserved under Witt equivalence — to the existence of a valuation of the field. The arguments in sections 5 and 6 are due to R. P e r l i s . The concluding section 7 gives corollaries and mentions some other relevant results.

Throughout the paper we consider only *global fields of characteristic different from* 2. Although our main question on Witt equivalence remains meaningful for function fields of characteristic 2, it has a simple and definitive answer: any two global function fields of characteristic 2 are Witt equivalent, and none of them is Witt equivalent to any global field of characteristic $\neq 2$. This can be easily seen by combining [2], Prop. (2.10) and [8], Thm. (5.10), p. 82 (cf. [13], Thm. (1.1)).

We use standard notation of the algebraic theory of quadratic forms (see [7]). For a quadratic form ϕ over a field F, the set of elements of $\dot{F}$ represented by ϕ is written $D_F(\phi)$. Here $\dot{F}$ is the multiplicative group of non-zero elements of F, and $\dot{F}^2$ will denote the subgroup of squares. We often use the same symbol to denote an element of $\dot{F}$ and the corresponding square class, i.e., an element of $\dot{F}/\dot{F}^2$.

A c k n o w l e d g m e n t. I want to thank Robert P e r l i s who has kindly permitted me to incorporate his results in sections 5 and 6.

1. Reciprocity equivalence

The following definition is due to P. E. C o n n e r . Two global fields F and E are said to be *reciprocity equivalent* when there is a bijective map

$$T\colon \Omega(F) \to \Omega(E) \tag{1.0.1}$$

between the sets of all primes of F and E (including infinite primes, if any), and a group isomorphism

$$t\colon \dot{F}/\dot{F}^2 \to \dot{E}/\dot{E}^2 \tag{1.0.2}$$

preserving Hilbert symbols:

$$(a,b)_P = (ta,tb)_{TP}$$

for all P in $\Omega(F)$ and all a,b in $\dot{F}/\dot{F}^2$.
The pair of maps (t,T) is said to be a *reciprocity equivalence* between F and E. The importance of reciprocity equivalence between global fields stems from the following fact:

Theorem (1.1). *Let F and E be global fields. Then F and E are Witt equivalent if and only if F and E are reciprocity equivalent.*

We will prove the sufficiency below (Prop. (1.3)) and the proof of necessity will occupy sections 2 through 6. The following result is well known and will be used several times to prove Witt equivalence of fields.

Harrison's Criterion (1.2). *Let F and E be arbitrary fields. The following statements are equivalent.*

(i) *F and E are Witt equivalent (i.e., the Witt rings WF and WE are isomorphic).*

(ii) *There is a ring isomorphism $i\colon WF \to WE$ sending 1-dimensional forms over F onto 1-dimensional forms over E.*

(iii) *There is a group isomorphism (1.0.2) such that $t(-1) = -1$ and $1 \in \mathrm{D}_F\langle a,b\rangle$ if and only if $1 \in \mathrm{D}_E\langle ta,tb\rangle$ for any $a,b \in \dot{F}$.*

(iv) *There is a group isomorphism (1.0.2) such that $t(-1) = -1$ and*
$$t(\mathrm{D}_F\langle a_1,\ldots,a_n\rangle) = \mathrm{D}_E\langle ta_1,\ldots,ta_n\rangle$$
for any quadratic form $\langle a_1,\ldots,a_n\rangle$ over F.

Proof. Harrison [6], p.21 (see also [11]). Recall that in this paper "arbitrary field" means "arbitrary field of characteristic $\neq 2$". Harrison's Criterion is also valid for fields of characteristic 2; for this see [2], Thm. (2.4).

In the sequel we call a map t satisfying (1.2)(iii), a *Harrison map*. Also, a ring isomorphism i satisfying (1.2)(ii) is said to be a *strong Witt ring isomorphism.*

Proposition(1.3). *Let F and E be global fields. If F and E are reciprocity equivalent, then F and E are Witt equivalent.*

Proof. Let (t,T) be a reciprocity equivalence between F and E. Then the computation
$$(tx,t(-1))_{TP} = (x,-1)_P = (x,x)_P = (tx,tx)_{TP} = (tx,-1)_{TP}$$
and non-degeneracy of the Hilbert symbol imply that $t(-1) = -1$. Also $(a,b)_P = (ta,tb)_{TP}$ means that $1 \in \mathrm{D}_{F_P}\langle a,b\rangle$ iff $1 \in \mathrm{D}_{E_{TP}}\langle ta,tb\rangle$, where F_P and E_{TP} are completions of F and E at P and TP, resp. By the Hasse Principle, $1 \in \mathrm{D}_F\langle a,b\rangle$ iff $1 \in \mathrm{D}_E\langle ta,tb\rangle$. Thus t is a Harrison map and F and E are Witt equivalent, by (1.2).

Proposition (1.4). *Let F and E be global fields and let t and T be a group isomorphism (1.0.2) and a bijective map (1.0.1), resp. The following are equivalent.*

(i) *(t,T) is a reciprocity equivalence between F and E.*

(ii) *For every prime P in $\Omega(F)$, the group isomorphism t induces an isomorphism of local Witt rings WF_P and WE_{TP}.*

P r o o f. This is proved in [11]. We omit the proof since the result will not be needed in the proof of Theorem (1.1). We state (1.4) mainly to motivate our point of view that reciprocity equivalence means matching Witts locally.

Proposition (1.5). *Let F and E be algebraic number fields of degrees n and m over the rational field $\mathbf{Q}$, resp. If F and E are reciprocity equivalent, then $n = m$.*

P r o o f. If (t,T) is a reciprocity equivalence between F and E, then, by Prop. (1.4), T sends infinite real (complex) primes of F onto infinite real (complex) primes of E. The result follows, since then

$$n = r_1(F) + 2r_2(F) = r_1(E) + 2r_2(E) = m,$$

where r_1 and $2r_2$ are the numbers of real and complex primes of the field, respectively.

2. Some valuation theory

Let F be an arbitrary field of characteristic $\neq 2$. We assume that v is a discrete valuation on F, and $A, M, \overline{F}, F_v$ are the valuation ring of v, the maximal ideal of A, the residue class field A/M and the completion of F with respect to v, respectively. We write t_v and θ_v for the natural morphisms

$$t_v\colon \dot{F}/\dot{F}^2 \to \dot{F}_v/\dot{F}_v^2 \qquad \text{and} \qquad \theta_v\colon WF \to WF_v.$$

Lemma (2.1). (1) *t_v and θ_v are surjective maps.*

(2) *$\operatorname{Ker}\theta_v$ is generated as an ideal by the forms $\langle 1,-a\rangle$, where $a \in \dot{F}\cap\dot{F}_v^2$.*

(3) *$\dot{F}\cap\dot{F}_v^2 = (1+4M)\cdot\dot{F}^2$.*

(4) *$[\dot{F}\colon (1+4M)\cdot\dot{F}^2] = [\dot{F}_v\colon \dot{F}_v^2]$.*

(5) *$[\dot{F}\colon (1+M)\cdot\dot{F}^2] = 2$ iff $\overline{F} = \overline{F}^2$.*

P r o o f. (1) Surjectivity of t_v is a familiar consequence of the *Local Square Theorem* (cf. [7], p. 160). Since the Witt ring is generated by 1-dimensional forms, the surjectivity of θ_v follows from that of t_v.

(2) Let $\phi \neq 0$ be an anisotropic form in $\operatorname{Ker}\theta_v$. Then $\dim\phi \geq 2$ and we may assume that $1 \in D_F(\phi)$. Thus if $\phi \cong \langle 1\rangle \perp \phi'$, then $-1 \in D_{F_v}(\phi')$. Choose a representation $-1 = \phi'(X_1,\ldots,X_k)$ with X_i in F_v and then pick up x_i in F close enough to X_i. Then $-a := \phi'(x_1,\ldots,x_k)$ is close to -1, hence in the same square class, i.e., $-a \in -\dot{F}_v^2 \cap D_F(\phi')$. It follows that $\phi \cong \langle 1,-a\rangle \perp \phi''$, where $a \in \dot{F}\cap\dot{F}_v^2$, and now induction completes the proof.

(3) Let $M_v := \{x \in \dot{F}_v \colon v(x) > 0\}$ and $U_v := \{x \in \dot{F}_v \colon v(x) = 0\}$. Then $1 + 4M_v \subseteq U_v^2$, by the Local Square Theorem, and

$$(1+4M)\cdot \dot{F}^2 \subseteq \dot{F} \cap (1+4M_v)\cdot \dot{F}_v^2 \subseteq \dot{F} \cap U_v^2 \cdot \dot{F}_v^2 = \dot{F} \cap \dot{F}_v^2.$$

Now let $x \in \dot{F} \cap \dot{F}_v^2$. Then $v(x) \equiv 0 \pmod 2$ and, up to a square in $\dot{F}$, we may assume that x is a unit. Now $x = y^2$ with $y \in U_v$. Pick up $a \in \dot{F}$ such that $y \equiv a \pmod{4M}$. Then $v(a) = 0$ and $x = y^2 \equiv a^2 \pmod{4M}$. Hence, for some $m \in M$,

$$x = a^2 + 4m = a^2(1 + 4ma^{-2}) \in (1+4M)\cdot \dot{F}^2.$$

(4) Using (3) and (4), we have

$$[\dot{F} \colon (1+4M)\cdot \dot{F}^2] = [\dot{F} \colon \dot{F} \cap \dot{F}_v^2] = [\dot{F}\cdot \dot{F}_v^2 \colon \dot{F}_v^2] = [\dot{F}_v \colon \dot{F}_v^2].$$

(5) Let $U := \{x \in F \colon v(x) = 0\}$ be the group of units in F. Observe that $(1+M)\cdot \dot{F}^2 \subseteq U \cdot \dot{F}^2$ and $[\dot{F} \colon U\cdot \dot{F}^2] = [v^{-1}(\mathbb{Z}) \colon v^{-1}(2\mathbb{Z})] = 2$. Hence $[\dot{F} \colon (1+M)\cdot \dot{F}^2] = 2$ iff $(1+M)\cdot \dot{F}^2 = U \cdot \dot{F}^2$ iff every unit in F becomes a square in the residue class field $\overline{F}$ iff $\overline{F} = \overline{F}^2$.

Remark (2.2). If v is non-dyadic (i.e., if $\operatorname{char}\overline{F} \neq 2$), then $4M = M$, and so $\dot{F} \cap \dot{F}_v^2 = (1+M)\cdot \dot{F}^2$ in this case.

Lemma (2.3). *Let v and w be two discrete valuations of the field F. Let M and N be the maximal ideals of their valuation rings in F and let $\theta_v \colon WF \to WF_v$ and $\theta_w \colon WF \to WF_w$ be the natural morphisms.*

(1) *If $(1+4M)\cdot \dot{F}^2 \subseteq (1+4N)\cdot \dot{F}^2$, then $v = w$.*
(2) *If $\operatorname{Ker}\theta_v \subseteq \operatorname{Ker}\theta_w$, then $v = w$.*

Proof. (1) Suppose $v \neq w$ and let $w(p) = 1$ for some $p \in F$. By the approximation theorem there exists $x \in F$ such that

$$v(x-1) \geq 2 + v(4), \qquad w(x-p) \geq 2.$$

Then $x \in 1 + 4M$, while $w(x) = w(x-p+p) = w(p) = 1$. Hence $x \notin (1+4N)\cdot \dot{F}^2$, since $w((1+4N)\cdot \dot{F}^2) \subseteq 2\mathbb{Z}$.
(2) By Lemma (2.1)(2), $\operatorname{Ker}\theta_v \subseteq \operatorname{Ker}\theta_w$ implies $\dot{F}\cap\dot{F}_v^2 \subseteq \dot{F}\cap\dot{F}_w^2$. Then Lemma (2.1)(3) and (2.3)(1) imply that $v = w$.

3. Rigid and basic elements

In this section we study the behaviour of rigid elements under Witt equivalence. Recall from [15] and [1] that, for a subgroup T of $\dot{F}$, we say that $x \in \dot{F}$ is *T-rigid* if $T + xT \subseteq T \cup xT$. And x is said to be *T-birigid* if both x and $-x$ are T-rigid. If x is not T-birigid, it is *T-basic*, and $B_F(T)$ denotes the set of all T-basic elements in $\dot{F}$.

Lemma (3.1). *Let F and E be two Witt equivalent fields and let t be a corresponding Harrison map. Let T be a subgroup of $\dot{F}$ containing $\dot{F}^2$ and put $S := t(T)$. Then*

(1) *$x \in \dot{F}^2$ is T-rigid iff $y := t(x)$ is S-rigid.*
(2) *$x \in \dot{F}^2$ is T-birigid iff $y = t(x)$ is S-birigid.*
(3) *$t(B_F(T)) = B_E(S)$.*

Proof. (1) Let x be T-rigid. We want $S + yS \subseteq S \cup yS$. So consider any $s_1 + ys_2$, where $s_1, s_2 \in S$. Let $s_1 = t(x_1)$, $s_2 = t(x_2)$, where $x_1, x_2 \in T$. Then

$$s_1 + ys_2 \in D_E\langle s_1, ys_2\rangle = t(D_F\langle x_1, xx_2\rangle) \subseteq t(T + xT) \subseteq \\ \subseteq t(T \cup xT) = S \cup yS.$$

(2) By (1), x and $-x$ are T-rigid iff $t(x)$ and $t(-x) = -t(x)$ are S-rigid.
(3) follows from (2).

Lemma (3.2). *Let F be a global field and let v be a valuation of F. Let $T := (1 + M) \cdot \dot{F}^2$, where M is the maximal ideal of the valuation ring of v.*

(1) *$B_F(T) = \pm T$.*
(2) *T is not additively closed*

Proof. (1) We have $1 + M \subseteq T$, hence v is T-compatible (cf. [1], Def. 1 7) and so, by [1], Prop. 1.9, $B_F(T) \subseteq U \cdot T$, where U is the group of units of the valuation ring of v. We have

$$T \subseteq \pm T \subseteq B_F(T) \subseteq U \cdot T. \tag{3.2.1}$$

Observe that $U \subseteq T$ iff every element in the re idue class field $\overline{F}$ is a square in $\overline{F}$ iff v is dyadic. Thus $T = U \cdot T$ if v is dyadic and (1) follows from (3.2.1). If v is non-dyadic, then

$$[U \cdot T \colon T] = [U \cdot U \cap T] - [U \cdot (1 + M) \cdot U^2] - 2$$

Now, if $-1 \notin T$, the result follows imm diately from (3.2.1).
It remains to con ider th case, where v i non-dyadic and $-1 \in T$. Observe

that then $-1 \in \overline{\dot{F}}^2$ and $B_F(\overline{\dot{F}}^2) = \overline{\dot{F}}^2$. Now according to [1], Prop 1.9(2), the isomorphism

$$U \cdot T/T \to \overline{\dot{F}}/\overline{U \cap T} = \overline{\dot{F}}/\overline{\dot{F}}^2$$

maps $B_F(T)/T$ onto $B_{\overline{F}}(\overline{T})/\overline{T}$, where $\overline{T} := \overline{U \cap T} = \overline{\dot{F}}^2$. It follows that $B_F(T) = T$, as needed.

(2) Contrary to (2), suppose that $T + T \subseteq T$. If $\operatorname{char} F = p \neq 0$, then $0 = p \cdot 1 \in T$, a contradiction. If $\operatorname{char} F = 0$ and p is a prime number such that $v(p) > 0$ and $f \in F$ satisfies $v(f) = 1$, then both $p \in T$ and $p + f = (p-1) + (1+f) \in T$. Now, if $v(p) > 1$, then $v(p+f) = 1$, so in either case T contains an element with odd value, a contradiction. This proves (2).

4. Matching non-dyadic primes

Proposition (4.1). *Let F and E be Witt equivalent global fields. Suppose $i: WF \to WE$ is a strong Witt ring isomorphism and let t be the corresponding Harrison map.*

(1) For every non-dyadic valuation v of F there exists a unique non-dyadic valuation w of E such that there is a commutative diagram

$$\begin{array}{ccc} WF & \xrightarrow{\ i\ } & WE \\ \downarrow & & \downarrow \\ WF_v & \longrightarrow & WE_w \end{array}$$

where the lower horizontal arrow is a ring isomorphism and the vertical arrows are the natural morphisms θ_v and θ_w.

(2) The assignment $v \mapsto w$ sets up a bijective map from the set of non-dyadic valuations F onto the set of non-dyadic valuations of E.

P r o o f. Put $T := (1 + M) \cdot \dot{F}^2$, where $M = \{x \in F : v(x) > 0\}$. By Lemma (2.1)(4) we have $[\dot{F} : T] = 4$. Now let $S := t(T)$. Then S is a subgroup of index 4 in $\dot{E}$ and S contains $\dot{E}^2$. By (3.1)(3) and Lemma (3.2), we have $B_E(S) = \pm S$. It follows that $[\dot{E} : B_E(S)] = 4$ when $-1 \in S$, or $[\dot{E} : B_E(S)] = 2$ when $-1 \notin S$. In the first case we apply [1], Theorem 2.16, with $H := B_E(S)$. It follows that there is a subgroup H^* of $\dot{E}$ such that $H \subseteq H^*$, $[H^* : H] \leq 2$ and $O_E(H^*, S)$ is an S-compatible valuation ring of E. Since $H \subseteq H^* \subseteq \dot{E}$, $[\dot{E} : H] = 4$ and $[H^* : H] \leq 2$, we conclude that $H^* \neq \dot{E}$, and so $O_E(H^*, S) \neq E$. Thus there is a non-trivial valuation w of E with the valuation ring $O_E(H^*, S)$.
In the second case, i.e., when $-1 \notin S$, we observe that S is not additively

closed, since otherwise T would be additively closed, contrary to Lemma (3.2). Thus, even though $B_E(S) = \pm S$, we have not the exceptional case in the sense of Definition 2.15 in [1] and, as above, there exists an S-compatible valuation ring $O_E(H^*, S) \neq E$ of E (here $H^* = H = B_E(S)$) and a non-trivial valuation w of E with the given valuation ring. We will show that, in either case, w satisfies the requirements of Proposition (4.1)(1). Let $N := \{x \in E\colon w(x) > 0\}$. S-compatibility of w means $1 + N \subseteq S$, hence we also have $(1+N)\cdot \dot{E}^2 \subseteq S$. Now $4 = [\dot{E}^2\colon S] \leq [\dot{E}^2\colon (1+N)\cdot \dot{E}^2]$, and so, by Lemma (2.1)(4)(5), we conclude that w is non-dyadic and $(1+N)\cdot \dot{E}^2 = S$.

Now let $\theta_v\colon WF \to WF_v$ and $\theta_w\colon WE \to WE_w$ be the natural morphisms. Then $t(T) = S$ and Lemma (2.1)(2)(3) imply that

$$i(\operatorname{Ker}\theta_v) = \operatorname{Ker}\theta_w.$$

Hence there exists a ring isomorphism $WF_v \to WE_w$ such that the diagram in (4.1)(1) commutes. If w_1 is another valuation of E making the diagram in (4.1)(1) commutative, we get $i(\operatorname{Ker}\theta_v) = \operatorname{Ker}\theta_{w_1}$. Hence $\operatorname{Ker}\theta_w = \operatorname{Ker}\theta_{w_1}$, and by Lemma (2.3)(2), we have $w = w_1$.
(2) The isomorphisms i^{-1} and t^{-1} give rise to the inverse map.

Proposition (4.2). *Let F and E be Witt equivalent algebraic number fields and let i be a strong Witt ring isomorphism.*
(1) *For every ordering P of the field F (if there are any) there exists a unique ordering Q of the field E such that there is a commutative diagram*

$$\begin{array}{ccc} WF & \xrightarrow{\ i\ } & WE \\ \downarrow & & \downarrow \\ WF_P & \longrightarrow & WE_Q \end{array}$$

where F_P and E_Q are real closures relative to P and Q, resp., the lower horizontal arrow is a ring isomorphism, and the vertical arrows are natural morphisms.
(2) *The assignment $P \mapsto Q$ sets up a bijective map from the set X_F of all orderings of F onto the set X_E of all orderings of E.*

P r o o f. It is known that for any ordering P of F there is a minimal prime ideal p of WF such that $P = \{a \in \dot{F}\colon \langle a\rangle \equiv \langle 1\rangle \pmod p\}$ (cf. [7], p. 246). Then $q := i(p)$ is a minimal prime ideal in WE and $Q\colon = \{b \in \dot{E}\colon \langle b\rangle \equiv \langle 1\rangle \pmod q\}$ is an ordering of E. The existence of a ring isomorphism $WF_P \to WE_Q$ and all the remaining claims can be proved with the arguments from the final part of proof of Proposition (4.1).

Lemma (4.3). *Let F and E be Witt equivalent global fields and let $i: WF \to WE$ be a strong Witt ring isomorphism and t a corresponding Harrison map. Let P be a prime of F (finite or infinite) and Q be a prime of E. The following are equivalent.*
(1) *There is a commutative diagram*

$$\begin{tikzcd} WF \arrow[r, "i"] \arrow[d, "\theta_P"'] & WE \arrow[d, "\theta_Q"] \\ WF_P \arrow[r, "i_P"'] & WE_Q \end{tikzcd}$$

where the lower horizontal arrow is a ring isomorphism and the vertical arrows are natural morphisms.
(2) $(a,b)_P = (ta,tb)_Q$ *for every* $a,b \in \dot{F}$.

Proof. (1)$\Rightarrow$(2). For $a,b \in \dot{F}$ we have

$$\begin{aligned}(a,b)_P = 1 &\iff \theta_P\langle\langle -a,-b\rangle\rangle = 0 \quad \iff i_P \circ \theta_P\langle\langle -a,-b\rangle\rangle = 0\\ &\iff \theta_Q \circ i\langle\langle -a,-b\rangle\rangle = 0 \iff \theta_Q\langle\langle -ta,-tb\rangle\rangle = 0\\ &\iff (ta,tb)_Q = 1.\end{aligned}$$

(2)$\Rightarrow$(1). This follows from Harrison's Criterion (1.2). Here i_P is induced by sending $\langle a\rangle$ into $\langle ta\rangle$.

Now we are in a position to make a step toward proving Theorem (1.1). To summarize our results we introduce almost reciprocity equivalence. For a global field F we write $\Omega_1(F)$ for the set consisting of all infinite real primes of F and of all finite non-dyadic primes of F.

Definition (4.4). *Two global fields F and E are said to be almost reciprocity equivalent if there is a bijective map*

$$T: \Omega_1(F) \to \Omega_1(E)$$

and a group isomorphism

$$t: \dot{F}/\dot{F}^2 \to \dot{E}/\dot{E}^2$$

such that

$$(a,b)_P = (ta,tb)_{TP}$$

for all primes P in $\Omega_1(F)$ and all elements a,b in $\dot{F}/\dot{F}^2$.

Theorem (4.5). *Let F and E be global fields. If F and E are Witt equivalent, then F and E are almost reciprocity equivalent. More precisely, if t is a Harrison map between F and E, then there is a map $T\colon \Omega_1(F) \to \Omega_1(E)$ such that (t,T) is an almost reciprocity equivalence between F and E.*

Proof. Combine Propositions (4.1), (4.2) and Lemma (4.3).

Corollary (4.6). *Let F and E be global function fields. Then F and E are Witt equivalent if and only if they are reciprocity equivalent.*

Proof. Function fields of characteristic $\neq 2$ have no dyadic primes, hence almost reciprocity equivalence is the same as reciprocity equivalence.

It is easy to show that an almost reciprocity equivalence of two Witt equivalent fields can be extended to a reciprocity equivalence when the fields have each a unique dyadic prime, or when the fields are quadratic number fields, This is also true in the general case but requires some extra work. We discuss the general case in the next two sections.

5. An exact sequence

Let F be an algebraic number field and let S be any finite, non-empty set of primes of F. In the application we have in mind S will be the set of all dyadic primes of F (see section 6). Given any $a \in \dot{F}$ we define the following 3 objects:

$h(a) :=$ the number of primes $P \in S$ at which $a \notin \dot{F}_P^2$,
$G(a) := \{b \in \dot{F} \colon (a,b)_P = 1$ for every prime P not in $S\}$,
$H(a) := \prod_{P \in S} \dot{F}_P / N_P(a)$.

Here F_P is the completion of F at P and $N_P(a) := D_{F_P}\langle 1, -a\rangle$ is the norm group from $F_P(\sqrt{a})/F_P$. Similarly, we will write $N(a)$ for $D_F\langle 1, -a\rangle$. Notice that

$$0 \le h(a) \le |S|. \tag{5.0.1}$$

$$G(a) = \bigcap_{P \notin S} N_P(a) \cap \dot{F}. \tag{5.0.2}$$

$$N(a) \subseteq G(a). \tag{5.0.3}$$

$$|H(a)| = 2^{h(a)}. \tag{5.0.4}$$

Here (5 0.2) follows from the fact that $(a,b)_P = 1$ iff $b \in N_P(a)$ and (5.0.3) follows from (5.0.2) and from $N(a) \subseteq N_P(a)$. For (5.0.4), use the fact that $N_P(a)$ has in $\dot{F}_P$ the index 1 or 2 depending on whether a is, or is not a square in $\dot{F}_P$.

Now we will consider the following sequence

$$1 \to N(a) \to G(a) \to H(a) \xrightarrow{\mu} \{\pm 1\} \to 1 \tag{5.0.5}$$

Here $N(a) \to G(a)$ is the inclusion map, $G(a) \to H(a)$ is the diagonal map sending $b \in G(a)$ onto the family $(b \cdot N_P(a))$. Finally, μ is defined as follows: $\mu((b_P \cdot N_P(a)) = \prod_{P \in S} (a, b_P)_P$.

Proposition (5.1). (i) *If* $h(a) \geq 1$, *then the sequence* (5.0.5) *is exact.* (ii) *If* $h(a) = 0$, *then* $N(a) = G(a)$.

Proof. $N(a) \subseteq \operatorname{Ker}(G(a) \to H(a))$ follows from $N(a) \subseteq N_P(a)$. Conversely, if $b \in G(a)$ and $b \in N_P(a)$ for every $P \in S$, then $b \in N_P(a)$ for every prime P. Hence $b \in N(a)$ by the Hasse Norm Theorem ([9], 65:23). This proves the exactness at $G(a)$.

For any $b \in G(a)$, we have $(a, b)_P = 1$ for all P not in S, hence

$$1 = \prod (a, b)_P = \prod_{P \in S} (a, b)_P = \mu((b \cdot N_P(a))),$$

the first product running over all primes P. Thus $\operatorname{Im} G(a) \subseteq \operatorname{Ker} \mu$. And if $(b_P \cdot N_P(a)) \in \operatorname{Ker} \mu$, i.e., if $\prod_{P \in S} (a, b_P)_P = 1$, then $(a, b_P)_P = -1$ for an even number of $P \in S$. Since a is not a square at the primes P, where $(a, b_P)_P = -1$, there exists a global element $b \in \dot{F}$ such that

$$\begin{aligned} &(a, b)_P = -1 \quad \text{whenever} \quad (a, b_P)_P = -1, \quad \text{and} \\ &(a, b)_P = 1 \quad \text{for all remaining primes } P \text{ of } F \ \text{([9], 71:19a).} \end{aligned}$$

Thus $b \in G(a)$ and, moreover, $(a, b)_P = (a, b_P)_P$ for every $P \in S$. Hence $b \in b_P \cdot N_P(a)$ for $P \in S$, i.e., $(b_P \cdot N_P(a)) \in \operatorname{Im} G(a)$. This proves the exactness at $H(a)$. Observe that the above proof is valid also when $h(a) = 0$, and then it proves (ii).

Now assume that $h(a) \geq 1$, so that $H(a)$ is a non-trivial group. We must prove the surjectivity of μ. If $a \notin \dot{F}_Q^2$, where $Q \in S$, pick up $b_Q \in \dot{F}$ such that $(a, b_Q)_Q = -1$ (non-degeneracy of the Hilbert symbol). Put $b_P = 1$ for $Q \neq P \in S$. Then $\mu((b_P \cdot N_P(a)) = (a, b_Q)_Q = -1$. Thus μ is surjective, as desired.

We draw a corollary to be used in the next section.

Corollary (5.2). *If* $h(a) \geq 1$, *then the group* $G(a)/N(a)$ *is finite of order* $2^{h(a)-1}$. *Hence* $N(a) = G(a)$ *if and only if* $h(a) = 0$ *or* 1.

6. Matching dyadic primes

In this section we conclude the proof of Theorem (1.1). So far, we have proved that Witt equivalence of F and E implies almost reciprocity equivalence (Theorem (4.5)), hence reciprocity equivalence if F and E are function fields (Corollary (4.6)). It remains to investigate the possibility of extending an almost reciprocity equivalence onto dyadic primes in number fields in a Hilbert-symbol-preserving way. The first thing is to prove that the numbers $g_2(F)$ and $g_2(E)$ of dyadic primes in F and E are equal. We will work in the following set-up.

$$F \text{ and } E \text{ are Witt equivalent number fields,} \tag{6.0.1}$$

$$t \text{ is a Harrison map between } F \text{ and } E, \tag{6.0.2}$$

$$(t,T) \text{ is an almost reciprocity equivalence between } F \text{ and } E. \tag{6.0.3}$$

We also use the notation of section 5. We will use the fact that, when (6.0.2) and (6.0.3) hold, then t^{-1} is a Harrison map between E and F and (t^{-1}, T^{-1}) is an almost reciprocity equivalence between E and F.

Proposition (6.1). *For any $a \in \dot{F}$, we have $h(a) = h(ta)$.*

P r o o f. Assume first that $h(a) \geq 1$. Since t preserves value sets of quadratic forms, it induces an isomorphism from $G(a)/N(a)$ onto $G(ta)/N(ta)$. Then $2^{h(a)-1} = 2^{h(ta)-1}$, by Corollary (5.2), hence $h(a) = h(ta)$. Now assume that $h(a) = 0$. If $h(ta) \geq 1$, then by symmetry, $h(a) = h(t^{-1}(ta)) \geq 1$, a contradiction. Hence $h(ta) = 0$, as desired.

Corollary (6.2). $g_2(F) = g_2(E)$.

P r o o f. Choosing a to be a non-square at every dyadic prime in F we see that $h(a) = g_2(F)$. By Proposition (6.1), $g_2(F) \leq g_2(E)$, and by symmetry, $g_2(F) = g_2(E)$.

Thus, so far, we have proved that Witt equivalence of F and E implies almost reciprocity equivalence and $g_2(F) = g_2(E)$. Now we want more. Fix a dyadic prime P of F and choose a to be a non-square at P but to be a square at the other dyadic primes of F. So $h(a) = 1$, and thus $h(ta) = 1$ also. So there is a unique dyadic prime P' of E at which ta is a local non-square. We will map P to P'.

First we show that P' does not depend on the element a we chose. So let a' be another non-square at P with $h(a') = 1$. Then ta' is a non-square at a unique dyadic prime P'' of E. If $P' \neq P''$, then $h(ta \cdot ta') = 2$, so $h(a \cdot a') = 2$, contradicting the fact that $h(a \cdot a') \leq 1$, since a and a' are local squares at every dyadic prime except P. Hence $P' = P''$.

Thus sending P to P' defines a map

$$T' : \Omega_2(F) \to \Omega_2(E)$$

between the sets of dyadic primes of F and E.

Claim 1. *T' is injective.*

If $P_1 \neq P_2$ in $\Omega_2(F)$ and both of them are mapped to P' in $\Omega_2(E)$, then choose a_1 and a_2 to be local non-squares at P_1 and at P_2, respectively, with $h(a_1) = h(a_2) = 1$. Then $h(a_1 \cdot a_2) = 2$, while $h(ta_1 \cdot ta_2) \leq 1$, since ta_1 and ta_2 are each square at dyadic primes of E different from P'. This contradicts (6.1).

Claim 2. *T' is bijective.*

This follows from Claim 1 and (6.2).

Claim 3. *If $h(a) = n$ and if $P_1, \dots, P_n$ are the dyadic primes where a is a local non-square, then $T'(P_1) = P'_1, \dots, T'(P_n) = P'_n$ are the dyadic primes of E, where ta is a non-square.*

This is obvious when $n = 0$ or 1. So fix $n \geq 2$ and suppose that $h(a) = n = h(ta)$. By the approximation theorem choose a global element x sufficiently close to a at P_1 to be in the same square class as a in $\dot{F}_{P_1}$, and simultaneously close to 1 at all dyadic primes $P \neq P_1$, to be a square in each $\dot{F}_P$, $P \neq P_1$. Then $xa \in \dot{F}^2_{P_1}$ and $xa \notin \dot{F}^2_{P_i}$ for $i = 2, 3, \dots, n$, hence $h(x) = 1$ and $h(xa) = n - 1$. By induction, $t(xa)$ is not a square at $T'(P_2), \dots, T'(P_n)$ and tx is not a square at $T'(P_1)$. Hence $ta = t(xa) \cdot tx$ is not a square at $T'(P_1), \dots, T'(P_n)$ only, as desired.

Claim 4. *The pair (t, T') preserves Hilbert symbols.*

Fix global elements a, b in $\dot{F}$ and fix a dyadic prime P of F. We want to show

$$(a, b)_P = (ta, tb)_{P'},$$

where $P' = T'(P)$. We proceed by induction on $h(a) = h(ta)$.
If this is 0, then a and ta are local squares at P and P', so both symbols are 1. If $h(a) = 1$, then a is a non-square at exactly one $P_1 \in \Omega_2(F)$. If $P \neq P_1$, then $(a, b)_P = (ta, tb)_{P'} = 1$, and if $P = P_1$, then using almost reciprocity, symbols at non-dyadic primes are preserved, and so the Hilbert reciprocity shows that these symbols at P and P' agree. Proceeding inductively, we assume that for any dyadic P we have $(c, b)_P = (tc, tb)_{P'}$ whenever $h(c) < n$. Pick an x as above, namely x lies in the square class of a locally at a given dyadic prime,

and x is a local square at the remaining dyadics. Then write $a = ax \cdot x$ and observe that $h(ax) < n$ and $h(x) < n$. Thus

$$(a,b)_P = (ax,b)_P \cdot (x,b)_P = (t(ax),tb)_{P'} \cdot (tx,tb)_{P'} = (ta,tb)_{P'}$$

as required.

Proof of Theorem (1.1). Given a strong Witt ring isomorphism and the corresponding Harrison map t, we construct an almost reciprocity equivalence (t,T) as in the proof of Theorem (4.5). Then we extend T onto dyadic primes using the map T' defined above. Call the extended map again T'. Now it remains to match infinite complex primes in F and E. It will be sufficient to show that $2r_2(F) = 2r_2(E)$, i.e., the numbers of complex primes in F and E are equal, since Hilbert symbols at complex primes are trivial, hence preserved under arbitrary matching of complex primes.

Let P be a dyadic prime of F and $P' = T'(P)$, the corresponding dyadic prime of E. By Lemma (4.3), the dyadic completions have the same number of square classes (cf. Harrison's Criterion (1.2)). Thus

$$2^{n_P+2} = [\dot{F}_P : \dot{F}_P^2] = [\dot{E}_{P'} : \dot{E}_{P'}^2] = 2^{n_{P'}+2},$$

where n_P is the degree of F_P over the dyadic field $\mathbf{Q}_2$, and $n_{P'}$ is the degree of $E_{P'}$ over $\mathbf{Q}_2$ (cf. [7], Cor. 2.23, p. 162). It follows that $n_P = n_{P'}$ whenever $P' = T'(P)$. Thus we have

$$n := [F : \mathbf{Q}] = \sum_{P|2} n_P = \sum_{P'|2} n_{P'} = [E : \mathbf{Q}] =: m$$

and also

$$2r_2(F) = n - r_1(F) = m - r_1(E) = 2r_2(E),$$

as desired.

Extending T' arbitrarily onto infinite complex primes we obtain a map $T'' : \Omega(F) \to \Omega(E)$ such that (t,T'') is a reciprocity equivalence. This finishes the proof of Theorem (1.1).

7. Conclusion

An important corollary to our main result is that the Witt equivalence of number fields preserves the degrees of number fields over the rational field $\mathbf{Q}$ (combine (1.1) and (1.5)). We do not know any direct proof of this, avoiding the use of reciprocity equivalence. Thus, for instance, $\mathbf{Q}$ and $\mathbf{Q}(\sqrt[3]{2})$ are not Witt equivalent, but we do not know how to prove this without using our result on almost reciprocity equivalence (Theorem (4.5)).

On the positive side, it follows that $\mathbf{Q}$ has a unique Witt ring in the class of all algebraic number fields. As to quadratic number fields, A. Czogała [4], [5] classified them up to reciprocity equivalence. He showed that there are exactly 7 classes of reciprocity equivalent quadratic number fields represented by

$$\mathbf{Q}(\sqrt{d}), \quad \text{where } d = -1, \pm 2, \pm 7, \pm 17.$$

Combined with Theorem (1.1), this implies that there are exactly 7 Witt equivalence classes (i.e., 7 distinct Witt rings) for all quadratic fields, represented by the above 7 fields.

Some further results in this direction are known. J. Carpenter [3], combining her results with [11], proves that for any given n, there are only finitely many Witt equivalence classes of algebraic number fields of degree n. For cubic fields ($n = 3$), a complete classification with respect to Witt equivalence is given in [13]. There are exactly 8 Witt equivalence classes for $n = 3$ and the 8 cubic fields generated by zeros of the following polynomials

$$X^3 + pX + q, \quad \text{where} \quad (p,q) = (1,1), \quad (1,8), \quad (1,4), \quad (-1,8),$$
$$(-3,1), \; (-7,2), \; (-4,1), \; (-17,8)$$

are pairwise Witt inequivalent.
Moreover, the number of Witt equivalence classes of number fields of an arbitrary degree n is determined in terms of some partition functions.

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