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# $r$–SYSTEMS OF UNARY ALGEBRAS I

## (On maximal and greatest $J$-classes of the direct product of unary algebras)

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ABSTRACT. The $J$-equivalence class in the direct product of an $r$-system of unary algebras is described by the $J$-equivalence classes of the separate components of the direct product. Also the greatest and the maximal $J$-classes in the direct product of an $r$-system of unary algebras are studied.

Various types of ideals can be considered in semigroups. There are minimal ideals, maximal ideals, prime ideals, completely prime ideals, and so on. In connection with this many authors studied the following problem: given a direct product of semigroups, which connection is between the ideals of this direct product and the ideals of its semigroup components? These problems are investigated in [1], [2], [10], [11], [12], [13], [14].

In this paper, we study a similar problem for unary algebras. In [3], the following theorem is proved:

Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra with proper subalgebras and $\emptyset \neq N \subset A$. Then $\langle N; F\rangle$ is a maximal subalgebra of $\mathbf{A}$ if and only if there exists a maximal $J$-class $[x]J$ of the partially ordered set $A/J$ of all $J$-classes of the algebra $\mathbf{A}$ such that $N = A \setminus [x]J$.

A similar theorem is valid for greatest subalgebras of unary algebras. From these results we can conclude that maximal (greatest) $J$-classes play a very important role in the description of maximal (greatest) subalgebras of a unary algebra. In the paper, we describe maximal (greatest) $J$-classes of a direct product of unary algebras. We make use of the concept of an $r$-system of unary algebras. From our results one can derive well-known results concerning maximal $\mathcal{L}$-classes of the direct product of semigroups.

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## 1. Introduction

We use the following notation. If $X$ is a subset of a set $Y$ and $X \neq Y$, then we write $X \subset Y$. $|X|$ stands for the cardinality of the set $X$. We denote the Cartesian product of sets $H_i$, $i \in I$, by $\prod(H_i \mid i \in I)$.

If $\sigma$ is an equivalence on a set $X$, then $X/\sigma$ is the set of all equivalence classes of the equivalence $\sigma$. An equivalence class of the equivalence $\sigma$ containing an element $x$ is denoted by $[x]\sigma$.

An algebra $\mathbf{A}$ is a pair $\langle A; F\rangle$, where $A$ is nonempty set and $F$ is a family of finitary operations on $A$. By $\mathcal{P}(\mathbf{A})$, we denote the set of all nonempty subsets $N$ of the set $A$ such that $\langle N; F\rangle$ is a subalgebra of the algebra $\mathbf{A}$. Let $x \in A$. We denote by $[x]$ the element of $\mathcal{P}(\mathbf{A})$ such that $\langle [x]; F\rangle$ is a subalgebra of the algebra $\mathbf{A}$ which is generated by $x$.

Let $\mathbf{A} = \langle A; F\rangle$ be an algebra. On the $A$, we define a binary relation $J$ in the following way: $xJy$ if and only if $[x] = [y]$. Evidently, this relation is reflexive, symmetric and transitive. Thus, it is an equivalence on $A$.

On the set $A/J$, we now define a binary relation $\leq$ in the following way: $[x]J \leq [y]J$ if and only if $[x] \subseteqq [y]$. Obviously, this relation is reflexive, antisymmetric and transitive. Thus, it is a partial order on the $A/J$, and $(A/J; \leq)$ is a partially ordered set. We briefly denote this set by $A/J$.

We concentrate only on unary algebras. A unary algebra is the pair $\mathbf{A} = \langle A; F\rangle$, where $F$ is a family of unary operations on the set $A$, i.e. $f\colon A \to A$ is a mapping of the set $A$ into $A$ for each $f \in F$.

Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra and $x \in A$. Let $\mathbf{N} = \langle N; F\rangle$ be a subalgebra of $\mathbf{A}$ such that $x \in N$. Then, for any $f_1 \in F$ we have $f_1(x) \in N$, and again, for any $f_2 \in F$, $f_2(f_1(x)) \in N$. Inductively, for any $f_1, f_2, \dots, f_k \in F$ we have $f_k(\dots f_2(f_1(x))\dots) \in N$.

**DEFINITION 1.1.** *Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra and $x \in A$. We define $F^+(x)$ to be the set of all $y \in A$ with the following property:*

*There exist $f_1, \dots, f_k \in F$ such that $y = f_k(\dots f_2(f_1(x))\dots)$. Thus,*

$$F^+(x) = \bigl\{y \in A \mid \text{there exist } f_1, \dots, f_k \in F \text{ such that } y = f_k(\dots f_2(f_1(x))\dots)\bigr\}.$$

Via this notation, we have for any $N \in \mathcal{P}(\mathbf{A})$ and $x \in N$: $F^+(x) \subseteqq N$. It is a matter of routine to check that $\langle F^+(x); F\rangle$ is a subalgebra of the $\mathbf{A}$. Also $\{x\} \cup F^+(x) \in \mathcal{P}(\mathbf{A})$, and for any $N \in \mathcal{P}(\mathbf{A})$, $x \in N$ we have $\{x\} \cup F^+(x) \subseteqq N$. Thus we proved the following lemma.

**LEMMA 1.1.** *Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra and $x \in A$. Then $[x] = \{x\} \cup F^+(x)$.*

The next lemma shows a relationship between the ordering of $J$-classes $[x]J$, $[y]J$ for $x \neq y$, and the property $x \in F^+(y)$.

**LEMMA 1.2.** *Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra and $x, y \in A$, $x \neq y$. Then $[x]J \leq [y]J$ if and only if $x \in F^+(y)$.*

Proof.

a) Let $[x]J \leq [y]J$. Thus $[x] \subseteqq [y]$, i.e. $\{x\} \cup F^+(x) \subseteqq \{y\} \cup F^+(y)$. Now $x \neq y$ implies $x \in F^+(y)$.

b) Let $x \in F^+(y)$. With respect to $F^+(y) \in \mathcal{P}(\mathbf{A})$, we have $[x] \subseteqq F^+(y) \subseteqq \{y\} \cup F^+(y) = [y]$ and $[x]J \leq [y]J$. □

In what follows, we often pay attention to the fact whether $x$ belongs or not to $F^+(x)$. The following lemma describes the difference between these two possibilities.

**LEMMA 1.3.** *Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra. Then for any $x, y \in A$,*

a) *if $|[x]J| \geq 2$, then $x \in F^+(y)$ for any $y \in [x]J$;*
b) *$x \in F^+(y)$ for any $y \in [x]J$ if and only if $x \in F^+(x)$;*
c) *if $x \notin F^+(x)$, then $|[x]J| = 1$.*

Proof.

a) Since $|[x]J| \geq 2$, there exists $y \in [x]J$ such that $y \neq x$. For any $y \in [x]J$ we have $[x] = [y]$. By Lemma 1.1, $\{x\} \cup F^+(x) = \{y\} \cup F^+(y)$. As $x \neq y$, we get $x \in F^+(y)$ and $y \in F^+(x)$. Part a) is proved in the case $x \neq y$. The rest we obtain under the following consideration. As $F^+(x) \in \mathcal{P}(\mathbf{A})$, $y \in F^+(x)$ implies $[y] \subseteqq F^+(x)$. Hence $x \in F^+(y) \subseteqq \{y\} \cup F^+(y) = [y] \subseteqq F^+(x)$. Thus, $x \in F^+(y)$ for every $y \in [x]J$.

b) If $x \in F^+(y)$ for any $y \in [x]J$, then also $x \in F^+(x)$. Conversely, let $x \in F^+(x)$. If there exists $y \in [x]J$, $y \neq x$, then from a) we have $x \in F^+(y)$. Thus, $x \in F^+(y)$ for any $y \in [x]J$.

The part c) follows directly from a) and b). □

## 2. $r$-Systems

In this section, we concentrate on a direct product of an $r$-system of unary algebras. First we give some definitions.

**DEFINITION 2.1.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be a system of unary algebras of the same type, $|I| \geq 2$. Set $A = \prod(A_i \mid i \in I)$ to be the Cartesian product of the sets $A_i$. For every $f \in F$ we define a unary operation on $A$ in the following way: $f(\alpha)(i) = f(\alpha(i))$ for any $i \in I$ and any $\alpha \in A$. Then $\langle A; F\rangle =$*

*$\langle \prod(A_i \mid i \in I); F \rangle$ is a unary algebra. This unary algebra will be called a direct product of $\mathbf{A}_i$, $i \in I$, and be denoted by $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$.* (See, for example, [7].)

Now our considerations will concern $J$-classes in the direct product of unary algebras.

**LEMMA 2.1.** *Let $\{\mathbf{A}_i = \langle A_i; F \rangle \mid i \in I\}$, $|I| \geq 2$, be a family of unary algebras of the same type. Let $\alpha, \beta \in A = \prod(A_i \mid i \in I)$, and $\alpha \in F^+(\beta)$. Then $\alpha(i) \in F^+(\beta(i))$ for every $i \in I$.*

P r o o f. By our assumption, there exist $f_1, \dots, f_k \in F$ such that $f_k(\dots f_2(f_1(\beta))\dots) = \alpha$. Then for any $i \in I$ we have $\alpha(i) = f_k(\dots f_2(f_1(\beta))\dots)(i) = f_k(\dots f_2(f_1(\beta(i)))\dots)$. Hence $\alpha(i) \in F^+(\beta(i))$ for any $i \in I$. □

We give an example which shows that the converse need not be true, i.e. $\alpha(i) \in F^+(\beta(i))$ for every $i \in I$ does not imply $\alpha \in F^+(\beta)$.

E x a m p l e 1. Let $A_1 = \{0, a, b\}$ and $f_0$, $f_1$ be unary operations defined on $A_1$ by the following tables:

| $f_0$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| | $0$ | $0$ | $0$ |

| $f_1$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| | $0$ | $b$ | $a$ |

Put $F = \{f_0, f_1\}$. Then $\mathbf{A}_1 = \langle A_1; F \rangle$ is a unary algebra. Let $A = A_1 \times A_1$, $\beta = (b, a) \in A_1 \times A_1$, $\alpha = (a, a) \in A_1 \times A_1$. We get $a = f_1(b)$, $a = f_1(f_1(a))$, $f_1(b, a) = (f_1(b), f_1(a)) = (a, b)$, $f_1(a, b) = (b, a)$, $f_0(b, a) = f_0(a, b) = f_0(0, 0) = (0, 0) = f_1(0, 0)$. So $a \in F^+(b)$, $a \in F^+(a)$, but $\alpha = (a, a) \notin F^+((b, a)) = F^+(\beta)$.

**DEFINITION 2.2.** *Let $\{\mathbf{A}_i = \langle A_i; F \rangle \mid i \in I\}$, $|I| \geq 2$, be a family of unary algebras of the same type. Suppose the direct product $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$ has the following property:*

> *If $\alpha, \beta \in A = \prod(A_i \mid i \in I)$ and $\alpha(i) \in F^+(\beta(i))$ for every $i \in I$, then $\alpha \in F^+(\beta)$.*

*In this case, the family $\{\mathbf{A}_i \mid i \in I\}$ will be called an $r$-system of unary algebras.*

The next lemma is a direct consequence of Lemma 2.1 and Definition 2.2.

**LEMMA 2.2.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras, and $\alpha, \beta \in A = \prod(A_i \mid i \in I)$. Then $\alpha \in F^+(\beta)$ if and only if $\alpha(i) \in F^+(\beta(i))$ for any $i \in I$.*

Remark 1. Let $S$ be a semigroup and $H$, $T$ be subsemigroups of $S$. Then a nonempty subsetset $A$ of $S$ such that $HA \subseteqq A$ ( $HA \subseteqq A$ and $AT \subseteqq A$ ) will be called $(H, \emptyset)$-ideal ( $(H, T)$-ideal) of $S$. In the case $H = S = T$, we have a left ideal (an ideal) of the semigroup $S$.

Now we give two examples. In the first one, we show a possibility how to assign an $r$-system of unary algebras to any systems of semigroups. This assignment induces a one-to-one correspondence between subalgebras of the unary algebras and left ideals ( $(H, \emptyset)$-ideals) of the single semigroups.

In the second example, any system of monoids is associated with an $r$-system of unary algebras such that to any subalgebra of the unary algebras an ideal ( $(H, T)$-ideal) of the monoids is assigned and vice versa.

Example 2. Let $|I| \geq 2$, $\mathbf{S}_i = \langle S_i; \cdot\rangle$ be a semigroup and let $\mathbf{H}_i = \langle H_i; \cdot\rangle$ be a subsemigroup of the semigroup $\mathbf{S}_i$ for any $i \in I$. Set $S = \prod(S_i \mid i \in I)$ and $H = \prod(H_i \mid i \in I)$. Define a binary operation on $S$ by $(\alpha \cdot \beta)(i) = \alpha(i) \cdot \beta(i)$ for any $\alpha, \beta \in S$ and every $i \in I$. Then the direct product $\mathbf{S} = \prod(\mathbf{S}_i \mid i \in I)$ is the semigroup $\langle S; \cdot\rangle$. Evidently, $\mathbf{H} = \prod(\mathbf{H}_i \mid i \in I)$ is a subsemigroup of the semigroup $\mathbf{S}$.

Now for any $\mathcal{H} \in H = \prod(H_i \mid i \in I)$ and every $i \in I$ we define a unary operation $f_{\mathcal{H}}$ on $S_i$ in the following way: $f_{\mathcal{H}}(y) = \mathcal{H}(i) \cdot y$ for every $y \in S_i$. Let $F_H = \{f_{\mathcal{H}} \mid \mathcal{H} \in H\}$. Then $\mathcal{S}_{(i,H)} = \langle S_i; F_H\rangle$ is a unary algebra for every $i \in I$, and all the unary algebras are of the same type.

Now, for any $A \subseteqq S_i$, $A \in \mathcal{P}(\mathcal{S}_{(i,H)})$ if and only if, for any $y \in A$ and $f_{\mathcal{H}} \in F_H$, $f_{\mathcal{H}}(y) \in A$. Hence $f_{\mathcal{H}}(y) = \mathcal{H}(i) \cdot y \in A$ for any $\mathcal{H} \in H$, and $y \in A$. However, for every $\mathcal{H} \in H$, $\mathcal{H}(i) \in H_i$. On the contrary, for every $h_i \in H_i$ there exists $\mathcal{H} \in H$ such that $h_i = \mathcal{H}(i)$. Therefore, $f_{\mathcal{H}}(y) \in A$ for any $f_{\mathcal{H}} \in F_H$ and every $y \in A$ if and only if $h_i \cdot y \in A$ for any $h_i \in H_i$ and every $y \in A$. So we have $A \in \mathcal{P}(\mathcal{S}_{(i,H)})$ if and only if $H_i \cdot A \subseteqq A$, i.e. $A$ is $(H_i, \emptyset)$-ideal of the semigroup $\mathbf{S}_i$. Moreover, in the case $H_i = S_i$, there exists a one-to one correspondence between subalgebras generated by one element and left ideals generated by the same element.

Now we show that a family $\{\mathcal{S}_{(i,H)} \mid i \in I\}$ is an $r$-system of unary algebras. Let $\alpha$, $\beta$ be arbitrary elements of $S = \prod(S_i \mid i \in I)$, and $\alpha(i) \in F_H(\beta(i))$ for any $i \in I$. On that account, for any $i \in I$ we can choose $f_{\alpha_1}, \dots, f_{\alpha_k} \in F_H$ such that $\alpha(i) = f_{\alpha_k}(\dots f_{\alpha_2}(f_{\alpha_1}(\beta(i)))\dots) = (\alpha_k(i) \cdot \dots \cdot \alpha_2(i) \cdot \alpha_1(i)) \cdot \beta(i)$. Put $z_i = (\alpha_k(i) \cdot \dots \cdot \alpha_2(i) \cdot \alpha_1(i))$. Obviously, $z_i \in H_i$ for any $i \in I$. Let $\tau \in H$

be such that $\tau(i) = z_i$ for every $i \in I$. Then $\alpha(i) = z_i \cdot \beta(i) = \tau(i) \cdot \beta(i) = f_\tau(\beta(i))$ for any $i \in I$. Consequently, $\alpha = \tau \cdot \beta = f_\tau(\beta)$ and $\alpha \in F_H(\beta)$. Thus, $\{\mathcal{S}_{(i,H)} \mid i \in I\}$ is an $r$-system of unary algebras. □

Example 3. Let $\mathbf{S}_i = \langle S_i; \cdot; 1 \rangle$ be a monoid, $\mathbf{H}_i$, $\mathbf{T}_i$ be submonoids of $\mathbf{S}_i$ for every $i \in I$, $|I| \geq 2$. Let $\mathbf{S} = \prod(\mathbf{S}_i \mid i \in I)$, $\mathbf{H} = \prod(\mathbf{H}_i \mid i \in I)$, $\mathbf{T} = \prod(\mathbf{T}_i \mid i \in I)$ be direct products of monoids. Let $S = \prod(S_i \mid i \in I)$, $H = \prod(H_i \mid i \in I)$, $T = \prod(T_i \mid i \in I)$ be Cartesian products.

For any $\mathcal{H} \in H$ and each $i \in I$, we define a unary operation $f_\mathcal{H}$ on the set $S_i$ in the following way: $f_\mathcal{H}(y) = \mathcal{H}(i) \cdot y$ for every $y \in S_i$. Similarly, for any $\tau \in T$, $i \in I$ we define on $S_i$ a unary operation $f_\tau$ by $f_\tau(y) = y \cdot \tau(i)$ for every $y \in S_i$. Let $F_H = \{f_\mathcal{H} \mid \mathcal{H} \in H\}$, $F_T = \{f_\tau \mid \tau \in T\}$. Then $\mathcal{S}_{(i,H,T)} = \langle S_i; F_H \cup F_T \rangle$ is a unary algebra for each $i \in I$. By the same way as in Example 2, to any subalgebra of this unary algebra is assigned $(H_i, T_i)$-ideal of $\mathbf{S}_i$ and vice versa. Moreover, in the case $H_i = S_i = T_i$, there exists a one-to-one correspondence between subalgebras generated by one element and ideals generated by the same element.

Unary algebras $\mathcal{S}_{(i,H,T)} = \langle S_i; F_H \cup F_T \rangle$ are of the same type. By the same token as in Example 2, we could show that $\{\mathcal{S}_{(i,H,T)} \mid i \in I\}$ is an $r$-system of unary algebras. □

Now we concentrate on a relationship of $J$-classes in a direct product of unary algebras and $J$-classes of components of this direct product.

**LEMMA 2.3.** *Let $\{\mathbf{A}_i = \langle A_i; F \rangle \mid i \in I\}$ be an arbitrary family of unary algebras, and $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$ be a direct product of these algebras. Let $\alpha$, $\beta$ be arbitrary elements of $A = \prod(A_i \mid i \in I)$. If $[\alpha]J \leq [\beta]J$, then $[\alpha(i)]J \leq [\beta(i)]J$ for any $i \in I$.*

Proof. Clearly, $[\alpha]J \leq [\beta]J$ implies $[\alpha] \subseteqq [\beta]$ and, consequently, $\alpha \in [\beta]$. Then either $\alpha = \beta$, or there exist $f_1, \dots, f_k \in F$ such that $\alpha = f_k(\dots f_2(f_1(\beta))\dots)$. Thus, for any $i \in I$, either $\alpha(i) = \beta(i)$ or $\alpha(i) = f_k(\dots f_2(f_1(\beta(i))\dots)$. In any case, for every $i \in I$ there holds $\alpha(i) \in [\beta(i)]$, and so $[\alpha(i)] \subseteqq [\beta(i)]$, i.e. $[\alpha(i)]J \leq [\beta(i)]J$ for any $i \in I$. □

If we consider an $r$-system of unary algebras, we prove, in a certain sense, the converse of Lemma 2.3.

**LEMMA 2.4.** *Let* $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ *be an* $r$*-system of unary algebras, and* $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. *Let* $\alpha \in A = \prod(A_i \mid i \in I)$, $\alpha \in F^+(\alpha)$. *Then for any* $\beta \in A$

a) $[\alpha]J \le [\beta]J$ *if and only if* $[\alpha(i)]J \le [\beta(i)]J$ *for each* $i \in I$,

b) $[\beta]J \le [\alpha]J$ *if and only if* $[\beta(i)]J \le [\alpha(i)]J$ *for every* $i \in I$.

P r o o f. By Lemma 2.3, we have only to prove that $[\alpha(i)]J \le [\beta(i)]J$ ($[\beta(i)]J \le [\alpha(i)]J$) for every $i \in I$ implies $[\alpha]J \le [\beta]J$ ($[\beta]J \le [\alpha]J$).

Let $\alpha, \beta \in A$, $\alpha \in F^+(\alpha)$ and $[\alpha(i)]J \le [\beta(i)]J$ ($[\beta(i)]J \le [\alpha(i)]J$) for any $i \in I$. From $\alpha \in F^+(\alpha)$, we have $\alpha(i) \in F^+(\alpha(i))$ for each $i \in I$, and, by Lemma 1.2, for any $i \in I$, either $\alpha(i) = \beta(i)$ or $\alpha(i) \in F^+(\beta(i))$ ($\beta(i) \in F^+(\alpha(i))$). So, for any $i \in I$ we have either $\alpha(i) = \beta(i) \in F^+(\alpha(i)) = F^+(\beta(i))$ or $\alpha(i) \in F^+(\beta(i))$ ($\beta(i) \in F^+(\alpha(i))$). It implies $\alpha(i) \in F^+(\beta(i))$ ($\beta(i) \in F^+(\alpha(i))$) for any $i \in I$. Then, by the definition of an $r$-system, $\alpha \in F^+(\beta)$ ($\beta \in F^+(\alpha)$). Hence, either $\alpha = \beta$, or $\alpha \ne \beta$, which implies (by Lemma 1.2) $[\alpha]J \le [\beta]J$ ($[\beta]J \le [\alpha]J$). □

The following is a direct consequence of Lemma 2.4.

**COROLLARY 2.1.** *Let* $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ *be an* $r$*-system of unary algebras,* $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. *Let* $\alpha \in A = \prod(A_i \mid i \in I)$ *and* $\alpha \in F^+(\alpha)$. *Then, for any* $\beta \in A$, $[\alpha]J = [\beta]J$ *if and only* $[\alpha(i)]J = [\beta(i)]J$ *for any* $i \in I$.

Now we are able to state the main result of this part.

**THEOREM 2.1.** *Let* $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ *be an* $r$*-system of unary algebras,* $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. *Let* $\alpha \in A = \prod(A_i \mid i \in I)$. *Then,*

a) *if* $\alpha \in F^+(\alpha)$, *then* $[\alpha]J = \prod([\alpha(i)]J \mid i \in I)$;

b) *if* $\alpha \notin F^+(\alpha)$, *then* $[\alpha]J = \{\alpha\}$.

P r o o f.

a) Suppose that $\alpha \in F^+(\alpha)$. Then $\beta \in [\alpha]J$ if and only if $[\beta]J = [\alpha]J$. By Corollary 2.1, $[\beta]J = [\alpha]J$ if and only if $[\beta(i)]J = [\alpha(i)]J$ for any $i \in I$. The last equality holds if and only if $\beta(i) \in [\alpha(i)]J$ for any $i \in I$. However, $\beta(i) \in [\alpha(i)]J$ for any $i \in I$ if and only if $\beta \in \prod([\alpha(i)]J \mid i \in I)$. Thus, $\beta \in [\alpha]J$ if and only if $\beta \in \prod([\alpha(i)]J \mid i \in I)$. This proves part a).

Part b) is a direct consequence of Lemma 1.3 c). □

Now we state a result in the case $\alpha \notin F^+(\alpha)$. In an $r$-system, there exists $i \in I$ such that $\alpha(i) \notin F^+(\alpha(i))$. Put $I_1 = \{i \in I \mid \alpha(i) \notin F^+(\alpha(i))\}$. Obviously, $I_1 \ne \emptyset$.

**THEOREM 2.2.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras, $\alpha \in A = \prod(A_i \mid i \in I)$, and $\alpha \notin F^+(\alpha)$. Let $I_1 = \{i \in I \mid \alpha(i) \notin F^+(\alpha(i))\}$, and $\beta$ be an element of $A$ such that there exists $j \in I_1$ with $\beta(j) = \alpha(j)$. Then*

a) $[\beta]J = \{\beta\}$,

b) *if $\beta \neq \alpha$, then $J$-classes $[\alpha]J$ and $[\beta]J$ are incomparable.*

Proof.

a) Clearly, $\beta(j) = \alpha(j) \notin F^+(\alpha(j)) = F^+(\beta(j))$. In an $r$-system of unary algebras, $\beta \notin F^+(\beta)$. By Lemma 1.3 c), $|[\beta]J| = 1$, and the first part of our theorem is proved.

b) Let $\beta \neq \alpha$, $\beta(j) = \alpha(j) \notin F^+(\alpha(j)) = F^+(\beta(j))$. Obviously, $\{\alpha\} = [\alpha]J \neq [\beta]J = \{\beta\}$. Assume that $[\alpha]J \leq [\beta]J$. By Lemma 1.2, $\alpha \in F^+(\beta)$. But it is true for an $r$-system if and only if $\alpha(i) \in F^+(\beta(i))$ for any $i \in I$. This is a contradiction to our assumption $\alpha(j) \notin F^+(\beta(j))$. By similar reasoning, we obtain a contradiction also in the case $[\beta]J \leq [\alpha]J$. Hence classes $[\alpha]J$ and $[\beta]J$ are incomparable. □

## 3. Main results

This part of the paper is devoted to maximal and greatest $J$-classes of the partially ordered set of all $J$-classes of a direct product of unary algebras.

**THEOREM 3.1.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras, and $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. Let $\alpha \in A = \prod(A_i \mid i \in I)$ be such that $\alpha \in F^+(\alpha)$. Then $[\alpha]J$ is a maximal element in $A/J$ if and only if $[\alpha(i)]J$ is a maximal element in $A_i/J$ for every $i \in I$.*

Proof.

a) $[\alpha]J$ is a maximal element of $A/J$ if and only if for any $\beta \in A$ the condition $[\alpha]J \leq [\beta]J$ implies $[\alpha]J = [\beta]J$. Let $j \in I$, and $\beta_j \in A_j$ be an arbitrary but fixed element. Let $[\alpha(j)]J \leq [\beta_j]J$. Let $\beta \in A$ be a fixed element such that $\beta(i) = \alpha(i)$ for each $i \in I$, $i \neq j$, and $\beta(j) = \beta_j$. Then $[\alpha(i)]J \leq [\beta(i)]J$ for any $i \in I$, and, by Lemma 2.4, $[\alpha]J \leq [\beta]J$. Assuming the maximality of an element $[\alpha]J$ we get $[\alpha]J = [\beta]J$. By Corollary 2.1, $[\alpha(i)]J = [\beta(i)]J$ for each $i \in I$, hence $[\alpha(j)]J = [\beta_j]J$. Thus $[\alpha(j)]J$ is a maximal element of $A_j/J$.

b) Let $\alpha \in A$, and $[\alpha(i)]J$ be a maximal element of $A_i/J$ for any $i \in I$. Let $\beta \in A$ and $[\alpha]J \leq [\beta]J$. By Lemma 2.3, $[\alpha(i)]J \leq [\beta(i)]J$ for any $i \in I$. Assuming the maximality of classes $[\alpha(i)]J$ we get $[\alpha(i)]J = [\beta(i)]J$ for each

$i \in I$. Further, by Corollary 2.1, $[\alpha]J = [\beta]J$. Thus $[\alpha]J$ is a maximal element of $A/J$. □

**LEMMA 3.1.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. Let $\alpha \in A = \prod(A_i \mid i \in I)$ be such that there exists $j_0 \in I$ with the property: $\alpha(j_0) \notin F^+(\alpha(j_0))$, and $[\alpha(j_0)]J$ is a maximal element in the set $A_{j_0}/J$. Then, for arbitrary $\beta \in A$ with $\beta(j_0) = \alpha(j_0)$, the class $[\beta]J = \{\beta\}$ is a maximal element of the set $A/J$.*

P r o o f. Let $\beta \in A$ be such that $\beta(j_0) = \alpha(j_0) \notin F^+(\alpha(j_0)) = F^+(\beta(j_0))$. As we consider an $r$-system, $\beta \notin F^+(\beta)$ and $[\beta]J = \{\beta\}$. Let us suppose that there exists $\tau \in A$, $\tau \neq \beta$, with $[\beta]J \leq [\tau]J$. Then $[\beta(i)]J \leq [\tau(i)]J$ for any $i \in I$. Since $\beta(j_0) = \alpha(j_0)$ and $[\alpha(j_0)]J$ is a maximal element of $A_{j_0}/J$, $[\alpha(j_0)]J = [\tau(j_0)]J$. Since $\alpha(j_0) \notin F^+(\alpha(j_0))$ and $[\alpha(j_0)]J = \{\alpha(j_0)\}$, we have $\alpha(j_0) = \beta(j_0) = \tau(j_0)$. Now, by Theorem 2.2 b) and the assumption $\tau \neq \beta$, classes $[\beta]J$ and $[\tau]J$ are incomparable. Thus $[\beta]J$ is a maximal element of $A/J$. □

Now we describe maximal $J$-classes of a direct product of an $r$-system of unary algebras in the case $\alpha \notin F^+(\alpha)$.

**THEOREM 3.2.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. Let $\alpha \in A = \prod(A_i \mid i \in I)$ be an element such that $\alpha \notin F^+(\alpha)$. Then $[\alpha]J$ is a maximal element of $A/J$ if and only if there exists at least one $j \in I$ such that $\alpha(j) \notin F^+(\alpha(j))$, and $[\alpha(j)]J$ is a maximal element of $A_j/J$.*

P r o o f.

a) Let us suppose $\alpha \notin F^+(\alpha)$, and $[\alpha]J$ is a maximal element of $A/J$. As we consider an $r$-system, there is $j \in I$ such that $\alpha(j) \notin F^+(\alpha(j))$. We denote by $I_1$ the set of all $j \in I$ with the property $\alpha(j) \notin F^+(\alpha(j))$. Obviously, $I_1 \neq \emptyset$. For any $j \in I_1$, by Lemma 1.3 c), we get $[\alpha(j)]J = \{\alpha(j)\}$. Now we show an existence of an element $j \in I_1$ such that $[\alpha(j)]J$ is a maximal element of $A_j/J$.

Let us suppose to the contrary that for any $j \in I_1$ there exists $\beta_j \in A$ such that $\alpha(j) \neq \beta_j$ and $[\alpha(j)]J \leq [\beta_j]J$. Let $\beta \in A$ be such that $\beta(j) = \beta_j$ for any $j \in I_1$, and $\beta(i) = \alpha(i)$ for any $i \in I \setminus I_1$. By the definition of the element $\beta$ and Lemma 1.2, we have $\alpha(i) \in F^+(\beta(i))$ for any $i \in I$. Therefore, in the $r$-system, $\alpha \in F^+(\beta)$, and thus $[\alpha]J \leq [\beta]J$. Since $\alpha \notin F^+(\alpha)$, $[\alpha]J = \{\alpha\}$. With respect to $\alpha \neq \beta$, we have $[\alpha]J \lneqq [\beta]J$. Therefore $[\alpha]J$ is not a maximal element of $A/J$, and this contradicts our assumption. Thus there exists at least one $j \in I_1$ such that $[\alpha(j)]J$ is a maximal element of $A_j/J$. (Clearly, $j \in I_1$ implies $\alpha(j) \notin F^+(\alpha(j))$).

b) If $j \in I_1$ and $[\alpha(j)]J$ is a maximal element of $A_j/J$, then, by Lemma 3.1, $[\alpha]J$ is a maximal element of $A/J$. □

Now we focus on the greatest $J$-class of the direct product of unary algebras.

**LEMMA 3.2.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an arbitrary system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. Let $\alpha \in A = \prod(A_i \mid i \in I)$. If $[\alpha]J$ is the greatest element of $A/J$, then $[\alpha(i)]J$ is the greatest element of $A_i/J$ for any $i \in I$.*

P r o o f. Let $\alpha \in A$ and $[\alpha]J$ be the greatest element of $A/J$. Let $j \in I$ be an arbitrary element of $I$, and $\beta_j$ be an arbitrary element of $A_j$. Suppose $\beta \in A$ has the property $\beta(i) = \alpha(i)$ for any $i \in I$, $i \neq j$, and $\beta(j) = \beta_j$. Since $[\alpha]J$ is the greatest element of $A/J$, we have $[\beta]J \leq [\alpha]J$. By Lemma 2.3, $[\beta(i)]J \leq [\alpha(i)]J$ for any $i \in I$. Therefore, also for $j \in I$ we have $[\beta_j]J = [\beta(j)]J \leq [\alpha(j)]J$, and, consequently, $[\alpha(j)]J$ is the greatest element of $A_j/J$. □

In the case of an $r$-system of unary algebras and $\alpha \in F^+(\alpha)$, also a converse of Lemma 3.2 is valid.

**THEOREM 3.3.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. If $\alpha \in A = \prod(A_i \mid i \in I)$ has the property $\alpha \in F^+(\alpha)$, then $[\alpha]J$ is the greatest element of $A/J$ if and only if $[\alpha(i)]J$ is the greatest element of $A_i/J$ for any $i \in I$.*

P r o o f. By Lemma 3.2, we have only to prove the case: If $[\alpha(i)]J$ is the greatest element of $A_i/J$ for any $i \in I$, then $[\alpha]J$ is the greatest element of $A/J$.

Let $\alpha \in F^+(\alpha)$ and $\beta \in A$ be an arbitrary element. If $[\alpha(i)]J$ is the greatest element of $A_i/J$ for any $i \in I$, then $[\beta(i)]J \leq [\alpha(i)]J$ for any $i \in I$. Then, by Lemma 2.4, we have $[\beta]J \leq [\alpha]J$. It means that $[\alpha]J$ is the greatest element of $A/J$. □

In what follows, we focus on the greatest $J$-class of the direct product of unary algebras in the case $\alpha \notin F^+(\alpha)$.

**LEMMA 3.3.** *Let $\langle A; F\rangle$ be a unary algebra and $\alpha \in A$ be such that $\alpha \notin F^+(\alpha)$. Then $[\alpha]J$ is the greatest element of $A/J$ if and only if $A = [\alpha] = \{\alpha\} \cup F^+(\alpha)$.*

Proof.

a) If $A = \{\alpha\} \cup F^+(\alpha)$, then for any $\beta \in F^+(\alpha)$ we have $[\beta] \subseteqq F^+(\alpha) \subseteqq [\alpha]$, and, consequently, $[\beta]J \leq [\alpha]J$ for every $\beta \in A$. Thus, $[\alpha]J$ is the greatest element of $A/J$.

b) Let $[\alpha]J$ be the greatest element of $A/J$, and $\alpha \notin F^+(\alpha)$. Then $[\alpha]J = \{\alpha\}$, and for any $\beta \in A$, $\beta \neq \alpha$, we have $[\beta]J \leq [\alpha]J$. By Lemma 1.2, $\beta \in F^+(\alpha)$. Hence $A \subseteqq F^+(\alpha) \cup \{\alpha\}$. The converse of this inclusion is obvious. Thus $A = \{\alpha\} \cup F^+(\alpha)$. □

**THEOREM 3.4.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$. Let $\alpha \in A = \prod(A_i \mid i \in I)$ and $\alpha \notin F^+(\alpha)$. Then $[\alpha]J$ is the greatest element of $A/J$ if and only if there exists $j \in I$ such that $\alpha(j) \notin F^+(\alpha(j))$, $[\alpha(j)]J$ is the greatest element of $A_j/J$, and, for any $i \in I$, $i \neq j$, $|A_i| = 1$.*

Proof. Let $\alpha \notin F^+(\alpha)$. Then there is $j \in I$ such that $\alpha(j) \notin F^+(\alpha(j))$.

a) Let $[\alpha]J$ be the greatest element of $A/J$. By Lemma 3.2, $[\alpha(i)]J$ is the greatest element of $A_i/J$ for any $i \in I$. By the definition of an $r$-system, we have $|I| \geq 2$. The condition $\alpha(j) \notin F^+(\alpha(j))$ implies $|A_j| \geq 2$. Let us suppose that there exists $i \in I$, $i \neq j$, such that $|A_i| \geq 2$. Now we consider $\beta \in A$ such that $\beta(j) = \alpha(j)$, and, for any $i \in I$, $i \neq j$, $\beta(i) \in A_i$ is an arbitrary element. As we assumed an existence of such $i \in I$, $i \neq j$, that $|A_i| \geq 2$, there exists at least one element $\beta \in A$, $\beta \neq \alpha$, satisfying our conditions. However, by Lemma 3.1, both $[\alpha]J = \{\alpha\}$ and $[\beta]J = \{\beta\}$ are maximal elements of $A/J$. Since $\alpha \neq \beta$, $[\alpha]J$ cannot be the greatest element of $A/J$. Thus, for any $i \in I$, $i \neq j$, we have $|A_i| = 1$.

b) If for any $i \in I$, $i \neq j$, there holds $|A_i| = 1$, then the direct product $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$ is isomorphic to a unary algebra $\mathbf{A}_j$. In this case, evidently, if $[\alpha(j)]J$ is the greatest element of $A_j/J$, then $[\alpha]J$ is the greatest element of $A/J$. □

By Lemma 3.2, if $[\alpha]J$ is the greatest $J$-class of the direct product of an arbitrary system $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ of unary algebras, then $[\alpha(i)]J$ is the greatest $J$-class of unary algebra $\mathbf{A}_i$ for any $i \in I$. If $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ is an $r$-system of unary algebras and $\alpha \in F^+(\alpha)$, then we have shown also the converse implication. If $\alpha \notin F^+(\alpha)$, we have proved the converse implication in the case of the existence of $j \in I$ such that $\alpha(j) \notin F^+(\alpha(j))$, $[\alpha(j)]J$ is the greatest element of $A_j/J$, and for any $i \in I$, $i \neq j$, $|A_i| = 1$. Our aim is to describe the case when there exists $i \in I$, $i \neq j$, such that $|A_i| \geq 2$. For the sake of convenience we introduce the following concept.

Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra and $\emptyset \ne M \subseteq A$. By $M/J$ we denote the set of all $J$-classes $[a]J \in A/J$ such that $[a]J \subseteq M$.

Let $\mathbf{A} = \langle A; F\rangle$ be a unary algebra and $N \in \mathcal{P}(\mathbf{A})$. Obviously, $N = \bigcup\{[a]J \mid a \in N\}$. Further, for any $N \in \mathcal{P}(\mathbf{A})$, $N/J = \{[a]J \in A/J \mid a \in N\}$.

Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras. $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$, $\alpha \in A = \prod(A_i \mid i \in I)$, and $\alpha \notin F^+(\alpha)$. To simplify the formulation of the next theorem, we denote by $I_1$ the set of all $j \in I$ such that $\alpha(j) \notin F^+(\alpha(j))$. Clearly, $I_1 = \{j \in I \mid \alpha(j) \in F^+(\alpha(j))\} \ne \emptyset$. In addition, let $N_0 = \prod(X_i \mid X_i = A_i \text{ for } i \in I \setminus I_1 \text{ and } X_i = A_i \setminus \{\alpha(i)\} \text{ for } i \in I_1)$.

**THEOREM 3.5.** *Let $\{\mathbf{A}_i = \langle A_i; F\rangle \mid i \in I\}$ be an $r$-system of unary algebras, $\mathbf{A} = \prod(\mathbf{A}_i \mid i \in I)$, and let there exist at least two $m, n \in I$, $m \ne n$, such that $|A_m| \ge 2$ and $|A_n| \ge 2$. Let $\alpha \in A = \prod(A_i \mid i \in I)$, $\alpha \notin F^+(\alpha)$, and $[\alpha(i)]J$ be the greatest element of $A_i/J$ for any $i \in I$. Then*

a) *$N_0 \cup \{\alpha\} \in \mathcal{P}(\mathbf{A})$, and $[\alpha]J$ is the greatest element in $(N_0 \cup \{\alpha\})/J$;*

b) *$(A \setminus N_0)/J = \{[\beta]J \mid \beta \in A \setminus N_0\}$, $|(A \setminus N_0)/J| \ge 2$, and $(A \setminus N_0)/J$ is the set of all maximal $J$-classes of $A/J$.*

Proof.

a) For any $j \in I_1$, $\alpha(j) \notin F^+(\alpha(j))$. In such a case, $|A_j| \ge 2$ and $[\alpha(j)]J = \{\alpha(j)\}$. According to our assumption, $[\alpha(j)]J$ is the greatest element of $A_j/J$. It is well known (see for example [3]) that then $A_j \setminus [\alpha(j)]J = A_j \setminus \{\alpha(j)\} \in \mathcal{P}(\mathbf{A}_j)$. Therefore $N_0 = \prod(X_i \mid X_i = A_i \text{ for } i \in I \setminus I_1, \text{ and } X_i = A_i \setminus \{\alpha(i)\} \text{ for } i \in I_1) \in \mathcal{P}(\mathbf{A})$. Now we prove that also $N_0 \cup \{\alpha\} \in \mathcal{P}(\mathbf{A})$.

We assumed $\alpha \notin F^+(\alpha)$. As $\langle N_0; F\rangle$ is a subalgebra of the algebra $\langle A; F\rangle$, we have only to prove that $f(\alpha) \in N_0$ for any $f \in F$.

For any $i \in I_1$, $\alpha(i) \notin F^+(\alpha(i))$. Thus, for any $i \in I_1$ and any $f \in F$, $f(\alpha(i)) \ne \alpha(i)$. Therefore, for any $i \in I_1$ and $f \in F$, $f(\alpha(i)) \in A_i \setminus \{\alpha(i)\}$. Obviously, for any $i \in I \setminus I_1$ and any $f \in F$ we have $f(\alpha(i)) \in A_i$. Thus, $f(\alpha) \in N_0 = \prod(X_i \mid X_i = A_i \text{ for } i \in I \setminus I_1, \text{ and } X_i = A_i \setminus \{\alpha(i)\} \text{ for } i \in I_1)$ for any $f \in F$. This proves $N_0 \cup \{\alpha\} \in \mathcal{P}(\mathbf{A})$.

In the following part we prove $[\beta]J \le [\alpha]J$ for any $\beta \in N_0 \cup \{\alpha\}$, $\beta \ne \alpha$.

From $\beta \ne \alpha$, $\beta \in N_0 \cup \{\alpha\}$, we get $\beta \in N_0$. Therefore $\beta(i) \in A_i \setminus \{\alpha(i)\}$ for any $i \in I_1$. This implies $\beta(i) \ne \alpha(i)$ for any $i \in I_1$. Since $[\alpha(i)]J$ is the greatest element of $A_i/J$ for any $i \in I$, we have $[\beta(i)]J \le [\alpha(i)]J$ for any $i \in I$. For any $i \in I \setminus I_1$, $\alpha(i) \in F^+(\alpha(i))$. On that account, from $[\beta(i)]J \le [\alpha(i)]J$ it follows that $[\beta(i)] = \{\beta(i)\} \cup F^+(\beta(i)) \subseteq [\alpha(i)] = \{\alpha(i)\} \cup F^+(\alpha(i)) = F^+(\alpha(i))$. Thus, $\beta(i) \in F^+(\alpha(i))$ for any $i \in I \setminus I_1$.

For any $i \in I_1$, $\beta(i) \neq \alpha(i)$ and $[\beta(i)]J \leq [\alpha(i)]J$. By Lemma 1.2, $\beta(i) \in F^+(\alpha(i))$.

We have proved $\beta(i) \in F^+(\alpha(i))$ for any $i \in I$. As we consider an $r$-system, we get $\beta \in F^+(\alpha)$, and, by Lemma 1.2, the condition $\alpha \neq \beta$ implies $[\beta]J \leq [\alpha]J$. Hence $[\alpha]J$ is the greatest element in $(N_0 \cup \{\alpha\})/J$.

b) By the definition of the set $N_0$, $A \setminus N_0 = \{\beta \in A \mid$ there exists such $j \in I_1$ that $\beta(j) = \alpha(j)\}$. With respect to $I_1 \neq \emptyset$ and the existence $m, n \in I$, $m \neq n$, such that $|A_m| \geq 2$, $|A_n| \geq 2$, we get $|A \setminus N_0| \geq 2$. Further, $\alpha(i) \notin F^+(\alpha(i))$ for any $i \in I_1$ and thus, $\beta \notin F^+(\beta)$ for any $\beta \in A \setminus N_0$. As $[\beta]J = \{\beta\}$ for any $\beta \in A \setminus N_0$, $(A \setminus N_0)/J = \{[\beta]J \mid \beta \in A \setminus N_0\}$ and $|(A \setminus N_0)/J| = |A \setminus N_0| \geq 2$.

For any $\beta \in A \setminus N_0$ there exists $j \in I_1$ such that $\beta(j) = \alpha(j)$, $\alpha(j) \notin F^+(\alpha(j))$, and $[\alpha(j)]J$ is the greatest element of $A_j/J$. By Lemma 3.1, $[\beta]J$ is a maximal element of $A/J$. Since $[\alpha]J = \{\alpha\}$ and $[\alpha]J$ is the greatest element of $(N_0 \cup \{\alpha\})/J$, it is clear that no element of $N_0/J$ is a maximal element of $A/J$. Thus $(A \setminus N_0)/J$ is the set of all maximal elements of $A/J$. Obviously, by $|(A \setminus N_0)/J| \geq 2$, there does not exist any greatest element of $A/J$. □

We have described maximal and greatest $J$-classes of the direct product of an $r$-system of unary algebras. Now we are able to describe the greatest and maximal subalgebras of this direct product. This work will be done in a forthcoming paper devoted to $J$-subalgebras of unary algebras.

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