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Mathematica Slovaca, Vol. 35 (1985), No. 2, 127--130

Persistent URL: http://dml.cz/dmlcz/133182

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ON VARIOUS NOTIONS OF REGULARITY IN ORDERED SPACES

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The purpose of this note is to present the relationships between some properties in σ-complete lattice ordered groups. The properties we deal with enable various constructions of measures and integrals with values in ordered spaces. They all substitute the usual ε-technique and are referred to as the concepts of regularity although one of them is a kind of distributivity. The notions and symbols not defined here will be used in the sense of [1] or [3].

We begin by recalling definitions. Some of them are known as the properties of vector lattices but we can work with appropriate formulations applicable also in the lattice group case.

Let G be a σ-complete lattice group. Let (a_{nk}) be a double sequence of elements in G^+ such that $a_{nk}\downarrow_k 0$ for every $n\in N$.

G is said to have the strong Egoroff property (see [3]) if there exist in G^+ a sequence $b_m\downarrow 0$ and $\Phi\colon N\times N\to N$ such that, for every pair $(m, n)\in N\times N$, we have $b_m\geqq a_{n,\Phi(m,n)}$.

G is weakly regular (see [4]) if, for every $b\in G^+$, there exists $\varphi\colon N\to N$ such that for every $m\in N$ $b\nleqq\sum_{n=1}^{m} a_{n,\varphi(n)}$ holds.

Luxemburg and Zaanen in [3] give an exhaustive discussion about various formulations of the strong Egoroff property in Archimedean Riesz spaces. They showed there that this is the same as the notion of complete regularity introduced by H. Nakano and is also related to Kantorovich's regular Riesz spaces.

All other notions of regularity or distributivity can be formulated in terms of ordered bounded double sequences in the lattice group G.

Let a, $a_{nk}\in G^+$, n, $k\in N$ such that $a\geqq a_{nk}\downarrow_k 0$ for each $n\in N$.

G is called a g-regular (see [8]) if and only if

$$\bigwedge\left\{\sum_{n=1}^{\infty} a_{n,\varphi(n)}\,\middle|\,\varphi\colon N\to N\right\}=0.$$

G is said to have the Egoroff property if there exist a sequence (v_m) in G, $v_m\downarrow 0$ and $\Phi\colon N\times N\to N$ such that $v_m\geqq a_{n,\Phi(m,n)}$ holds for every $(m, n)\in N\times N$.

G is weakly σ-distributive (see [9]) if

$$\bigwedge\left\{\bigvee_{n=1}^{\infty} a_{n,\varphi(n)} \,\middle|\, \varphi\colon N \to N\right\} = 0.$$

These concepts were used and studied in [3, 6, 4, 3, 8, 9, 5], respectively. In some contexts they are sufficient conditions of the constructions in others necessary conditions as well. The following result details the relationships between the conditions mentioned above.

Proposition. *The following implications hold in the class of all σ-complete lattice groups.*

Strong Egoroff propery $\Rightarrow$ g-regularity $\Rightarrow$ Egoroff property $\Rightarrow$ weak σ-distributivity.
Every σ-complete lattice group is weak regular. No other implications hold, except those resulting from above.

Proof. 1. Strong Egoroff property implies g-regularity. Let a, $a_{nk} \in G$, $a \geqq a_{nk} \downarrow_k 0$ for all $n \in N$. By [1] theorem 4 there exist a complete vector lattice F and a lattice group isomorphism $h\colon G \to h(G) \subset F$ preserving all suprema and infima in F, i.e.

$$\text{if } x_0 = \bigvee x_\gamma \text{ in } G, \text{ then } h(x_0) = \bigvee h(x_\gamma) \text{ in } F.$$

It is clear that $(2^n a_{nk}) \downarrow_k 0$ for all $n \in N$ and by the assumption there exist (b_m) in G, $b_m \downarrow 0$ and $\Phi\colon N \times N \to N$ such that $b_m \geqq 2^n a_{n,\Phi(m,n)}$ for every (m, n). We have $h(b_m) \geqq 2^n h(a_{n,\Phi(m,n)})$ in F and

$$2^{-n} h(b_m) \geqq h(a_{n,\Phi(m,n)}) \text{ for every pair } (m, n).$$

Keeping m fixed for the moment, we have

$$h(b_m) = \sum_{n=1}^{\infty} 2^{-n} h(b_m) \geqq \sum_{n=1}^{\infty} h(a_{n,\Phi(m,n)}) = h\Big(\sum_{n=1}^{\infty} a_{n,\Phi(m,n)}\Big).$$

Since h is an isomorphism, $b_m \geqq \sum_{n=1}^{\infty} a_{n,\Phi(m,n)}$. Hence for all $m \in N$ we have $\varphi_m\colon N \to N$ such that $b_m \geqq \sum_{n=1}^{\infty} a_{n,\varphi_m(n)}$.

Since $b_m \downarrow 0$, we have $\bigwedge\left\{\sum_{n=1}^{\infty} a_{n,\varphi(n)} \,\middle|\, \varphi\colon N \to N\right\} = 0$.

2. A g-regular lattice group G has the Egoroff property. Let $a \geqq a_{nk} \downarrow_k 0$ $n = 1, 2, \ldots$. Setting $b_{nk} = \bigvee_{i=1}^{n} a_{ik}\, n$, $k = 1, 2, 3, \ldots$, we have $a \geqq b_{nk} \downarrow_k 0$ and so by the hypothesis there exist $\varphi\colon N \to N$ and $b \in G$ such that

$$b=\sum_{n=1}^{\infty} b_{n,\varphi(n)}=\bigvee_{m}\left(\sum_{n=1}^{m} b_{n,\varphi(n)}\right).$$

Let $c_m^{\varphi}=b-\sum_{n=1}^{m-1} b_{n,\varphi(n)}$ for every $m=2, 3, 4, \ldots$. It is evident that $c_m^{\varphi}\downarrow 0$ and $0\leqq b_{n,\varphi(n)}=c_n^{\varphi}-c_{n+1}^{\varphi}$. We have $0\leqq b_{n,\varphi(n)}+c_{n+1}^{\varphi}=c_n^{\varphi}$ and $0\leqq b_{n,\varphi(n)}\leqq c_n^{\varphi}\downarrow 0$. Now we can define $\Phi\colon N\times N\to N$ and (v_m), setting $v_m=c_m^{\varphi}$ for all $m\in N$ and

$$\Phi(m, n)=\begin{cases}\varphi(m) & \text{if } n\leqq m\\ \varphi(n) & \text{if } n>m.\end{cases}$$

Now $v_m\geqq a_{n,\Phi(m,n)}$ for all $n, m\in N$.

3. If G has the Egoroff property, then G is weakly σ-distributive.

Since for all m we can construct $\varphi^m\colon N\to N$ such that $b_m\geqq\bigvee_{i=1}^{\infty} a_{i,\varphi^m(i)}$, this part of the proposition is evident.

It is a suprising fact (see [7]) that Archimedean lattice groups are weakly regular. The following proof is included for the convience of the reader. We do not know how to avoid the method of representation of the Archimedean lattice group by a lattice of almost finite continuous functions on a Stone space. Again we turn attention to [1] theorem 4 and note that a vector lattice F consists of almost finite continuous functions on a compact, Hausdorff extremally disconnected space E. Since $a_{nk}\downarrow_k 0$ for every $n\in N$, $b\in G^+$ and $h\colon G\to h(G)\subset F$ is a lattice group isomorphism, we have $h(a_{nk})\downarrow_k 0$ in F and $A_n=\{x\in E\mid h(a_{nk})(x)\not\downarrow_k 0\}$ is a set of the first category (see [9]).

Since $h(b)>0$, there exist a clopen set U and an $\varepsilon>0$, ε real such that $(h(b))(x)>\varepsilon$ on U. By the Baire category theorem there exists $x_0\in U-\bigcup_{n=1}^{\infty} A_n$, and so $(h(a_{nk}))(x_0)\downarrow_k 0$ for every $n\in N$.

It is easy to define $\varphi\colon N\to N$ such that

$$\sum_{n=1}^{m} h(a_{n,\varphi(n)})(x_0)<\varepsilon \quad \text{for all } m\in N.$$

Since the ordering is pointwise, we have $\sum_{n=1}^{m} h(a_{n,\varphi(n)})\ngeqq h(b)$ for all $m\in N$. However, h is an isomorphism, hence we have $\sum_{n=1}^{m} a_{n,\varphi(n)}\ngeqq b$ for $m=1, 2, 3, \ldots$.

Counterexamples.

1. The space $L(X)$ of all real functions on a countable point set X, with pointwise ordering, has the strong Egoroff property ([3] § 68). Let us consider the subspace F of $L(X)$ consisting of all functions with only a finite number of nonzero coordinates. It is easy to see that F is σ-complete and g-regular and it is known that

this space fails to have the σ-property, thus it has not the strong Egoroff property (fot the relation between the σ-property and the strong Egoroff property see [3] theorem 70.2).

2. Let us consider the subspace m of $L(X)$ consisting of all bounded functions. m has the Egoroff property ([3] §67) but we can construct bounded double sequences $a_{nk} \downarrow_k 0$ in a such way that for each φ: $N \to N$ the sequence $\left(\sum_{n=1}^{k} a_{n,\varphi(n)}\right)_k$ is unbounded and so m fails to be g-regular.

3. Let $X = N^N$ and $L(X)$ the Riesz space of all real functions on X. It is shown in [3] §67 that $L(X)$ does not have the Egoroff property. $L(X)$ is Dedekind complete and suprema and infima of any bounded sets are ordinary pointwise. As space of the real numbers is a weak σ-distributive space, $L(X)$ is also one.

REFERENCES

[1] BERNAU S. J.: Uniwue representation of Archimedean lattice groups and normal Archimedean lattice rings. Proc. London Math. Soc. 3, 15, 1965, 599—631.

[2] FREMLIN D. H.: A direct proof of the Matthes-Wright integral extension theorem. J. London Math. Soc. 11, 1975, 276—284.

[3] LUXEMBURG W. A.—ZAANEN A. C.: Riesz Spaces I. North-Holland, Amsterdam 1971.

[4] RIEČAN B.: On the lattice group valued measures. Čas. pěst. mat. 101, 1976, 343—349.

[5] RIEČAN B.: On measures and integrals with values in ordered groups. Math. Slovaca 33, 1983, 2, 153—163.

[6] RIEČAN B.: On the extension of operators with values in linear semiordered spaces (Russian). Čas. pěst. mat. 93, 1968, 450—471.

[7] RIEČAN B.—VOLAUF P.: On a technical lemma in lattice ordered groups. Acta Math. Univ. Comen., to appear.

[8] VOLAUF P.: On extension of maps with values in ordered spaces. Math. Slovaca 30, 1980, 4, 351—361.

[9] WRIGHT J. D. M.: The measure extension problem for vector lattices. Ann. Inst. Fourier, Grenoble 21, 4, 1971, 65—85.

Received November 11, 1982

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РАЗНЫЕ ПОНЯТИЯ РЕТУЛЯРНОСТИ В ПОЛУУПОРЯДОЧЕННЫХ ПРОСТРАНСТВАХ

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Резюме

В работе устоновливается отношение между свойствами регулярности σ-полных l-групп. Свойства, которые изучаются, играют центральную роль в теории меры и интеграла со значениями в полуупорядоченных пространствах.