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RANDOM SETS AND THEIR ASYMPTOTIC MEASURE

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ABSTRACT. The paper continues the investigation begun in [5] and [7] by presenting a probabilistic model for random sets suitable to handle the weak convergence of processes given by

$$\operatorname{Meas}[X_n \cap I(t)] - t \operatorname{Meas}(X_n), \qquad t \in [0,1],$$

where X_n are random sets and $I(t)$ is a non-random set valued process indexed by $[0,1]$. Moreover, sufficient conditions are found to ensure the asymptotic normality of r.v's

$$\operatorname{Meas}[X_n \cap Y_n] - \operatorname{Meas}[X_n] \cdot \operatorname{Meas}[Y_n],$$

where X_n and Y_n are independent random sets.

1. Introduction

We shall fix a complete probability space $(\Omega, \mathcal{A}, P)$ and a measure algebra $\mathbb{E} = \mathbb{E}(m)$ associated with a nonatomic countably generated probability space $(E, \mathcal{E}, m)$. We shall keep the notation $\mathbb{E}(\lambda)$ for the algebra generated by the Lebesgue interval $\big([0,1], \operatorname{Borel}[0,1], \lambda\big)$ and fix an isomorphism $i : \mathbb{E}(\lambda) \to \mathbb{E}(m)$ ([4, p. 173]). The algebra $\mathbb{E}(m)$ is assumed to be topologized as a subspace in $L_1(m)$. To avoid a complicated notation we agree not to distinguish a set $F \in \mathcal{E}$ from its equivalence class $[F] \in \mathbb{E}$. Similarly we shall keep the notation m even for the measure $[m]$ defined on $\mathbb{E}$ by $[m]([F]) = m(F)$.

A Borel measurable r.v. $X \colon \Omega \to \mathbb{E}$ will be called a *random set* (RS), its measure $m(X)$ the *size of* X.

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Further denote by $\mathbb{A} = \mathbb{A}(m)$ the group of all automorphisms acting on $\mathbb{E}(m)$ and topologize it by the coarsest topology that makes maps $a \mapsto aF$ from $\mathbb{A}$ into $\mathbb{E}$ to be continuous for all $F \in \mathbb{E}$. Recall ([3, p. 88–89]) that, endowed with such a topology, $\mathbb{A}$ is a complete separable topological group.

A Borel measurable r.v. $A\colon \Omega \to \mathbb{A}(m)$ will be called a *random automorphism* (RA), the net $\{I(t)\,,\ t \in [0,1]\} \subset \mathbb{E}(m)$ defined by $I(t) = i[0,t]$, *inspection of* $\mathbb{E}(m)$.

Consider, now, a random automorphism A and a r.v. $p\colon \Omega \to [0,1]$ to define a random set X with the size p by

$$X = AI\big(m(X)\big) \qquad \text{everywhere on} \quad \Omega\,. \tag{1}$$

The following theorem states the existence as such a representation generally (see also [6] and [7]).

THEOREM 1. *For any random set X there exists a X-measurable random automorphism A such that* (1) *holds.*

Proof. Define a map $F\colon \mathbb{A} \times [0,1] \to \mathbb{E}$ by $F(a,t) = aI(t)$ for $(a,t) \in \mathbb{A} \times [0,1]$. Such a map is obviously continuous and surjective by [3, p. 104]. By von Neumann measurable selection theorem ([8, p. 128–9]) there is a universally measurable map $S\colon \mathbb{E} \to \mathbb{A} \times [0,1]$ such that $F \circ S$ is the identity on $\mathbb{E}$. Hence, $S(X)$ is a Borel measurable r.v. (P is a complete measure!) with values in $\mathbb{A} \times [0,1]$, $A = \mathrm{pr}_A\, S(X)$ is then a X-measurable RA satisfying (1). □

Remark that the joint distribution of a pair A, $m(X)$ is far from being uniquely determined by the distribution of X in relation (1). Hence, we may choose the most suitable one to produce X by (1). If it happens that there is a solution A independent of $m(X)$ we shall say that X is an *orthogonal random set.* Let us agree that whenever we write (1) for an orthogonal set X we mean A and $m(X)$ to be independent. An orthogonal set given by (1) is said to be a *Haar random set* if $\mathcal{L}(A)$ is a Haar measure on a compact subgroup of $\mathbb{A}(m)$.

We shall make use of representation (1) in Section 3. Another obvious way how to produce a random set X is to consider a measurable $0-1$ valued stochastic process $\{Y(x)\,,\ x \in E\}$ and to put $X = \big[\{Y = 1\}\big]$. Again, any RS may be constructed in this way (see also [6], [7]).

THEOREM 2. *For any random set X there exists a $0-1$ valued, $\sigma X \times \mathcal{E}$-measurable process $\{Y(x),\ x \in E\}$ such that*

$$\{x \in E:\ Y(x) = 1\} \in X \qquad a.s. \quad [P]. \tag{2}$$

If Z is another process satisfying (2), *then*

$$Y(x) = Z(x) \quad a.s. \quad [P] \qquad for \quad x \in N,$$

where $N \in \mathcal{E}$ is a m-zero set.

Proof. The uniqueness part is obvious as $Y = Z$ a.e. $[P \times m]$. Put

$$R(D) = \int_\Omega m\big(X(\omega) \cap D_\omega\big)\ P(\mathrm{d}\omega), \qquad D \in \sigma X \times \mathcal{E}$$

and define a $\sigma X \times \mathcal{E}$-measurable process $M = \{M(x),\ x \in E\}$ by $\mathrm{d}R = M\,\mathrm{d}(m \times P)$. It follows from the separability of $\mathcal{E}$ that

$$P\left[\int_F M(x)\ m(\mathrm{d}x) = m(X \cap F), \quad F \in \mathcal{E}\right] = 1,$$

and therefore $\{M = 1\} \in X$ a.s. $[P]$.

Putting $Y = \mathrm{Indicator}_{\{0,1\}}(M)$ we obtain a process with properties (2). □

Having a random set X and a process Y constructed to X in Theorem 2, we denote

$$P[X \ni x_1, \dots, x_n] = P\big[Y(x_j) = 1,\ 1 \le j \le n\big], \\ (x_1, \dots, x_n) \in E^n, \quad n \in \mathbb{N},$$

and call it a *probability of inclusion of $x_1, \dots, x_n$ into X*.

If n is fixed, then all probabilities of inclusion

$$(x_1, \dots, x_n) \mapsto P[X \ni x_1, \dots, x_n]$$

considered as functions defined on E^n are equivalent w.r.t. the product measure m^n according to Theorem 2. Obviously

$$\int_F P[X \ni x_1, \dots, x_n]\ m^n(\mathrm{d}x_1, \dots, \mathrm{d}x_n) = Em^n(X^n \cap F) \tag{3}$$

holds for all $F \in \mathcal{E}^n$. Remark that $X^n = X \times \ldots \times X$ is a correctly defined random set with values in the measure algebra associated with $(E^n, \mathcal{E}^n, m^n)$.

In addition to random automorphisms and 0–1-measurable processes as tools when describing probability distributions of a random set there is one which makes it possible to study their asymptotic behaviour by means of continuous stochastic processes. If X is a RS, then a continuous stochastic process defined by $t \mapsto m(X \cap I(t))$ for $t \in [0,1]$ will be called an *inspection process* of the random set X.

As the map $F \mapsto \{m(F \cap I(t)),\ t \in [0,1]\}$ is a bicontinuous injection of $\mathbb{E}$ into $C[0,1]$ no loss would occur when studying RS' by means of their inspection processes as far as the weak convergence of probability distributions is concerned. Most of the results to be presented in Section 2 and 3 relate to sequences of random sets which have their points distributed "almost uniformly" over E in the sense that $\max\limits_{t \in [0,1]} \big(m(X_n \cap I(t)) - tm(X_n)\big) \to 0$ in probability or even

$$\sqrt{n}\big(m(X_n \cap I(t)) - tm(X_n)\big) \to \sigma W_0(t) \qquad \text{in distribution on} \quad C[0,1], \quad (4)$$

where W_0 is a Brownian bridge and σ a positive constant. Note that only trivial RS' with $X(\Omega) \subset \{\emptyset, E\}$ are allowed to have the inspection process with trajectories $t \mapsto m(X) \cdot t$. We shall say that random sets X_n are *asymptotically uniformly distributed* (AUD) if (4) holds for a $\sigma > 0$.

Example. Denote for a fixed $n \in \mathbb{N}$

$$I_{n,j} = I\Big(\frac{j}{n}\Big) - I\Big(\frac{j-1}{n}\Big), \qquad 1 \le j \le n. \qquad (5)$$

Consider (5) as a finite statistical population and perform a simple random sampling procedure on it with a fixed size $1 \le k \le n$ to get a sample, say, $\{I_{n,j_1}, \ldots, I_{n,j_k}\}$ and define an RS by

$$X_n = I_{n,j_1} \cup \cdots \cup I_{n,j_{r_k}}, \qquad \Big(m(X_n) = \frac{k}{n} = p\Big).$$

Such an RS will be referred to as a *random permutation* (of order n), the terminology is justified since X_n is in fact Haar RS associated with the finite group $\mathcal{P}_n \subset \mathbb{A}$ of all permutations of the intervals $I_{n,j}$ in (5).

Remark (see [6], [7]) that random permutations X_n have the AUD property with $\sigma^2 = p \cdot (1-p)$ provided that $m(X_n) \to p \in (0,1)$.

2. Asymptotically uniformly distributed RS'

Consider a sequence of random sets X_n. Searching for reasons that make random permutations obey the AUD property we easily arrive at the following conditions:

(6) $m(X_{n,j})$, $1 \le j \le n$ are exchangeable r.v.'s, when $X_{n,j} = X_n \cap I_{n,j}$,

(7) $\operatorname*{ess\,sup}_{(t,s) \in I^2_{n,1}} \left| P[X_n \ni t,s \mid X_n \ni s] - 1 \right| \to 0, \quad n \to \infty,$

(8) $\operatorname*{ess\,sup}_{(t,s,u,v) \in I^2_{n,1} \times I^2_{n,2}} \left| P[X_n \ni t,s \mid X_n \ni u,v] - P[X_n \ni t,s] \right| \to 0, \quad n \to \infty,$

where we have used notation (5) and the essential suprema in (7) and (8) are defined w.r.t. m^2 or m^4-measure, respectively. Let us agree to call such a sequence of RS' a *symmetric corrosion.*

THEOREM 3. *Let $\{X_n\}$ be a symmetric corrosion such that*

$$m(X_n) \to p \in (0,1) \qquad \textit{in probability.} \tag{9}$$

Then the random sets X_n are AUD with $\sigma^2 = p \cdot (1-p)$.

Note that (9) may be easily verified under the presence of corrosion properties (6) and (9), only.

LEMMA. *If (6) and (9) hold, then (9) is equivalent to the convergence $Em(X_n) \to p$.*

To see that, use successively (6), (3) and (8) to get

$$Em(X_n)^2 = n^2 \int\limits_{I_{n,1} \times I_{n,2}} P[X_n \ni x,y]\, m^2(\mathrm{d}x, \mathrm{d}y) + o(n^{-1}) = \big(Em(X_n)\big)^2 + o(1),$$

hence $E\big(m(X_n) - p\big)^2 = \big(Em(X_n) - p\big)^2 + o(1)$.

Proof of Theorem 3. Denote

$$U_n(t) = \sqrt{n}\big(m(X_n \cap I(t)) - tm(X_n)\big), \qquad t \in [0,1],$$
$$\xi_{n,k} = \sqrt{n}\big(m(X_{n,k}) - n^{-1} m(X_n)\big), \qquad 1 \le k \le n,$$
$$V_n(t) = \sum_{k=1}^{[nt]} \xi_{n,k}, \qquad t \in [0,1].$$

As $\max\limits_{t\in[0,1]} |U_n(t)-V_n(t)| = o(n^{-1/2})$, convergence (4) is equivalent to the convergence in distribution on Skorochod space $D[0,1]$ of the processes V_n to σW_0. Observing that $\xi_{n,k}$, $1 \le k \le n$ are exchangeable r.v.'s, $\sum\limits_{k=1}^{n} \xi_{n,k} = 0$ and $\max\limits_{1\le k\le n} |\xi_{n,k}| = o(n^{1/2})$, the convergence would be verified according to Theorem in [2, p. 212] if we could prove that

$$\sum_{k=1}^{n} \xi_{n,k}^2 \to p\cdot(1-p) \qquad \text{in probability.} \tag{10}$$

An easy calculation shows that

$$\sum_{k=1}^{n} \xi_{n,k}^2 = n\sum_{k=1}^{n} m(X_{n,k})^2 + B_n\,, \qquad B_n \to -p^2 \quad \text{in probability (by (9))}.$$

Hence, we are to prove that

$$A_n = n\sum_{k=1}^{n} m(X_{n,k})^2 \to p \qquad \text{in probability.} \tag{11}$$

It follows from (7) using (3) that

$$\begin{aligned} &|n^2Em(X_{n,1})^2 - nEm(X_{n,1})| \\ &\qquad = n^2\Big|\int\limits_{I_{n,1}^2} P[X_n \ni x,y] - P[X_n \ni x]\ m^2(\mathrm{d}x,\mathrm{d}y)\Big| = o(1) \end{aligned}$$

as $n\to\infty$. Therefore, by (6) and (9), we get

$$n^2Em(X_{n,1})^2 \to p\,. \tag{12}$$

Now, denote

$$V_n = n^4Em(X_{n,1})^2m(X_{n,2})^2\,, \qquad D_n = n^4\big(Em(X_{n,1})^2\big)^2\,.$$

Using (3) once more we calculate

$$V_n = n^4 \int\limits_{I_{n,1}^2\times I_{n,2}^2} P[X_n \ni x,y,u,v]\ m^4(\mathrm{d}x,\mathrm{d}y,\mathrm{d}u,\mathrm{d}v)\,,$$

$$D_n = n^4 \int\limits_{I_{n,1}^2\times I_{n,2}^2} P[X_n \ni x,y]P[X_n \ni u,v]\ m^4(\mathrm{d}x,\mathrm{d}y,\mathrm{d}u,\mathrm{d}v)\,.$$

Therefore

$$V_n \to p^2 \qquad \text{in probability} \tag{13}$$

by (8) and (12). Finally, it follows from (6) and (13) that

$$EA_n^2 = n^3 Em(X_{n,1})^4 + n(n-1)n^2 Em(X_{n,1})^2 \cdot Em(X_{n,2})^2 \to p^2 .$$

Thus, using (12) we get that $E(A_n - p)^2 \to 0$ which is equivalent to (11). □

Applying suitable continuous functionals to the processes U_n and W_0 we may get assertions of the following type.

COROLLARY 1. *Let $\{X_n\}$ be a symmetric corrosion satisfying* (9). *Then*

$$\sqrt{n} \max_{t\in[0,1]} \left| m\big(X_n \cap I(t)\big) - t \cdot m(X_n) \right| \to \sqrt{p\cdot(1-p)} \max_{t\in[0,1]} \left| W_0(t) \right|$$

and

$$\lambda\big\{t \in [0,1] : \; m\big(X_n \cap I(t)\big) > t \cdot m(X_n)\big\} \to U \qquad \textit{in distribution,}$$

where U is a r.v. uniformly distributed on $[0,1]$, (*see* [2, p. 85]).

A slightly more sophisticated argument has to be used to prove:

THEOREM 4. *Let X_n be AUD random sets and $f \in L_1(E, \mathcal{E}, m)$ such that*

(14) $G(f,t) = \int_{I(t)} f(x)\ m(\mathrm{d}x)$ *is a function with the second derivation continuous on* $[0,1]$.

Denote

$$U_n(f,t) = \sqrt{n}\left(\int_{X_n \cap I(t)} f(x)\ \mathrm{d}x - m(X_n) \int_{I(t)} f(x)\ \mathrm{d}x \right),$$

$$W_0(f,t) = \int_0^t G'(f,s)\ W_0(\mathrm{d}s) \qquad (\textit{Itô's integral})$$

for $t \in [0,1]$. *Then*

$$U_n(f,t) \to \sigma W_0(f,t) \qquad \textit{in distribution on } C[0,1]. \tag{15}$$

Let a preliminary remark precede the proof of Theorem 4. The isomorphism $i^{-1}\colon \mathbb{E}(m) \to \mathbb{E}(\lambda)$ is naturally extended to an isomorphism $S\colon L_1(m) \to L_1(\lambda)$ by

$$\int_F f \, \mathrm{d}m = \int_{i^{-1}F} S(f) \, \mathrm{d}\lambda, \qquad F \in \mathcal{E}, \quad f \in L_1(m). \tag{16}$$

Obviously

$$S(f \cdot g) = S(f) \cdot S(g) \tag{17}$$

holds for $f, g \in L_2(m)$.

P r o o f. First assume that $\mathbb{E}(m) = \mathbb{E}(\lambda)$, f has the first derivation continuous, $I(t) = [0,t]$.

Let Y_n be a process constructed to X_n in Theorem 2 and put

$$Z_n(x) = \sqrt{n}\big(Y_n(x) - \lambda(X_n)\big) \qquad \text{for} \quad x \in [0,1].$$

Obviously, denoting $U_n(t) = U_n(1,t)$,

$$U_n(f,t) = \int_0^t f(x) Z_n(x) \, \mathrm{d}x = \int_0^t f(x) \, U_n(\mathrm{d}x) = f(t) \cdot U_n(t) - \int_0^t U_n(x) f'(x) \, \mathrm{d}x$$

holds on $\Omega \times [0,1]$. Putting

$$H(u)(t) = f(t)u(t) - \int_0^t u f' \, \mathrm{d}x \qquad \text{for} \quad u \in C[0,1] \quad \text{and} \quad t \in [0,1],$$

we define a continuous functional $H\colon C[0,1] \to C[0,1]$ such that $U_n(f) = H(U_n)$. Hence $U_n(f) \to \sigma H(W_0)$ in distribution, which concludes the proof, as $H(W_0)$ is easily seen to be the diffusion $t \mapsto \int_0^t f(x) \, W_0(\mathrm{d}x)$.

Now, we shall handle the general formulation of our theorem. As $G'(f) \in S(f)$, where $S(f)$ is defined by (16), we have

$$U_n(f,t) = \sqrt{n} \left[\int_{i^{-1}X_n \cap I(t)} G'(f,u) \, \mathrm{d}u - \lambda(i^{-1}X_n) \cdot \int_0^t G'(f,u) \, \mathrm{d}u \right],$$

$$t \in [0,1].$$

According to the first part of the proof, $U_n(f) \to \sigma W_0(f)$ in distribution because $i^{-1}X_n$ are AUD random sets in $\mathbb{E}(\lambda)$. □

It is obvious enough that one may use the invariance principle stated by Theorem 4 to receive some information about an asymptotic behaviour of the statistics $\hat{I}_n(f) = m(X_n)^{-1} \cdot \int_{X_n} f \, dm$, which may be chosen as estimators for $I(f) = \int_E f \, dm$, when f is an unknown function observable on X_n, only. Denote $r^2(f) = I(f^2) - I(f)^2$ for $f \in L_2(m)$.

COROLLARY 2. *Let X_n's and f be such as in Theorem 4. Further assume that*

$$m(X_n) > 0 \quad \textit{almost surely,} \quad \sqrt{n} \cdot m(X_n) \to \infty \quad \textit{in probability,}$$
$$f \in L_2(m), \quad r^2(f) > 0 .$$

Then

$$\hat{I}_n(f) \to I(f) \qquad \textit{in probability,} \tag{18}$$

$$\frac{\sqrt{n} \cdot m(X_n)\big(\hat{I}_n(f) - I(f)\big)}{\sigma \cdot \hat{r}_n(f)} \to N(0,1) \qquad \textit{in distribution,} \tag{19}$$

where $\hat{r}_n^2(f) = \hat{I}_n(f^2) - \hat{I}_n^2(f)$.

Proof. Convergence (18) follows from Theorem 4 directly. Observe that $G'(f) \in S(f)$, G and S being defined by (14) and (17), respectively. Hence

$$\hat{I}_n(f^2) \to I(f^2) \qquad \text{in probability.} \tag{20}$$

Moreover, by Theorem 4 we get

$$\sqrt{n} \cdot m(X_n) \cdot \big(\hat{I}_n(f) - I(f)\big) \to N\big(0, r^2(f)\big) \qquad \text{in distribution}$$

because $\operatorname{Var} W_0(f,1) = r^2(f)$. Hence, (19) is a consequence of (18) and (20). □

3. Asymptotic measure of independent intersections

A natural question arises in connection with the AUD property of random sets. Is this property closed under independent intersections? The answer is positive when corrosions are concerned due to Theorem 3. We need of course a more strict symmetry property than proposed by (6). We shall say that $\{X_n\}$ is a *strongly symmetric corrosion* if (7), (8), and

$$\mathcal{L}(aX_n) = \mathcal{L}(X_n), \qquad a \in \mathcal{P}_n, \quad n \in \mathbb{N}, \tag{21}$$

hold. One needs only arguments to show that (21) implies (6).

THEOREM 5. *Let* $\{X_n\}$ *and* $\{Y_n\}$ *be i.i.d. strongly symmetric corrosion. Then* $Z_n = X_n \cap Y_n$ *is a strongly symmetric corrosion. If, moreover,*

$$P[X_n \ni x] \to p \qquad \text{in } m\text{-measure for some} \quad p \in (0,1), \tag{22}$$

then Z_n*'s are* AUD *sets with* $\sigma^2 = p^2(1-p^2)$.

Proof. Symmetry property (21) is fulfilled trivially for Z_n. According to Theorem 2 and relation (3) we have

$$\begin{gathered} P[Z_n \ni x] = P^2[X_n \ni x], \qquad P[Z_n \ni x,y] = P^2[X_n \ni x,y] \\ \text{a.e. } [m] \text{ or a.e. } [m^2], \text{ respectively.} \end{gathered} \tag{23}$$

Now, it follows from (22) that limiting procedures (7), (8) are satisfied, too. Thus, $\{Z_n\}$ is a strongly symmetric corrosion such that $P[Z_n \ni x] \to p^2$ in m-measure by (22) and (23). Finally, the lemma attached to Theorem 3 implies that $m(Z_n) \to p^2$ in probability and we arrive to (4) with $\sigma^2 = p^2(1-p^2)$ by Theorem 3. □

Remark. The following simple observation shows that assumption (22) cannot be removed when applying Theorem 3 to independent intersections: If X_n and Y_n are i.i.d. random sets enjoying properties (8) and (21) then

$m(X_n) \to p$, $m(X_n \cap Y_n) \to p^2$ in probability if and only if $P[X_n \ni x] \to p$ in m-measure.

The assertion follows readily using Lemma both for the sequences $\{X_n\}$ and $\{X_n \cap Y_n\}$, observing that

$$m(X_n \cap Y_n) - p^2 = \int_E |P[X_n \ni x] - Em(X_n)|^2 \, m(\mathrm{d}x) + o(1) \qquad \text{as } Em(X_n) \to p.$$

Note that Theorem 5 is of no value when searching for a limit of a sequence $\sqrt{n}\big(m(X_n \cap Y_n) - p^2\big)$, X_n, Y_n being i.i.d. random sets such that $m(X_n) \to p$. For these purposes we shall employ automorphism "technology" developed by Theorem 1.

THEOREM 6. *Let $X_n = T_n I(m(X_n))$ and $Y_n = A_n I(m(Y_n))$ be independent orthogonal random sets such that*

$$\mathcal{L}(A_n^{-1}) * \mathcal{L}(T_n) = \mathcal{L}(T_n). \tag{24}$$

If moreover X_n's are AUD random sets and $m(Y_n) \to q$ in probability for some $q \in [0,1]$, then

$$\sqrt{n}\big(m(X_n \cap Y_n) - m(X_n) \cdot m(Y_n)\big) \to \sigma W_0(q) \qquad \text{in distribution.} \tag{25}$$

P r o o f. As A_n, T_n, $m(X_n)$, $m(Y_n)$ are independent it follows by (24) that the left-hand side variable in (25) and

$$U_n(m(Y_n)) = \sqrt{n}\Big(m\big(X_n \cap I(m(Y_n))\big) - m(X_n) \cdot m(Y_n)\Big)$$

have an identical probability distribution (having denoted by $U_n(t)$ the process considered in (4)). The rest follows easily by the AUD property of the sequence $\{X_n\}$. □

Note that (24) is the condition that imposes the most serious restrictions when trying to apply the preceding theorem. It is a known fact ([2], [9], for instance) that equation (24) holds if and only if

$$\mathcal{L}(T_n) = H * P \quad \text{and} \quad \text{support}\, \mathcal{L}(A_n) \subset G,$$

where H is the Haar measure on a compact group $G \subset \mathbb{A}$ and P an arbitrary probability distribution on $\mathbb{A}$. It follows that if X_n, Y_n in Theorem 6 are also identically distributed, we are left with Haar random sets, only, to satisfy (24). The next statement is thus about the best we may gain via Theorem 6.

COROLLARY 3. *If X_n and Y_n are i.i.d. Haar random sets, such that $\{Y_n\}$ is a AUD sequence and $m(Y_n) \to p$ in probability, then*

$$\sqrt{n}\big(m(X_n \cap Y_n) - m(X_n) \cdot m(Y_n)\big) \to N\big(0, \sigma^2 p(1-p)\big).$$

The statement proposes some problems to be solved. To find a property that would make a sequence of Haar random sets to be AUD is one of them. The main obstacle when trying to prove the statement of Corollary 3 for more than two sequences stems from the fact that an intersection of i.i.d. Haar RS' need not be a Haar one. The difficulty does not arise in case of random permutations (see [5]).

A somewhat more sophisticated version of Theorem 6 is given below.

THEOREM 7. *Let X_n be AUD random sets such that*

$$\sqrt{n}\big(m(X_n) - p\big) \to 0 \qquad \text{in probability for some} \quad p \in [0,1]. \tag{26}$$

Then

$$D_n = \sqrt{n}\big(m(X_n \cap Y_n) - p \cdot q\big) \to \sigma W_0(q) + p \cdot Z \qquad \text{in distribution,} \tag{27}$$

where Z is r.v. independent of W_0 and $q \in [0,1]$, for any sequence of orthogonal random sets $Y_n = A_n I\big(m(Y_n)\big)$ such that

$$X_n \quad \text{and} \quad Y_n \quad \text{are independent,} \qquad \mathcal{L}(A_n^{-1} X_n) = \mathcal{L}(X_n), \tag{28}$$

and

$$Z_n = \sqrt{n}\big(m(Y_n) - q\big) \to Z \qquad \text{in distribution.} \tag{29}$$

Proof. Denoting $q_n = m(Y_n)$ it follows from (26) and the orthogonality of Y_n that D_n is distributed as $C_n = \sqrt{n}\big(m(X_n \cap I(q_n)) - p \cdot q\big)$. Obviously, we have $C_n = U_n(q_n) + p \cdot Z_n$, where U_n is defined by (4). As the process U_n and the variable Z_n are independent, as $U_n \to \sigma W_0$ by (26); (27) is a standard consequence of (29). □

Note that (28), which is a slightly relaxed from of (24), is equivalent to support $\mathcal{L}(A_n^{-1}) \subset G(X_n)$, where $G(X_n)$ is a closed subgroup in $\mathbb{A}$ defined by $G(X_n) = \{a \in A : \mathcal{L}(aX_n) = \mathcal{L}(X_n)\}$.

Hence, by Theorem 3 and Theorem 7 we get:

COROLLARY 4. *If $\{X_n\}$ is a strongly symmetric corrosion satisfying (26) and Y_n are random permutations of a fixed size $q \in [0,1]$, then*

$$\sqrt{n}\big(m(X_n \cap Y_n) - p \cdot q\big) \to N\big(0,\, p \cdot q(1-p)(1-q)\big) \qquad \text{in distribution.}$$

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