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HAUSDORFF COMPACTIFICATIONS OF COMPLETELY REGULAR SPACES

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ABSTRACT. We study a technique to obtain compact spaces from two fixed subsets of a semigroup with unit element. We apply that technique to obtain compactifications of completely regular Hausdorff topological spaces. We establish the relationship between our compactification technique with other known ones. In particular, we study the Stone-Čech, Wallman, k points ($k \leq \omega$), and 0-dimensional compactifications. Certain types of compactifications introduced here are always of Wallman type but the converse is not known to us. Finally P-spaces and F-spaces are characterized.

Introduction

Let X be a completely regular space and let $C(X)$ be the ring of all continuous real-valued functions on X. The set of maximal ideals in $C(X)$ endowed with the Stone topology, which is called the structure space for $C(X)$, is homeomorphic to βX, the Stone-Čech compactification of X. The concept of maximal z-filter is closely related to the concept of maximal ideal: the Stone-Čech compactification of X can also be obtained by means of the maximal z-filters on X. This compactification could also be obtained by means of the minimal ideals of zero-sets of X.

One of the aims of this paper is to introduce a concept that is dual to the concept of maximal ideal on $\mathcal{C}(X)$. This is achieved by introducing the concept of $P\mathcal{L}$-filter on a semigroup with unit.

In Section 1, the family $P(\mathcal{L})$ of all minimal $P\mathcal{L}$-filters is endowed with a topology $T_{P,\mathcal{L}}$ such that $\big(P(\mathcal{L}), T_{P,\mathcal{L}}\big)$ is compact. In Section 2, we obtain the conditions that a family of continuous functions $\mathcal{L}$ must satisfy in order that $P(\mathcal{L})$ be a Hausdorff compactification of X. The compactifications of Alexandroff, Stone-Čech and many others, in the following sections, can be obtained

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by using different families $\mathcal{L}$. Finally, P-spaces and F-spaces are characterized by using our techniques.

The real-compactifications of X are obtained in [10] by using some rings of continuous functions. Although the way followed in [10] is rather different to ours, both techniques use adequate subsets of $C(X)$ to obtain compactifications of X.

1. $P\mathcal{L}$-filters

Let A be a non-empty set endowed with an associative operation $+$, whose zero element will be denoted by 0. Let P be a set such that $P \cap A \neq \emptyset$ and $0 \notin P$. Let $\mathcal{L} \subset A$ be such that $0 \in \mathcal{L}$ and $P \cap \mathcal{L} \neq \emptyset$.

Definition 1.1. Let p be a non-empty subset of $\mathcal{L}$. We will say that p is a *$P\mathcal{L}$-filter* if p satisfies:

1) $0 \notin p$;
2) If $\{a_1, \dots, a_n\} \subset \mathcal{L}$ and $a_1 + \cdots + a_n \in P$, then there exists $i \in \{1, \dots, n\}$ such that $a_i \in p$.

It is clear that if p is a $P\mathcal{L}$-filter and $a \in \mathcal{L} \cap P$, then $a \in p$.

Let p be a $P\mathcal{L}$-filter. We will say that p is a *minimal $P\mathcal{L}$-filter* if there does not exist a $P\mathcal{L}$-filter q such that $q \subset p$ and $p \neq q$. By Zorn's lemma, if p is a $P\mathcal{L}$-filter then there exists a minimal $P\mathcal{L}$-filter contained in p.

Theorem 1.2. *Let p be a minimal $P\mathcal{L}$-filter. If $\{a_1, \dots, a_n\} \subset \mathcal{L}$ and $a_1 + \cdots + a_n \in p$, then there exists $i \in \{1, \dots, n\}$ such that $a_i \in p$.*

Proof. Let us suppose that $a_1 + \cdots + a_n \in p$ and that $\{a_1, \dots, a_n\} \cap p = \emptyset$. Since p is a minimal $P\mathcal{L}$-filter, we have that $q = p \setminus \{a_1 + \cdots + a_n\}$ is not a $P\mathcal{L}$-filter. If $q = \emptyset$ and $a \in P \cap \mathcal{L}$, then $a = a_1 + \cdots + a_n$ and there exist $i \in \{1, \dots, n\}$ such that $a_i \in p$. Therefore, q is non-empty. Since q is not a $P\mathcal{L}$-filter, there exists $\{b_1, \dots, b_m\} \subset \mathcal{L} \setminus q$ such that $b_1 + \cdots + b_m \in P$. We can suppose that $b_1 \in p$, then we have that $b_1 = a_1 + \cdots + a_n$. This iterative reasoning allow us to conclude that $(a_1 + \cdots + a_n) \overset{(m)}{+ \cdots +} (a_1 + \cdots + a_n) \in P$, which contradicts that $\{a_1, \dots, a_n\} \cap p = \emptyset$. □

Let us denote by $P(\mathcal{L})$ the set of all minimal $P\mathcal{L}$-filters. We also use the notation $o(a) = \{p \in P(\mathcal{L}) : a \in p\}$ and $c(a) = \{p \in P(\mathcal{L}) : a \notin p\}$ for every $a \in \mathcal{L}$.

Let $T_{P,\mathcal{L}}$ be the topology for $P(\mathcal{L})$ determined by the subbase $\{o(a) : a \in \mathcal{L}\}$. We have that $\{c(a) : a \in \mathcal{L}\}$ is a closed subbase for $T_{P,\mathcal{L}}$.

THEOREM 1.3. *$\big(P(\mathcal{L}), T_{P,\mathcal{L}}\big)$ is compact.*

Proof. Let $\{c(a_i)\}_{i\in I}$ be a family of elements of the closed subbase which has the finite intersection property. For every finite subset $J \subset I$, there exists $p \in P(\mathcal{L})$ such that $p \in \bigcap_{i\in J} c(a_i)$. Therefore $\sum_{i\in J} a_i \notin P$ and $q = \mathcal{L} \setminus \big(\{a_i\}_{i\in I} \cup \{0\}\big)$ is a $P\mathcal{L}$-filter. Let $q' \in P(\mathcal{L})$ be such that $q' \subset q$. For every $i \in I$ we have that $a_i \notin q'$ and therefore $q' \in \bigcap_{i\in I} c(a_i)$. $\square$

Remark 1.4. If $B \subset P(\mathcal{L})$, then the closure of B in $\big(P(\mathcal{L}), T_{P,\mathcal{L}}\big)$ is $\mathrm{cl}(B) = \Big\{a \in P(\mathcal{L}) :\ a \subset \bigcup_{b\in B} b\Big\}$.

THEOREM 1.5. *Let us suppose that, for $\{a,b\} \subset A$, the condition $a+a+b \in P$ implies that $a+b \in P$. If p is a $P\mathcal{L}$-filter, then p is minimal if and only if for every $a \in p$ there exists $b \in \mathcal{L}\setminus p$ such that $a+b \in P$.*

Proof. Let us suppose that $a \in p$ and that $a+b \notin P$ for every $b \in \mathcal{L}\setminus p$. We have that $q = p\setminus\{a\}$ is not a $P\mathcal{L}$-filter and there exists $\{b_1,\dots,b_m\} \subset \mathcal{L}\setminus q$ such that $b_1 + \dots + b_m \in P$. We can suppose that $b_1 \in p$. Therefore $b_1 = a$, $a + (b_2 + \dots + b_m) \in P$ and $b_2 + \dots + b_m \in p$. By Theorem 1.2, we can assume that $b_2 \in p$. This iterative reasoning allow us to conclude that $a + a \in P$ and that $a + 0 \in P$. This is a contradiction, because $0 \in \mathcal{L}\setminus p$.

If p is not a minimal $P\mathcal{L}$-filter, then there exists a $P\mathcal{L}$-filter q such that $q \subset p$ and $q \neq p$. Let $a \in p\setminus q$. By hypothesis, there exists $b \in \mathcal{L}\setminus p$ such that $a+b \in P$ Hence, $a \in q$ or $b \in q$. $\square$

EXAMPLE 1.6. Let $(\mathcal{F}, \vee, \wedge, ')$ be a Boolean algebra with unit element and zero element, which will be denoted by x and 0 respectively. We consider $\mathcal{F}$ endowed with the operation $\vee$ as the operation $+$. This operation is associative and its unit element is 0 Let $P = \{x\}$ and let $\mathcal{L} = \mathcal{F}$. Clearly, $\big(P(\mathcal{F}), T_{P,\mathcal{F}}\big)$ is a 0-dimensional compact Hausdorff space and the Boolean algebra of its clopen subsets is isomorphic to the Boolean algebra $\mathcal{F}$. Hence $\big(P(\mathcal{F}), T_{P,\mathcal{F}}\big)$ is the Stone space of $\mathcal{F}$ ([11]). It should be noticed that every minimal $P\mathcal{F}$-filter is a maximal filter of the Boolean algebra $\mathcal{F}$ and conversely.

2. $P(\mathcal{L})$ spaces in $\mathcal{C}_+(X)$

Let (X, T) be a completely regular Hausdorff space and let $\mathcal{C}(X)$ be the ring of all real-valued continuous functions on X. It is clear that $f \in \mathcal{C}(X)$ is invertible in the ring $\mathcal{C}(X)$ if and only if $f(x) \neq 0$ for every $x \in X$. If $a \in \mathbb{R}$, then we will denote by $a \in \mathcal{C}(X)$ the corresponding constant function. A zero set in X is a set of the form $z(f) = \{x \in X : f(x) = 0\}$ for some $f \in C(X)$. The collection of all zero sets in X will be denoted by $Z(X)$. The complement of a zero set is called a cozero set and is denoted by $\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$. Let

$$\mathcal{C}_+(X) = \{f \in \mathcal{C}(X) : f(x) \geq 0, \ x \in X\},$$

endowed with the $+$ operation, and let P be the set of the functions $f \in \mathcal{C}_+(X)$ that are invertible in $\mathcal{C}(X)$. If $\mathcal{L} \subset \mathcal{C}_+(X)$ is such that $0 \in \mathcal{L}$ and $\mathcal{L} \cap P \neq \emptyset$, then we will consider the topological space $(P(\mathcal{L}), T_{P,\mathcal{L}})$. We will try to find conditions on the set $\mathcal{L}$ in order that $(P(\mathcal{L}), T_{P,\mathcal{L}})$ be a Hausdorff compactification of X.

THEOREM 2.1.

1) *If $p \subset \mathcal{L}$ is a $P\mathcal{L}$-filter, then p is minimal if and only if for every $f \in p$ there exists $g \in \mathcal{L} \setminus p$ such that $f + g \in P$.*
2) *If $p \in P(\mathcal{L})$ and q is a $P\mathcal{L}$-filter such that $p \not\subset q$, then there exists $\{f, g\} \subset \mathcal{L}$ such that $f \in p \setminus q$, $g \in q \setminus p$ and $f + g \in P$.*
3) *Let $p \in P(\mathcal{L})$. If $\{f, g\} \subset \mathcal{L}$ and $f \cdot g \in p$, then $f \in p$ and $g \in p$.*
4) *Let $p \in P(\mathcal{L})$. If $f \in p$, $g \in \mathcal{L}$ and $\mathrm{coz}(f) \subset \mathrm{coz}(g)$, then $g \in p$.*

Proof. It is clear that 1) is a consequence of Theorem 1.5.

If $f \in p \setminus q$, then there exists $g \in \mathcal{L} \setminus p$ such that $f + g \in P$, and so $g \in q \setminus p$. This proves 2).

For 3), if $f \cdot g \in p$, then there exists $h \in \mathcal{L} \setminus p$ such that $f \cdot g + h \in P$. Hence $f + h \in P$, $g + h \in P$ and $\{f, g\} \subset p$.

For 4), let $h \in \mathcal{L} \setminus p$ be such that $f + h \in P$. We have that $g + h \in P$ and therefore, $g \in P$. $\square$

DEFINITION 2.2. We will say that $\mathcal{L}$ *has the* (A) *property* if for every $\{f, g\} \subset \mathcal{L}$ we have that $f + g \in \mathcal{L}$ and $f \cdot g \in \mathcal{L}$.

THEOREM 2.3. *Let us suppose that $\mathcal{L}$ has the* (A) *property. Then:*

1) *If $p \in P(\mathcal{L})$ and $\{f, g\} \subset p$, then $f \cdot g \in p$.*
2) *If $p \in P(\mathcal{L})$ and $f \in p$, then $f + g \in p$ for every $g \in \mathcal{L}$.*
3) *For every $\{f, g\} \subset \mathcal{L}$,*
   - 3a) $o(f) \cap o(g) = o(f \cdot g)$,
   - 3b) $o(f) \cup o(g) = o(f + g)$,
   - 3c) $c(f) \cap c(g) = c(f + g)$,
   - 3d) $c(f) \cup c(g) = c(f \cdot g)$,
   - 3e) $\{o(f) : f \in \mathcal{L}\}$ *is an open basis for* $T_{P,\mathcal{L}}$.

Proof.

1) There exists $\{j,h\} \subset \mathcal{L} \setminus p$ such that $f+j \in P$ and $g+h \in P$. Hence $f \cdot g + j + h \in P$ and $f \cdot g \in p$.

2) There exists $h \in \mathcal{L} \setminus p$ such that $f+h \in P$. Hence $(f+g)+h \in P$ and $f+g \in p$.

3) is a consequence of 1) and 2). □

DEFINITION 2.4. If $\mathcal{L}$ has the (A) property, then we will say that a $P\mathcal{L}$-filter p is *multiplicative* if for every $\{f,g\} \subset p$ we have that $f \cdot g \in p$.

Remark 2.5. Note that if $x \in X$, then $p_x = \{f \in \mathcal{L} : f(x) \neq 0\}$ is a $P\mathcal{L}$-filter. If, for every $x \in X$, p_x is a minimal $P\mathcal{L}$-filter, then we can define the mapping $\varphi\colon X \to P(\mathcal{L})$ by $\varphi(x) = p_x$ for every $x \in X$.

THEOREM 2.6. *If $x \in X$, then p_x is minimal if and only if for every $f \in \mathcal{L}$ such that $x \in \mathrm{coz}(f)$ there exists $g \in \mathcal{L}$ such that $x \in z(g) \subset \mathrm{coz}(f)$.*

Proof.

Necessity.

Let $f \in \mathcal{L}$ be such that $x \in \mathrm{coz}(f)$. We have that $f \in p_x$ and so there exists $g \in \mathcal{L} \setminus p$ such that $f+g \in P$. Therefore $x \in z(g) \subset \mathrm{coz}(f)$.

Sufficiency.

Let $f \in p_x$. Since $x \in \mathrm{coz}(f)$, there exists $g \in \mathcal{L}$ such that $x \in z(g) \subset \mathrm{coz}(f)$. This proves that $f+g \in P$ and $g \notin p_x$. □

DEFINITION 2.7. We will say that $\mathcal{L}$ *has the* (B) *property* if for every $f \in \mathcal{L}$ and every $x \in \mathrm{coz}(f)$ there exists $g \in \mathcal{L}$ such that $x \in z(g) \subset \mathrm{coz}(f)$.

THEOREM 2.8. *Let us suppose that $\mathcal{L}$ has the properties* (A) *and* (B). *Let $\varphi\colon X \to P(\mathcal{L})$ be the mapping defined by $\varphi(x) = p_x$ for every $x \in X$. Then, $(P(\mathcal{L}), \varphi)$ is a compactification of X if and only if $\{\mathrm{coz}(f) : f \in \mathcal{L}\}$ is an open basis of X.*

Proof.

Necessity.

Let U be an open set in X and let $x \in U$. Since φ is a homeomorphism of X onto $\varphi(X)$, then there exists a $f \in \mathcal{L}$ such that $\varphi(x) \in o(f) \cap \varphi(X) \subset \varphi(U)$ and, consequently, $x \in \mathrm{coz}(f) \subset U$.

Sufficiency.

Let $\{x,y\} \subset X$, $x \neq y$. Since $\{\mathrm{coz}(f) : f \in \mathcal{L}\}$ is an open basis of X, there exists $f \in \mathcal{L}$ such that $x \in \mathrm{coz}(f) \subset X \setminus \{y\}$, and so $\varphi(x) \neq \varphi(y)$. Since $\varphi(\mathrm{coz}(f)) = o(f) \cap \varphi(X)$ for every $f \in \mathcal{L}$, φ is a homeomorphism of X onto $\varphi(X)$. We claim that $\varphi(X)$ is dense in $P(\mathcal{L})$. Let G be a non-empty open subset of $P(\mathcal{L})$ and let $p \in G$. There exists $f \in \mathcal{L}$ such that $p \in o(f) \subset G$. Since $0 \notin p$, there exists $x \in X$ such that $f(x) \neq 0$. This proves that $\varphi(x) \in o(f) \subset G$. □

DEFINITION 2.9. We will say that $\mathcal{L}$ *has the* (C) *property* if $\{\mathrm{coz}(f) : f \in \mathcal{L}\}$ is an open basis of X.

DEFINITION 2.10. Let p be a $P\mathcal{L}$-filter. We will say that p is *fixed* if $\bigcap\limits_{f\in p} \mathrm{coz}(f) \neq \emptyset$. We will say that p is *free* if $\bigcap\limits_{f\in p} \mathrm{coz}(f) = \emptyset$.

Remark 2.11. Let us suppose that $\mathcal{L}$ satisfies the properties (B) and (C).

1) For every closed set $F \subset X$ and every $x \in X \setminus F$ there exists $g \in \mathcal{L}$ such that $x \in z(g)$ and $z(g) \cap F = \emptyset$.

2) If p is a $P\mathcal{L}$-filter, then p is fixed if and only if $p = p_x$ for some $x \in X$. Let us prove that, in this case, $\bigcap\limits_{f\in\mathcal{L}\setminus p} z(f) = \{x\}$. If p is fixed and $\{x, y\} \in \bigcap\limits_{f\in p} \mathrm{coz}(f)$, $x \neq y$, then, by the (C) property, there exists $g \in \mathcal{L}$ such that $g(x) = 0$ and $g(y) = a \neq 0$, $a \in \mathbb{R}$. Let $F = \{z \in X : g(z) \leq \frac{a}{2}\}$. There exists $h \in \mathcal{L}$ such that $y \in z(h)$, $z(h) \cap F = \emptyset$, and so $h + g \in P$ This contradicts that $h \notin p$ and $g \notin p$. Hence, there exists an unique $x \in X$ such that $\{x\} = \bigcap\limits_{f\in p} \mathrm{coz}(f)$. Since p_x is minimal, $p = p_x$. Finally, if $p = p_x$, then it is clear that $\bigcap\limits_{f\in\mathcal{L}\setminus p} z(f) = \{x\}$.

This proves that every fixed $P\mathcal{L}$-filter is minimal.

3) If p is a minimal $P\mathcal{L}$-filter, then we have either $\bigcap\limits_{f\in\mathcal{L}\setminus p} z(f) = \emptyset$ or $\bigcap\limits_{f\in\mathcal{L}\setminus p} z(f) = \{x\}$ for some $x \in X$. In the latter case $p = p_x$.

4) If, in addition, $\mathcal{L}$ satisfies the property (A) then X is compact if and only if every minimal $P\mathcal{L}$-filter is fixed.

Our next aim is to find conditions on $\mathcal{L}$ in order to $(P(\mathcal{L}), \varphi)$ be a Hausdorff compactification of X.

THEOREM 2.12. *If $\mathcal{L}$ satisfies the properties* (A), (B) *and* (C), *then $P(\mathcal{L})$ is a Hausdorff space if and only if for every $\{f, g\} \subset \mathcal{L}$ such that $f + g \in P$ there exists $\{f', g'\} \subset \mathcal{L}$ such that $\{f + f', g + g'\} \subset P$ and $f' \cdot g' = 0$*

P r o o f . Let us suppose that $P(\mathcal{L})$ is a Hausdorff space. Let $\{f, g\} \subset \mathcal{L}$ be such that $f + g \in P$. We have that $c(f) \cap c(g) = \emptyset$ and there exist two disjoint open subsets G_1 and G_2 of $P(\mathcal{L})$ such that $c(f) \subset G_1$ and $c(g) \subset G_2$. For every $p \in c(f)$, there exists $h_p \in \mathcal{L}$ such that $p \in o(h_p) \subset G_1$. Since $c(f)$ is compact, there exists $\{p_1, \dots, p_n\} \subset c(f)$ such that $c(f) \subset o(h_{p_1}) \cup \cdots \cup o(h_{p_n}) \subset G_1$. If $f' = h_{p_1} + \cdots + h_{p_n}$, then $c(f) \subset o(f') \subset G_1$. Similarly there exists $g' \in \mathcal{L}$ such that $c(g) \subset o(g') \subset G_2$. Since $o(f') \cap o(g') = \emptyset$, we have that $f' \cdot g' = 0$ and $\{f + f', g + g'\} \subset P$, because $c(f) \subset o(f')$ and $c(g) \subset o(g')$.

Conversely, let $\{p,q\} \subset P(\mathcal{L})$, $p \neq q$. If $f \in p \setminus q$ and $g \in \mathcal{L} \setminus p$ are such that $f+g \in P$, then there exists $\{f',g'\} \subset \mathcal{L}$ such that $\{f+f', g+g'\} \subset P$ and $f' \cdot g' = 0$. Consequently, $q \in o(f')$, $p \in o(g')$ and $o(f') \cap o(g') = \emptyset$. □

DEFINITION 2.13. We will say that $\mathcal{L}$ *has the* (D) *property* if for every $\{f,g\} \subset \mathcal{L}$ such that $f+g \in P$ there exists $\{f',g'\} \subset \mathcal{L}$ such that $\{f+f', g+g'\} \subset P$ and $f' \cdot g' = 0$.

Remark 2.14.

1) Let us suppose that $\mathcal{L}$ satisfies the (D) property and that $\{p,q\} \subset P(\mathcal{L})$, $p \neq q$. There exist $f \in p \setminus q$ and $g \in q \setminus p$ such that $f \cdot g = 0$. This proves that every multiplicative $P\mathcal{L}$-filter contains an unique minimal $P\mathcal{L}$-filter.

2) If $\mathcal{L} = \mathcal{C}_+(X)$, then $\mathcal{L}$ satisfies the properties (A), (B), (C) and (D).

THEOREM 2.15 (STONE-ČECH COMPACTIFICATION). *Let X be a completely regular Hausdorff space. The compactification $\big(P(\mathcal{C}_+(X)), \varphi\big)$ of X is equivalent to the Stone-Čech compactification $(\beta X, e)$ of X.*

Proof. Let $z(f)$ and $z(g)$ be two disjoint zeros in X. We have that $\varphi(z(f)) \subset c(f)$, $\varphi(z(g)) \subset c(g)$ and that $\mathrm{cl}\big(\varphi(z(f))\big) \cap \mathrm{cl}\big(\varphi(z(g))\big) \subset c(f) \cap c(g) = c(f+g) = \emptyset$, because $f+g \in P$. □

In the next theorem we determine the homeomorphism between the compactifications $(\beta X, e)$ and $\big(P(\mathcal{C}_+(X)), \varphi\big)$ of X.

THEOREM 2.16. *$(\beta X, e)$ and $\big(P(\mathcal{C}_+(X)), \varphi\big)$ are equivalent through the mapping*

$$\begin{aligned} \phi\colon \beta X &\to P\big(\mathcal{C}_+(X)\big), \\ t &\mapsto \phi(t) = \left\{ f \in \mathcal{C}_+(X) :\ t \notin \overline{e\big(z(f)\big)}^{\beta X} \right\}. \end{aligned}$$

Proof. We first show that $\phi(t)$ is a $P\mathcal{C}_+(X)$-filter for every $t \in \beta X$. It is clear that $0 \notin \phi(t)$. If $\{f_1, \dots, f_n\} \subset \mathcal{C}_+(X)$ satisfies $f_1 + \dots + f_n \in P$, then

$$\overline{e\big(z(f_1 + \dots + f_n)\big)}^{\beta X} = \bigcap_{i=1}^{n} \overline{e\big(z(f_i)\big)}^{\beta X} = \emptyset.$$

Hence there exists $i \in \{1, \dots, n\}$ such that $f_i \in \phi(t)$.

We now show that $\phi(t)$ is minimal for every $t \in \beta X$. Let $f \in \phi(t)$, then $t \notin \overline{e\big(z(f)\big)}^{\beta X}$ and there exist $h^\beta \in \mathcal{C}_+(\beta X)$ and a closed neighborhood $V_t \subset \beta X$ of t such that $h^\beta(V_t) = \{0\}$ and $h^\beta\left(\overline{e\big(z(f)\big)}^{\beta X}\right) = \{1\}$.

Let $h = h^\beta \circ e$. We have that $f + h \in P$ and $t \in \overline{e(z(h))}^{\beta X}$. Thus $h \in \mathcal{C}_+(X) \setminus \phi(t)$.

Let us check that ϕ is one to one. Let $\{t_1, t_2\} \subset \beta X$ be such that $t_1 \neq t_2$. There exists $\{h_1^\beta, h_2^\beta\} \subset \mathcal{C}_+(\beta X)$ such that $\operatorname{coz}(h_1^\beta) \cap \operatorname{coz}(h_2^\beta) = \emptyset$, $t_1 \in \operatorname{coz}(h_1^\beta)$ and $t_2 \in \operatorname{coz}(h_2^\beta)$. Then $h_1 \in \phi(t_1)$, $h_2 \in \phi(t_2)$, where $h_1 = h_1^\beta \circ e$ and $h_2 = h_2^\beta \circ e$. Since $h_1 \cdot h_2 = 0$, $\phi(t_1) \neq \phi(t_2)$.

We now show that ϕ is continuous. Let $(t_\alpha)_{\alpha \in I}$ be a net in βX that converges to $t \in \beta X$. Since $\phi(t) \in o(f)$ for some $f \in \mathcal{C}_+(X)$, there exists $\alpha_0 \in I$ such that $t_\alpha \notin \overline{e(z(f))}^{\beta X}$ for every $\alpha \geq \alpha_0$, $\alpha \in I$. Hence, $\phi(t_\alpha) \in o(f)$ for $\alpha \geq \alpha_0$.

It is clear that $\phi \circ e = \varphi$. □

Remark 2.17.

a) It is easy to prove that the subring $\mathcal{C}_+^*(X)$ of all bounded functions $f \in \mathcal{C}_+(X)$ satisfies the properties (A), (B), (C) and (D) and that also $(P(\mathcal{C}_+^*(X)), \varphi)$ is equivalent to $(\beta X, e)$ by means of the homeomorphism:

$$\phi \colon \beta X \to P(\mathcal{C}_+^*(X)),$$
$$t \mapsto \phi(t) = \left\{ f \in \mathcal{C}_+^*(X) : \ t \notin \overline{e(z(f))}^{\beta X} \right\}.$$

b) Let us observe that every minimal $P\mathcal{C}_+(X)$-filter is a family of continuous functions and that it is a point in the compact space $P(\mathcal{C}_+(X))$. The proof of Theorem 2.16 lets us check that if $p \in P(\mathcal{C}_+(X))$, then $p = \{f \in \mathcal{C}_+(X) : \ p \notin \operatorname{cl}(\varphi(z(f)))\}$ and that if $\overline{f} \in \mathcal{C}_+(P(\mathcal{C}_+(X)))$ and $\overline{f}(p) \neq 0$, then $f = \overline{f} \circ \varphi \in p$.

THEOREM 2.18 (ALEXANDROFF COMPACTIFICATION). *Let X be a locally compact Hausdorff space which is not compact. Let us denote by $\mathcal{L}$ the set of functions $f \in \mathcal{C}_+(X)$ such that $f(X \setminus K)$ is a singleton for some compact set $K \subset X$. Then:*

1) *$\mathcal{L}$ satisfies the properties* (A), (B), (C) *and* (D).
2) *The compactification $(P(\mathcal{L}), \varphi)$ is equivalent to the Alexandroff compactification of X.*

Proof.

1) It is obvious that $\mathcal{L}$ is a subring of $\mathcal{C}(X)$ and that $\mathcal{L}$ satisfies the properties (B) and (C), because X is locally compact. If $\{h_1, h_2\} \subset \mathcal{L}$ and $h_1 + h_2 \in P$, then we consider $h = \frac{h_1}{h_1 + h_2} \in \mathcal{L}$, $g_1 = (1/2 - h) \vee 0 \in \mathcal{L}$, $g_2 = (h - 1/2) \vee 0 \in \mathcal{L}$. We have $\{h_1 + g_1, h_2 + g_2\} \subset P$ and $g_1 \cdot g_2 = 0$.

2) Let us prove that $P(\mathcal{L}) \setminus \varphi(X)$ is unitary. Suppose that there exist two different elements p and q in $P(\mathcal{L}) \setminus \varphi(X)$. Let $\{f, g\} \subset \mathcal{L}$ be such that $\operatorname{cl}(o(f)) \cap \operatorname{cl}(o(g)) = \emptyset$, $p \in o(f)$ and $q \in o(g)$. Since $f \in \mathcal{L}$, there exist a compact

$K \subset X$ and a real number $b \in \mathbb{R}$ such that $f(X \setminus K) = \{b\}$. Let us observe that if $b = 0$, then $\mathrm{cl}\big(o(f)\big) \subset \mathrm{cl}\big(\varphi(K)\big) \subset \varphi(X)$, because K is compact, which is impossible. Hence $b \neq 0$ and $P(\mathcal{L}) \setminus \varphi(X) \subset \mathrm{cl}\big(\varphi(X \setminus K)\big) \subset \mathrm{cl}\big(o(f)\big)$. Therefore $q \in \mathrm{cl}\big(o(f)\big)$, which is a contradiction. □

Remark 2.19. Under the hypotheses of Theorem 2.18, it is easy to check that the following results hold:

- The set p_0 of functions $f \in \mathcal{L}$ such that $(X \setminus K) \not\subset z(f)$ for every compact set $K \subset X$, is a multiplicative $P\mathcal{L}$-filter.
- For every $p \in P(\mathcal{L}) \setminus \varphi(X)$, every compact set K in X and every $f \in \mathcal{L}$, we have that $z(f) \cap (X \setminus K) \neq \emptyset$.
- $P(\mathcal{L}) \setminus \varphi(X)$ contains an unique point: the unique minimal $P\mathcal{L}$-filter contained in p_0.

In the sequel, we will prove that the compactifications $\big(P(\mathcal{L}), \varphi\big)$ are Wallman-type ([2]). Let us recall the main characteristics of this type of compactification. Let C be a collection of subsets of X; we will say that C is a *lattice* on X if $\{\emptyset, X\} \subset C$ and C is closed under finite intersections and joins. A lattice C will be called a *Wallman base* on X if:

(1) C is base for the closed subsets of X,
(2) C is a *normal* lattice on X (i.e. for every $\{a, b\} \subset C$ such that $a \cap b = \emptyset$, there exists $\{c, d\} \subset C$ such that $a \subset X \setminus c$, $b \subset X \setminus d$, and $c \cup d = X$),
(3) C is a *disjunctive lattice* on X (i.e. for every closed F in X and every $x \in X \setminus F$ there exists $a \in C$ such that $x \in a$ and $a \cap F = \emptyset$).

The set of all C*-ultrafilters* will be denoted by $w(C)$.

In order to topologize $w(C)$, let us denote $c^* = \big\{\mathcal{F} \in w(C) : c \in \mathcal{F}\big\}$ for every $c \in C$. Then, $\{c^*\}_{c \in C}$ is a base for the closed sets of some topology for $w(C)$. It has the property that $w(C)$ is a compactification of X if and only if C is a Wallman base on X (with respect to the embedding map $w\colon X \to w(C)$ defined by $w(x) = \{a \in C : x \in a\}$). The space $\big(w(C), w\big)$ is called, in ([3]), the *Wallman-Frink compactification* and, in ([9]), the *Wallman-Shanin compactification.* A compactification (X', α) of X will be called a *Wallman-type compactification* if there exists a Wallman base C on X such that (X', α) and $\big(w(C), w\big)$ are equivalent.

THEOREM 2.20. *If X is a completely regular Hausdorff space and $\mathcal{L} \subset C_+(X)$ satisfies the properties* (A), (B), (C) *and* (D), *then $C = \{z(f)\}_{f \in \mathcal{L}}$ is a Wallman-base on X and the compactifications $\big(P(\mathcal{L}), \varphi\big)$ and $\big(w(C), w\big)$ are equivalent.*

Proof. If $f \in \mathcal{L}$ and $g \in \mathcal{L}$, then $f \cdot g \in \mathcal{L}$ and $f + g \in \mathcal{L}$. Hence $z(f) \cap z(g) \in C$ and $z(f) \cup z(g) \in C$. Since C is a closed basis of X, by

Remark 2.11, it is clear that C is disjunctive. Likewise, if $f_1 \in \mathcal{L}$, $f_2 \in \mathcal{L}$ and $z(f_1) \cap z(f_2) = \emptyset$, then we deduce, by (D), that there exist $g_1 \in \mathcal{L}$, $g_2 \in \mathcal{L}$ such that $z(f_1) \subset X \setminus z(g_1)$, $z(f_2) \subset X \setminus z(g_2)$ and $z(g_1) \cup z(g_2) = X$.

We now show that the compactifications $\big(P(\mathcal{L}), \varphi\big)$ and $\big(w(C), w\big)$ are equivalent through the mapping $h\colon P(\mathcal{L}) \to w(C)$ defined by $h(p) = \big\{z(f) : f \in \mathcal{L} \setminus p\big\}$ for every $p \in P(\mathcal{L})$.

We first show that h is well defined. If $p \in P(\mathcal{L})$, then $h(p)$ is a C-ultrafilter. If $f \notin p$, then f is not invertible, hence $z(f) \neq \emptyset$ and $\emptyset \notin h(p)$. If $z(f_1) \in h(p)$ and $z(f_2) \in h(p)$, then $f_1 \in \mathcal{L} \setminus p$ and $f_2 \in \mathcal{L} \setminus p$. Therefore $f_1 + f_2 \in \mathcal{L} \setminus p$ and $z(f_1) \cap z(f_2) \in h(p)$. Since p is minimal, if $z(f) \subset z(g)$ and $z(f) \in h(p)$, then $z(g) \in h(p)$. In order to prove the maximality of $h(p)$, we consider $g \in \mathcal{L}$ where $z(g) \notin h(p)$. We have that $g \in p$ and there exists $f \in \mathcal{L} \setminus p$ such that $f + g \in P$. Therefore $z(f) \cap z(g) = \emptyset$ and $z(f) \in h(p)$.

Let us check that h is one to one. If $\{p_1, p_2\} \subset P(\mathcal{L})$, $p_1 \neq p_2$, then it is clear that there exists $f \in p_1 \setminus p_2$ and $z(f) \in h(p_2) \setminus h(p_1)$.

Since

$$h\big(c(f)\big) = h\big(\{p \in P(\mathcal{L}) :\ f \notin p\}\big) = \big\{h(p) :\ z(f) \in h(p)\big\} = z(f)^*$$

for every $f \in \mathcal{L}$, it is clear that h is continuous.

Finally, $h \circ \varphi = w$ because we have that

$$w(x) = \big\{z(f) \in C :\ x \in z(f)\big\} = \big\{z(f) \in C :\ p_x \in c(f)\big\} = h\big(\varphi(x)\big)$$

for every $x \in X$. □

Remark 2.21.

a) If a Wallman base C on X is given, then we can ask if there exists some $\mathcal{L} \subset \mathcal{C}_+(X)$ such that $\big(P(\mathcal{L}), \varphi\big)$ and $\big(w(C), w\big)$ are equivalent. Let us observe that if C is contained in $Z(X)$, then $\mathcal{L} = \big\{f \in \mathcal{C}_+(X) :\ z(f) \in C\big\}$ satisfies the properties (A), (B), (C) and (D). We also have, as in Theorem 2.20, that the compactifications $\big(P(\mathcal{L}), \varphi\big)$ and $\big(w(C), w\big)$ are equivalent.

b) Let A be a subring of $\mathcal{C}(X)$. An ideal I in A is a filter ideal ([2]) in A when $Z[I]$ is a $Z[A]$-filter. It is clear that an ideal I is a filter ideal in A if and only if $z(f) \neq \emptyset$ for every $f \in I$. The set of maximal filter ideals is denoted by $F[A]$ and $\{f^* :\ f \in A\}$, where $f^* = \big\{I \in F[A] :\ f \in I\big\}$, is a closed basis of some topology for $F[A]$. $F[A]$ is a Hausdorff compactification of X if and only if $Z[A]$ is a Wallman-base on X. In this case the compactifications $F[A]$ and $w\big(Z[A]\big)$ of X are equivalent.

c) Let us suppose that A is a subring of $\mathcal{C}_+(X)$ such that the properties (B), (C) and (D) hold. We have:

1) If I is a filter ideal in A, then $\{f \in A :\ f \notin I\}$ is a PA-filter.

2) If p is a minimal PA-filter, then $\{f \in A : f \notin q\}$ is a maximal filter ideal in A.
3) If I is a maximal filter ideal in A, then $\{f \in A : f \notin I\}$ is a minimal PA-filter.

From these results it is easy to deduce that the compactifications $F[A]$, $P[A]$ and $w(Z[A])$ of X are equivalent. Moreover, the mapping $\phi : F[A] \to P(A)$ defined by $\phi(I) = \{f \in A : f \notin I\}$ for every $I \in F[A]$, is a homeomorphism and $\phi \circ \psi = \varphi$, where ψ is the mapping that corresponds to the compactification $F[A]$.

4) In the appendix we characterize P-spaces and F-spaces through some results that evoke Gillman–Jerison [4].

3. $P_*(\mathcal{L})$ spaces in $\mathcal{C}_+(X)$

Let X be a completely regular Hausdorff space and let $\mathcal{C}^*(X)$ be the subring of bounded functions in $\mathcal{C}(X)$. Let $\mathcal{C}_+^*(X) = \mathcal{C}^*(X) \cap \mathcal{C}_+(X)$. We denote by P_* the set of $f \in \mathcal{C}_+(X)$ such that there exists $a > 0$, $a \in \mathbb{R}$, such that $f(x) > a$ for every $x \in X$.

It is clear that $P_* \cap \mathcal{C}_+^*(X)$ is the set of $f \in \mathcal{C}_+^*(X)$ such that f is invertible in the ring $\mathcal{C}^*(X)$.

Let $\mathcal{L} \subset \mathcal{C}_+(X)$ be such that $0 \in \mathcal{L}$ and $\mathcal{L} \cap P_* \neq \emptyset$ and let us consider the compact space $(P_*(\mathcal{L}), T_{P_*,\mathcal{L}})$. We will denote $o^*(f) = \{p \in P_*(\mathcal{L}) : f \in p\}$, $c^*(f) = \{p \in P_*(\mathcal{L}) : f \in \mathcal{L} \setminus p\}$ for every $f \in \mathcal{L}$.

Remark 3.1.

1) Conditions 1), 2) and 3) in Theorem 2.1 hold if we replace P by P_*. Condition 4) does not hold, see Example 3.14.
2) Theorem 2.4 also holds if we replace P by P_*, $o(f)$ by $o^*(f)$, $c(f)$ by $c^*(f)$ and $T_{P,\mathcal{L}}$ by $T_{P_*,\mathcal{L}}$.
3) For every $x \in X$, we have that $p_x = \{f \in \mathcal{L} : f(x) \neq 0\}$ is a $P_*\mathcal{L}$-filter.
4) The concept of multiplicative $P_*\mathcal{L}$-filter can be defined in a similar way to Definition 2.4.

It easy to check that the following result holds:

THEOREM 3.2. *Let $x \in X$. Then, p_x is a minimal $P_*\mathcal{L}$-filter if and only if for every $f \in \mathcal{L}$ such that $x \in \operatorname{coz}(f)$ there exists $g \in \mathcal{L}$ such that $x \in z(g)$ and $f + g \in P_*$.*

DEFINITION 3.3. We will say that $\mathcal{L}$ *has the* (B') *property* if for every $f \in \mathcal{L}$ and every $x \in \mathrm{coz}(f)$ there exists $g \in \mathcal{L}$ such that $f + g \in P_*$ and $x \in z(g)$.

It is clear that if $\mathcal{L}$ satisfies (B'), then it also satisfies (B). As in Theorem 2.8, the following result can be proved:

THEOREM 3.4. *Let us suppose that $\mathcal{L}$ has the properties* (A) *and* (B') *and that $\varphi_* : X \to P_*(\mathcal{L})$ is the mapping defined for every $x \in X$ by $\varphi_*(x) = p_x$. Then $(P_*(\mathcal{L}), \varphi_*)$ is a compactification of X if and only if $\mathcal{L}$ has the* (C) *property.*

DEFINITION 3.5. Let p be a $P_*\mathcal{L}$-filter. We will say that p is *fixed* if $\bigcap\limits_{f \in p} \mathrm{coz}(f) \neq \emptyset$. We will say that p is *free* if $\bigcap\limits_{f \in p} \mathrm{coz}(f) = \emptyset$.

Remark 3.6. If $\mathcal{L}$ satisfies the properties (B') and (C), then:

1) For every closed set $F \subset X$ and every $x \in X \setminus F$ there exist $g \in \mathcal{L}$ and $a \in \mathbb{R}$, $a > 0$, such that $x \in z(g)$ and $g(F) \subset (a, +\infty)$.
2) If p is a $P_*\mathcal{L}$-filter, then p is fixed if and only if $p = p_x$ for some $x \in X$. In this case $\bigcap\limits_{f \in \mathcal{L} \setminus p} z(f) = \{x\}$.
3) If p is a minimal $P_*\mathcal{L}$-filter, then either $\bigcap\limits_{f \in \mathcal{L} \setminus p} z(f) = \emptyset$ or $\bigcap\limits_{f \in \mathcal{L} \setminus p} z(f) = \{x\}$ for some $x \in X$. In the latter case $p = p_x$.
4) If $\mathcal{L}$ also satisfies (A), then X is compact if and only if every minimal $P_*\mathcal{L}$-filter is fixed.

DEFINITION 3.7. We will say that $\mathcal{L}$ *satisfies the* (D') *property* if for every $\{f, g\} \subset \mathcal{L}$ such that $f + g \in P_*$ there exists $\{f', g'\} \subset \mathcal{L}$ such that $\{f + f', g + g'\} \subset P_*$ and $f' \cdot g' = 0$.

THEOREM 3.8. *If $\mathcal{L}$ satisfies the properties* (A), (B'), (C) *and* (D'), *then $(P_*(\mathcal{L}), \varphi_*)$ is a Hausdorff compactification of X.*

Proof. If $\{p, q\} \subset P(\mathcal{L})$, $p \neq q$, then there exists some $f \in p \setminus q$. Since $f \in p$, there exists $g \in \mathcal{L} \setminus p$ such that $f + g \in P_*$. By hypothesis, there exists $\{f', g'\} \subset \mathcal{L}$ such that $\{f + f', g + g'\} \subset P_*$ and $f' \cdot g' = 0$. Therefore $q \in o^*(f')$, $p \in o^*(g')$ and $o^*(f') \cap o^*(g') = \emptyset$. □

Remark 3.9.

a) Let us suppose that $\mathcal{L}$ satisfies (D'). If $\{p, q\} \subset P_*(\mathcal{L})$ and $p \neq q$, then there exist $f \in p \setminus q$ and $g \in q \setminus p$ such that $f \cdot g = 0$. Hence, every multiplicative $P_*\mathcal{L}$-filter contains an unique minimal $P_*\mathcal{L}$-filter.

b) It is easy to prove that $C_+^*(X)$ satisfies the properties (A), (B'), (C) and (D').

THEOREM 3.10 (STONE-ČECH COMPACTIFICATION). $\big(P_*\big(\mathcal{C}^*_+(X)\big), \varphi_*\big)$ *is a Hausdorff compactification of X which is equivalent to $(\beta X, e)$.*

Proof. Let $z(f)$ and $z(g)$ be two disjoint zero-sets of X. We can suppose that $\{f,g\} \subset \mathcal{C}^*_+(X)$. If we consider the functions $h = \frac{f}{f+g}$, $h_1 = \big(h - \frac{1}{3}\big) \vee 0$, $h_2 = \big(\frac{2}{3} - h\big) \vee 0$, then we have that $h_1 + h_2 \in P_*$ and that $c^*(h_1) \cap c^*(h_2) = c^*(h_1 + h_2) = \emptyset$. Therefore $\varphi_*\big(z(f)\big) \subset c^*(h_1)$, $\varphi_*\big(z(g)\big) \subset c^*(h_2)$ and $\mathrm{cl}\big(\varphi_*\big(z(f)\big)\big) \cap \mathrm{cl}\big(\varphi_*\big(z(g)\big)\big) \subset c^*(h_1 + h_2) = \emptyset$. □

Remark 3.11. It is clear that $\big(P_*\big(\mathcal{C}_+(X)\big), \varphi_*\big)$ is equivalent to $(\beta X, e)$.

Now, we will give an explicit homeomorphism between the compactifications that appear in Theorem 3.10.

THEOREM 3.12. *The compactifications $(\beta X, e)$ and $\big(P_*\big(\mathcal{C}^*_+(X)\big), \varphi_*\big)$, of X are equivalent by means of the mapping $\phi\colon \beta X \to P_*\big(\mathcal{C}^*_+(X)\big)$ defined by $\phi(t) = \big\{f \in \mathcal{C}^*_+(X) :\ f^\beta(t) \neq 0\big\}$ for every $t \in \beta X$, where $f^\beta \in \mathcal{C}(\beta X)$ is such that $f^\beta \circ e = f$.*

Proof. We first show that $\phi(t) \in P_*\big(\mathcal{C}^*_+(X)\big)$ for every $t \in \beta X$. It is clear that $0 \notin \phi(t)$. Let $\{f_1, \dots, f_n\} \subset \mathcal{C}^*_+(X)$ be such that $f_1 + \dots + f_n \in P_*$. Clearly $f_1^\beta + \dots + f_n^\beta$ is invertible in $\mathcal{C}_+(\beta X)$ and therefore, there exists $i \in \{1, \dots, n\}$ such that $f_i^\beta(t) \neq 0$.

If $f \in \phi(t)$, then $f \notin z(f^\beta)$ and there exists some $h^\beta \in \mathcal{C}_+(\beta X)$ such that $h^\beta(t) = 0$ and $h^\beta\big(z(f^\beta)\big) = \{1\}$. Therefore, $h^\beta + f^\beta$ is invertible in $\mathcal{C}(\beta X)$. If $h = h^\beta \circ e$ and $f = f^\beta \circ e$, then we have that $f + h \in P_*$. Hence $\phi(t)$ is minimal.

Let us prove that ϕ is continuous. Let $(t_\alpha)_{\alpha \in I}$ be a net in βX that converges to $t \in \beta X$ and let $f \in \mathcal{C}^*_+(X)$ be such that $\phi(t) \in o^*(f)$. We have that $t \notin z(f^\beta)$ and therefore there exists $\alpha_0 \in I$ such that if $\alpha \geq \alpha_0$, $\alpha \in I$, then $t_\alpha \notin z(f^\beta)$. This proves that $\phi(t_\alpha) \in o^*(f)$ for every $\alpha \geq \alpha_0$, $\alpha \in I$.

Finally, it is clear that $\phi \circ e = \varphi_*$. □

The following theorem give us a method of obtaining any Hausdorff compactification of a completely regular space.

THEOREM 3.13. *Let (X', α) be a Hausdorff compactification of X and let $\mathcal{L} = \big\{f \in \mathcal{C}_+(X) :\ f = \overline{f} \circ \alpha,\ \overline{f} \in \mathcal{C}_+(X')\big\}$. Then $\big(P_*(\mathcal{L}), \varphi_*\big)$ is a Hausdorff compactification of X, that is equivalent to (X', α) by means of the mapping $\phi\colon X' \to P_*(\mathcal{L})$ defined by $\phi(x') = \big\{f \in \mathcal{L} :\ \overline{f}(x') \neq 0\big\}$ for every $x' \in X'$.*

Proof. It is clear that $\mathcal{L}$ satisfies the property (A).

Let us prove that $\mathcal{L}$ satisfies (B'). Let $f \in \mathcal{L}$, $x \in \mathrm{coz}(f)$ and $b \in \mathbb{R}$, $b > 0$, be such that $f(x) = b$. It is clear that $g = (b - f) \vee 0 \in \mathcal{L}$, $x \in z(g)$ and $f + g \in P_*$.

In order to prove that $\mathcal{L}$ has the property (C), let $G \subset X$ be an open set and let $x \in G$. There exists an open set $V \subset X'$ such that $\alpha(G) = V \cap \alpha(X)$ and there exists $\overline{f} \in \mathcal{C}_+(X')$ such that $\overline{f}(\alpha(x)) = 1$ and $\overline{f}(X' \setminus V) = \{0\}$. Therefore $f = \overline{f} \circ \alpha \in \mathcal{L}$ and $x \in \operatorname{coz}(f) \subset G$.

Let us check that $\mathcal{L}$ satisfies the property (D'). Let $\{f, g\} \subset \mathcal{L}$ be such that $f + g \in P_*$. There exists $a \in \mathbb{R}$, $a > 0$, such that $(f+g)(x) > a$ for every $x \in X$. Then $f' = \left(\frac{a}{3} - f\right) \vee 0 \in \mathcal{L}$, $g' = \left(\frac{a}{3} - g\right) \vee 0 \in \mathcal{L}$ and we have that $f' \cdot g' = 0$ and $\{f + f', g + g'\} \subset P_*$.

Finally, it is clear that ϕ is a homeomorphism and $\phi \circ \alpha = \varphi_*$. □

In the next example we will establish the essential differences between $P_*(\mathcal{L})$ and $P(\mathcal{L})$.

Example 3.14.

1) If p is a minimal $P_*\mathcal{L}$-filter, then $P_* \cap \mathcal{L} \subset p$. We will see that $f \in \mathcal{L}$ can be invertible in $\mathcal{C}(X)$, that is $f \in P \cap \mathcal{L}$, and, however, $f \notin p$. We consider ω with the discrete topology. It is easy to prove that the function $f \colon \omega \to \mathbb{R}$ defined for every $i \in \omega$ by $f(i) = 1/i$ does not belong to any free filter of $P_*(\mathcal{C}_+(\omega))$.

2) If A is a finite subset of ω and $p \in P_*(\mathcal{C}_+(\omega))$ is a free filter, then $\operatorname{coz}(\chi_{A^c}) \subset \operatorname{coz}(f)$, $\chi_{A^c} \in p$ and $f \notin p$, thus condition 4) in Theorem 2.1 does not hold.

3) Let (X', α) be a Hausdorff compactification of X and let $\mathcal{L} = \{f \in \mathcal{C}_+(X) : \overline{f} \circ \alpha = f,\ \overline{f} \in \mathcal{C}_+(X')\}$. Theorem 3.13 shows that $\mathcal{L}$ satisfies the properties (A), (B'), (C) and (D') and that $P_*(\mathcal{L})$ is equivalent to X'. On the other hand, it is easy to prove that $\mathcal{L}$ satisfies (D). Therefore, $P(\mathcal{L})$ is a compactification of X which, in general, does not coincide with X'. For instance, if we consider the spaces ω with the discrete topology $\gamma\omega = \omega \cup \{\infty\}$, the Alexandroff compactification of ω, and $\mathcal{L} = \{f \in \mathcal{C}_+(\omega) : \overline{f} \circ \gamma = f,\ \overline{f} \in \mathcal{C}_+(\gamma\omega)\}$, then it can be proved that $P(\mathcal{L})$ is equivalent to $\beta\omega$.

Let us suppose that $\mathcal{L} \subset \mathcal{C}_+(X)$ satisfies the properties (A), (B'), (C), (D) and (D'). The next theorem shows the relationship between the compactifications $(P(\mathcal{L}), \varphi)$ and $(P_*(\mathcal{L}), \varphi_*)$ of X.

Theorem 3.15. *If* $\mathcal{L} \subset \mathcal{C}_+(X)$ *satisfies properties* (A), (B'), (C), (D) *and* (D'), *then* $P_*(\mathcal{L}) \leq P(\mathcal{L})$.

Proof. If $p \in P(\mathcal{L})$, then p is a multiplicative $P_*\mathcal{L}$ filter. Let us consider the mapping $G \colon P(\mathcal{L}) \to P_*(\mathcal{L})$ such that, for every $p \in P(\mathcal{L})$, $G(p)$ is the unique minimal $P_*\mathcal{L}$-filter contained in p. For every $x \in X$, p_x is a minimal $P\mathcal{L}$-filter and a minimal $P_*\mathcal{L}$-filter. Therefore $G \circ \varphi = \varphi_*$. Finally, we will prove that if $\{\varphi(x_\alpha)\}_{\alpha \in I}$ is a net in $\varphi(X)$ that converges to $p \in P(\mathcal{L}) \setminus \varphi(X)$, then the net $\{\varphi_*(x_\alpha)\}_{\alpha\in I}$ converges to $G(p)$. Let $f \in \mathcal{L}$ be such that $G(p) \in o^*(f)$. Since $G(p) \subset p$, then $f \in p$ and $p \in o(f)$. Hence there exists $\alpha_0 \in I$ such that $\varphi(x_\alpha) \in o(f)$ for every $\alpha \geq \alpha_0$, $\alpha \in I$. Therefore $f(x_\alpha) \neq 0$ and $\varphi_*(x_\alpha) \in o^*(f)$ for every $\alpha \geq \alpha_0$, $\alpha \in I$. □

Remark 3.16.

1) In the proof of Theorem 3.15 we can check that if there exists a continuous mapping $g\colon P(\mathcal{L}) \to P_*(\mathcal{L})$ such that $g \circ \varphi = \varphi_*$, then $g(p)$ is the minimal $P_*\mathcal{L}$-filter contained in p for every $p \in P(\mathcal{L})$.

2) If $\mathcal{L} \subset \mathcal{C}_+(X)$ satisfies properties (A), (B'), (C), (D) and (D'), and if $P \cap \mathcal{L} = P_* \cap \mathcal{L}$, then $P(\mathcal{L}) = P_*(\mathcal{L})$ and $T_{P,\mathcal{L}} = T_{P_*,\mathcal{L}}$.

3) Let X be a locally compact Hausdorff space and let $\mathcal{L}$ be the set of the mappings $f \in \mathcal{C}_+(X)$ such that $f(X \setminus K)$ is a singleton for some compact $K \subset X$. We know that $\mathcal{L}$ satisfies (A), (B), (C) and (D) and it is easy to prove that $\mathcal{L}$ satisfies (B') and (D'). By Theorem 3.15, $\big(P_*(\mathcal{L}),\varphi_*\big) \leq \big(P(\mathcal{L}),\varphi\big)$ and $\big(P_*(\mathcal{L}),\varphi_*\big)$ is equivalent to the Alexandroff compactification of X.

In the next section we will try to obtain compactifications $\big(P_*(\mathcal{L}),\varphi_*\big)$ of X such that:

1) If $f \in \mathcal{L}$, then there exists $\overline{f} \in \mathcal{C}_+\big(P_*(\mathcal{L})\big)$ such that $\overline{f} \circ \varphi_* = f$.
2) If $\mathcal{L}_1 \subset \mathcal{L}_2$, then $\big(P_*(\mathcal{L}_1),\varphi_*^1\big) \leq \big(P_*(\mathcal{L}_2),\varphi_*^2\big)$.

Definition 3.17. We will say that $\mathcal{L} \subset \mathcal{C}_+(X)$ is a *c-set* of X if it satisfies:

i) $\mathcal{L} \subset \mathcal{C}_+^*(X)$.
ii) $\mathcal{L}$ satisfies the properties (A) and (C).
iii) If $f \in \mathcal{L}$ and $a \in \mathbb{R}$, then $(f-a)\vee 0 \in \mathcal{L}$ and $(a-f)\vee 0 \in \mathcal{L}$.

4. $P_*(\mathcal{L})$ spaces, where $\mathcal{L}$ is a c-set of X

If $\mathcal{L}$ is a c-set of X, then:

(a) $\mathcal{L}$ satisfies (D') because if $\{h_1,h_2\} \subset \mathcal{L}$ and $h_1 + h_2 \in P_*$, then there exists $a \in \mathbb{R}$, $a > 0$, such that $(h_1+h_2)(x) > a$ for every $x \in X$. Therefore $g_1 = \big(\frac{a}{2}-h_1\big)\vee 0 \in \mathcal{L}$, $g_2 = \big(\frac{a}{2}-h_2\big)\vee 0 \in \mathcal{L}$ and we have that $\{h_1+g_1, h_2+g_2\} \subset P_*$, $g_1 \cdot g_2 = 0$.

(b) $\mathcal{L}$ satisfies (B') because if $f \in \mathcal{L}$ and $f(x) = a$, $a \in \mathbb{R}$, $a > 0$, then $x \notin F = \{z \in X : f(z) \leq \frac{a}{2}\}$ and, by the properties of $\mathcal{L}$, there exist $g \in \mathcal{L}$ and $b \in \mathbb{R}$, $b > 0$, such that $g(x) = 0$, $g(F) = \{b\}$, and $f + g \in P_*$. Therefore, if $\mathcal{L}$ is a c-set of X, then $\big(P_*(\mathcal{L}),\varphi_*\big)$ is a Hausdorff compactification of X. Finally, it is clear that $\mathcal{C}_+^*(X)$ is a c-set of X.

THEOREM 4.1. *Let $\mathcal{L}$ be a c-set of X and let $(P_*(\mathcal{L}), \varphi_*)$ be the corresponding Hausdorff compactification of X. Then:*

a) *For every $f \in \mathcal{L}$ there exists $\overline{f} \in \mathcal{C}_+(P_*(\mathcal{L}))$ such that $\overline{f} \circ \varphi_* = f$.*

b) *For every $p \in P_*(\mathcal{L})$ we have that $f \in p$ if and only if $\overline{f}(p) \neq 0$.*

Proof.

a) We will prove that if $(\varphi_*(x_\alpha))_{\alpha \in I}$ is a net in $\varphi_*(X)$ that converges to $p \in P_*(\mathcal{L})$, then $(f(x_\alpha))_{\alpha \in I}$ converges in $\mathbb{R}$. If the net $(f(x_\alpha))_{\alpha \in I}$ does not converge in $\mathbb{R}$, then there exist two subnets $(f(x_\alpha^1))_{\alpha \in I_1}$ and $(f(x_\alpha^2))_{\alpha \in I_2}$ of $(f(x_\alpha))_{\alpha \in I}$ and two positive real numbers $m_1 < m_2$ such that $(f(x_\alpha^1))_{\alpha \in I_1}$ converges to m_1 and $(f(x_\alpha^2))_{\alpha \in I_2}$ converges to m_2. Let us consider the functions $g_1 = \left(f - \frac{m_1 + 2m_2}{3}\right) \vee 0$, $g_2 = \left(\frac{2m_1 + m_2}{3} - f\right) \vee 0$. These two functions are in $\mathcal{L}$. We can suppose that $g_1(x_\alpha) = 0$ for every $\alpha \in I_1$, and then $\varphi_*(x_\alpha) \in c^*(g_1)$. We also have that $\varphi_*(x_\alpha) \in c^*(g_2)$ for every $\alpha \in I_2$. Hence $p \in c^*(g_1) \cap c^*(g_2) = c^*(g_1 + g_2)$, which is false because $g_1 + g_2$ is invertible in $\mathcal{C}^*(X)$ and, therefore, $c^*(g_1 + g_2) = \emptyset$.

b) Since $f \in \mathcal{L}$ is invertible in $\mathcal{C}^*(X)$ if and only if $\overline{f}$ is invertible in $\mathcal{C}(P_*(\mathcal{L}))$, it is easy to prove that the set $H = \{f \in \mathcal{L} : \overline{f}(p) \neq 0\}$ is a minimal $P_*\mathcal{L}$-filter contained in p for every $p \in P_*(\mathcal{L})$. Hence, $H = p$. □

Remark 4.2.

1) If $\mathcal{L} \subset \mathcal{C}_+^*(X)$ is a c-set of X with the property (D), then we can consider the compactification $(P(\mathcal{L}), \varphi)$ of X and, as in part (a) of Theorem 4.1, it can be proved that if $f \in \mathcal{L}$, then there exists $\overline{\overline{f}} \in \mathcal{C}_+(P(\mathcal{L}))$ such that $\overline{\overline{f}} \circ \varphi = f$. We also have that if $g \colon P(\mathcal{L}) \to P_*(\mathcal{L})$ is the continuous mapping such that $g \circ \varphi = \varphi_*$ and $\overline{f} \in \mathcal{C}_+(P_*(\mathcal{L}))$ is such that $\overline{f} \circ \varphi_* = f$, then $\overline{f} \circ g = \overline{\overline{f}}$.

2) Let us observe that part (b) of Theorem 4.1 does not hold, in general, for the compactification $(P(\mathcal{L}), \varphi)$. To see this, let $(\gamma\omega, \gamma)$ be the Alexandroff compactification of ω and let $\mathcal{L} = \{f \in \mathcal{C}_+(\omega) : \overline{f} \circ \gamma = f,\ \overline{f} \in \mathcal{C}_+(\gamma\omega)\}$. We have that $\mathcal{L}$ is a c-set of ω that satisfies the property (D), $(P(\mathcal{L}), \varphi)$ is equivalent to $\beta\omega$ and there exists $\overline{\overline{f}} \in \mathcal{C}_+(P(\mathcal{L}))$ such that $\overline{\overline{f}} \circ \varphi = f$ for every $f \in \mathcal{L}$. Let $g \colon \omega \to \mathbb{R}$ be the function defined for every $i \in \omega$ by $g(i) = \frac{1}{i}$. For every $p \in P(\mathcal{L}) \setminus \varphi(\omega)$ we have that $g \in p$, because g is invertible in $\mathcal{C}(\omega)$, but $\overline{\overline{g}}(p) = 0$.

It is important to know if $(P_*(\mathcal{L}), \varphi_*)$ is equivalent to $(P(\mathcal{L}), \varphi)$ because in this case we would have that $(P_*(\mathcal{L}), \varphi_*)$ is a Wallman-type compactification.

THEOREM 4.3. *Let $\mathcal{L}$ be a c-set of X that satisfies the property* (D). *Then, $(P(\mathcal{L}), \varphi)$ is equivalent to $(P_*(\mathcal{L}), \varphi_*)$ if and only if for every $\{p_1, p_2\} \subset P(\mathcal{L})$,*

$p_1 \neq p_2$, there exists $\{f_1, f_2\} \subset \mathcal{L}$ such that $f_1 + f_2$ is invertible in $C^*(X)$, $f_1 \in p_1 \setminus p_2$ and $f_2 \in p_2 \setminus p_1$.

Proof. Let us suppose that both compactifications are equivalent and let $G\colon P(\mathcal{L}) \to P_*(\mathcal{L})$ be a continuous and bijective mapping such that $G \circ \varphi = \varphi_*$. If $\{p_1, p_2\} \subset \mathcal{L}$, $p_1 \neq p_2$, then $G(p_1) \neq G(p_2)$ and there exists $f \in \mathcal{L}$ such that $G(p_1) \in o^*(f)$ and $G(p_2) \notin o^*(f)$. Let $\overline{f} \in C_+(P_*(\mathcal{L}))$ be such that $\overline{f} \circ \varphi_* = f$. Clearly, $\overline{f}(G(p_2)) = 0$ and $\overline{f}(G(p_1)) = a > 0$. If we define $f_1 = (f - \frac{a}{3}) \vee 0$, $f_2 = (\frac{a}{2} - f) \vee 0$, then $\{f_1, f_2\} \subset \mathcal{L}$, $f_1 + f_2$ is invertible in $C^*(X)$, $f_1 \in G(p_1) \subset p_1$ and $f_2 \in G(p_2) \subset p_2$. Let us prove that $f_1 \notin p_2$ (similarly, we could prove that $f_2 \notin p_1$). If $f_1 \in p_2$, then $p_2 \in o(f_1)$ and there exists a net $\{\varphi(x_\alpha)\}_{\alpha \in I} \subset o(f_1)$ that converges to p_2. Hence, $f(x_\alpha) > \frac{a}{3}$ for every $\alpha \in I$, and $\overline{f}(G(p_2)) \geq \frac{a}{3}$, because $\{G(\varphi(x_\alpha))\}_{\alpha \in I} = \{\varphi_*(x_\alpha)\}_{\alpha \in I}$ converges to $G(p_2)$.

By Theorem 3.15, we know that $(P_*(\mathcal{L}), \varphi_*) \leq (P(\mathcal{L}), \varphi)$. Let $G\colon P(\mathcal{L}) \to P_*(\mathcal{L})$ be a surjective continuous mapping such that $G \circ \varphi = \varphi_*$. Let us check that G is one to one: let $\{p_1, p_2\} \subset P(\mathcal{L})$, $p_1 \neq p_2$; there exist $f_1 \in \mathcal{L}$ and $f_2 \in \mathcal{L}$ such that $f_1 \in p_1 \setminus p_2$, $f_2 \in p_2 \setminus p_1$ and $f_1 + f_2$ is invertible in $C^*(X)$. We have that $f_2 \notin G(p_1)$ and $f_1 \notin G(p_2)$. Hence, $f_1 \in G(p_1)$ and $f_2 \in G(p_2)$. □

The following theorem will allow us to compare the compactifications obtained through the c-sets of X.

THEOREM 4.4. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two c-sets of X. If for every $\{p_1, p_2\} \subset P_*(\mathcal{L}_1)$, $p_1 \neq p_2$, there exists $f \in \mathcal{L}_2$ such that $f \in p_1 \setminus p_2$, then $(P_*(\mathcal{L}_1), \varphi_*^1) \leq (P_*(\mathcal{L}_2), \varphi_*^2)$.*

Proof. We define $h\colon \varphi_*^2(X) \to P_*(\mathcal{L}_1)$ by $h(\varphi_*^2(x)) = \varphi_*^1(x)$ for every $x \in X$. Let $p \in P_*(\mathcal{L}_2) \setminus \varphi_*^2(X)$ and let $(\varphi_*^2(x_\alpha))_{\alpha \in I}$ be a net that converges in $P_*(\mathcal{L}_2)$ to p. If $(h(\varphi_*^2(x_\alpha)))_{\alpha \in I} = (\varphi_*^1(x_\alpha))_{\alpha \in I}$ does not converge in $P_*(\mathcal{L}_1)$, there are two subnets $(\varphi_*^1(x_\alpha^1))_{\alpha \in I_1}$ and $(\varphi_*^1(x_\alpha^2))_{\alpha \in I_2}$ of $(\varphi_*^1(x_\alpha))_{\alpha \in I}$ and two different elements q_1 and q_2 in $P_*(\mathcal{L}_1)$ such that $(\varphi_*^1(x_\alpha^1))_{\alpha \in I_1}$ converges to q_1 and $(\varphi_*^1(x_\alpha^2))_{\alpha \in I_2}$ converges to q_2. By hypothesis, there exists $f \in \mathcal{L}_2$ such that $f \in q_1 \setminus q_2$. Let $\overline{f} \in C(P_*(\mathcal{L}_1))$ be such that $\overline{f} \circ \varphi_*^1 = f$. There exists $a \in \mathbb{R}$, $a > 0$, such that $\overline{f}(q_1) = a$ and $f(x_\alpha^1) > 2a/3$ for every $\alpha \in I_1$. Since $f \in \mathcal{L}_2$, there exists $\overline{\overline{f}} \in C_+(P_*(\mathcal{L}_2))$ such that $\overline{\overline{f}} \circ \varphi_*^2 = f$. Then $\overline{\overline{f}}(p) \geq 2a/3$. Since $f(q_2) = 0$, we can suppose that $f(x_\alpha^2) < a/3$ and $\overline{\overline{f}}(p) \leq a/3$ for every $\alpha \in I_2$. □

Remark 4.5. The following statements are consequences of Theorem 4.4.

(a) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are c-sets of X and $\mathcal{L}_1 \subseteq \mathcal{L}_2$, then $P_*(\mathcal{L}_1) \leq P_*(\mathcal{L}_2)$.

(b) If $\mathcal{L}$ is a c-set of X, then $(P_*(\mathcal{L}), \varphi_*)$ is equivalent to $(\beta X, e)$ if and only if for every $\{q_1, q_2\} \subset P_*(\mathcal{C}_+^*(X))$, $q_1 \neq q_2$, there exists $f \in \mathcal{L}$ such that $f \in q_1 \setminus q_2$.

(c) Let (X', α) be a Hausdorff compactification of X. Let $\mathcal{L}$ be a c-set of X. The compactifications $(P_*(\mathcal{L}), \varphi_*)$ and (X', α) are equivalent if and only if the two following conditions are satisfied:

(1) For every $f \in \mathcal{L}$ there exists $\overline{f} \in \mathcal{C}(X')$ such that $\overline{f} \circ \alpha = f$.

(2) $\{z(\overline{g})\}_{g \in \mathcal{L}}$ is a closed basis of X'.

If $(P_*(\mathcal{L}), \varphi_*)$ is equivalent to $(\beta X, e)$, then $(P_*(\mathcal{C}_+^*(X)), \varphi_*^o)$ is equivalent to $(P_*(\mathcal{L}), \varphi_*)$ and there exists a homeomorphism $\Psi\colon P_*(\mathcal{C}_+^*(X)) \to P_*(\mathcal{L})$ such that $\Psi \circ \varphi_*^o = \varphi_*$. If $\{q_1, q_2\} \subset P_*(\mathcal{C}_+^*(X))$, $q_1 \neq q_2$, then $\Psi(q_1) \neq \Psi(q_2)$ and there exists $f \in \mathcal{L}$ such that $f \in \Psi(q_1) \setminus \Psi(q_2)$. Let $\overline{f} \in \mathcal{C}_+(P_*(\mathcal{L}))$ and $\overline{\overline{f}} \in \mathcal{C}_+(P_*(\mathcal{C}_+^*(X)))$ be such that $\overline{f} \circ \varphi_* = f$ and $\overline{\overline{f}} \circ \varphi_*^o = f$. We have that $\overline{f}(\Psi(q_1)) \neq 0$ and $\overline{f}(\Psi(q_2)) = 0$. Since $\overline{f} \circ \Psi = \overline{\overline{f}}$, then $\overline{\overline{f}}(q_1) \neq 0$ and $\overline{\overline{f}}(q_2) = 0$ and therefore $f \in q_1 \setminus q_2$. This proves (b).

For (c), let us assume that (1) and (2) hold, then for every $x' \in X'$ we can check that $\phi(x') = \{f \in \mathcal{L} : \overline{f}(x') \neq 0\}$ is a minimal $P_*\mathcal{L}$-filter and then we can consider the mapping $\phi\colon X' \to P_*(\mathcal{L})$ defined for every $x' \in X'$ by $\phi(x')$. It is plain to prove that ϕ is an one to one continuous map and that $\phi \circ \alpha = \varphi_*$.

Conversely, let us suppose that (X', α) and $(P_*(\mathcal{L}), \varphi_*)$ are equivalent. There exists a homeomorphism Ψ of X' onto $P_*(\mathcal{L})$ such that $\Psi \circ \alpha = \varphi_*$. For every $f \in \mathcal{L}$ there exists $\overline{\overline{f}} \in \mathcal{C}_+(P_*(\mathcal{L}))$ such that $\overline{\overline{f}} \circ \varphi_* = f$, $\overline{\overline{f}} \circ \Psi \in \mathcal{C}(X')$ and $\overline{\overline{f}} \circ \Psi \circ \alpha = f$. On the other hand, let F be a closed set in X' and let $x' \in X' \setminus F$. Then $\Psi(x') \notin \Psi(F)$ and there exists $f \in \mathcal{L}$ such that $\Psi(x') \notin c^*(f)$ and $\Psi(F) \subset c^*(f)$. By Theorem 4.1, we deduce that $\overline{\overline{f}}(\Psi(x')) \neq 0$ and that $\overline{\overline{f}}(\Psi(F)) = \{0\}$. Therefore $x' \notin z(\overline{\overline{f}} \circ \Psi)$ and $F \subset z(\overline{\overline{f}} \circ \Psi)$.

Belley and Lessard [1], [5] have studied a method of obtaining a compact space by using a non-empty set X and a set V of bounded continuous real-valued functions on X. An outline of the method is the following: for every $x \in X$, they consider the mapping $\delta_x\colon V \to \mathbb{R}$ defined by $\delta_x(f) = f(x)$ for every $f \in V$. Let $D_f \subset \mathbb{R}$ be a compact set such that $\mathrm{Im}(f) \subset D_f$. The set $\prod_{f \in V} D_f$, endowed with the product topology, is a compact space. Moreover, the space $X_V^o = \overline{\{\delta_x,\ x \in X\}} \subset \prod_{f \in V} D_f$ is a compact Hausdorff space with the induced topology. In these papers they prove that X_V^o is the set of functions $\nu\colon V \to \mathbb{R}$ such that there exists a net $(x_\alpha)_{\alpha \in I} \subset X$ such that $(\delta_{x_\alpha}(f))_{\alpha \in I}$ converges to $\nu(f)$ for every $f \in V$. If $j\colon X \to X_V^o$ is the mapping defined by

$j(x) = \delta_x$ for every $x \in X$, then $j(X)$ is dense in X_V^o and (X_V^o, j) is a Hausdorff compactification of X if and only if V satisfies the three following conditions:

(a) $V \subset \mathcal{C}^*(X)$,
(b) V separates the points of X,
(c) for every $x \in X$ and every neighborhood U of x there exists $\{f_1, \dots, f_n\} \subset V$ such that $(f_1(x), \dots, f_n(x))$ is not in the closure of the set $\{(f_1(y), \dots, f_n(y)) : y \in X \setminus U\}$.

THEOREM 4.6. *Let $\mathcal{L}$ be a c-set of X. Then:*

1) *$(X_{\mathcal{L}}^o, j)$ is a Hausdorff compactification of X.*
2) *The compactifications $(X_{\mathcal{L}}^o, j)$ and $(P_*(\mathcal{L}), \varphi_*)$ are equivalent.*

Proof.

1) Clearly, j is one-one. Let us show that j is continuous: if $(x_\alpha)_{\alpha \in I}$ is a net in X that converges to $x \in X$, then the net $(f(x_\alpha))_{\alpha \in I}$ converges to $f(x)$ for every $f \in \mathcal{L}$, and then $(\delta_{x_\alpha})_{\alpha \in I}$ converges to δ_x. The mapping j^{-1} is also continuous because if $(\delta_{x_\alpha})_{\alpha \in I}$ is a net that converges to δ_x, $x \in X$ and V_x is a neighborhood of x in X, then there exists $f \in \mathcal{L}$ and $a \in \mathbb{R}$, $a > 0$, such that $f(x) = 0$ and $f(V_x^c) = \{a\}$. Since $(f(x_\alpha))_{\alpha \in I}$ converges to $f(x)$, there exists $\alpha_0 \in I$ such that $f(x_\alpha) < a$ for every $\alpha \geq \alpha_0$. Therefore $x_\alpha \in V_x$ if $\alpha \geq \alpha_0$, $\alpha \in I$.

2) Let us consider the mapping $l \colon X_{\mathcal{L}}^o \to P_*(\mathcal{L})$ defined for every $\nu \in X_{\mathcal{L}}^o$ by $l(\nu) = p_\nu$, where $p_\nu = \{f \in \mathcal{L} : \nu(f) \neq 0\}$. Let us show that p_ν is a minimal $P_*\mathcal{L}$-filter. Clearly $0 \notin p_\nu$ and if $\{f_1, \dots, f_n\} \subset \mathcal{L}$, $f_1 + \dots + f_n \in P_*$, then there exists $a \in \mathbb{R}$, $a > 0$, such that $(f_1 + \dots + f_n)(x) > a$ for every $x \in X$. Since $\nu \in X_{\mathcal{L}}^o$ there exists a net $(x_\alpha)_{\alpha \in I}$ in X such that $(f_1(x_\alpha) + \dots + f_n(x_\alpha))_{\alpha \in I}$ converges to $\nu(f_1 + \dots + f_n)$. Then $\nu(f_1 + \dots + f_n) = \nu(f_1) + \dots + \nu(f_n) \geq a$ and there exists $i \in \{1, \dots, n\}$ such that $\nu(f_i) \neq 0$. To prove the minimality of p_ν, let $f \in p_\nu$ be such that $\nu(f) = a$, $a \in \mathbb{R}$, $a > 0$. Then $g = \left(\frac{a}{2} - f\right) \vee 0 \in \mathcal{L}$ and we have that $f + g \in P_*$ and $\nu(g) = 0$. Therefore $g \notin p_\nu$.

3) Let us prove that l is one-one. Let $\{\nu, \nu'\} \subset X_{\mathcal{L}}^o$ be such that $\nu \neq \nu'$. We can assume the existence of $\{a, b\} \subset \mathbb{R}$ and $f \in \mathcal{L}$ such that $\nu(f) < a < b < \nu'(f)$. Let $t = \left(f - \frac{a+b}{2}\right) \vee 0$. Then $t \in \mathcal{L}$, $\nu(t) = 0$ and $\nu'(t) \neq 0$. Therefore $t \in p_{\nu'} \setminus p_\nu$.

4) In order to prove that l is continuous, let us consider a net $(\nu_\alpha)_{\alpha \in I}$ in $X_{\mathcal{L}}^o$ converging to $\nu \in X_{\mathcal{L}}^o$ and let $f \in \mathcal{L}$ be such that $l(\nu) \in o^*(f)$. Clearly, $\nu(f) \neq 0$ and there exists $\alpha_0 \in I$ such that $\nu_\alpha(f) \neq 0$ for every $\alpha \in I$ such that $\alpha \geq \alpha_0$. Therefore, $l(\nu_\alpha) \in o^*(f)$. Finally, it is clear that $l \circ j = \varphi_*$. □

5. Countable compactifications

Let X be a Hausdorff space. It was shown, in [6], that the two following statements are equivalent (Magill's theorem):

(a) X has a n-point Hausdorff compactification.

(b) X is locally compact and contains n mutually disjoint, open sets $\{G_1, \dots, G_n\}$ such that $X \setminus \bigcup_{i=1}^{n} G_i$ is compact but $X \setminus \bigcup_{j \neq i} G_j$ is not compact for every $i \in \{1, \dots, n\}$.

A family of open sets $\{G_1, \dots, G_n\}$ with these properties is called an n*-star* of X.

Now, we will give a brief description of this compactification. Let $\{G_1, \dots, G_n\}$ be an n-star of X and let $K = X \setminus \bigcup_{i=1}^{n} G_i$. Let $\{z_1, \dots, z_n\}$ be a set with n distinct elements that are not in the space X. Let $Z = X \cup \{z_1, \dots, z_n\}$ and

$$\begin{aligned} B = \{H \subset X :\ & H \text{ is open in } X\} \\ \cup \big\{H \cup \{z_i\} :\ & i \in \{1, \dots, n\},\ H \subset X \text{ is open in } X \\ & \text{and } (K \cup G_i) \cap (X \setminus H) \text{ is compact in } X\big\}, \end{aligned}$$

then B is an open basis for a topology for Z. Let $\alpha \colon X \to Z$ be the mapping defined by $\alpha(x) = x$ for every $x \in X$. Then (Z, α) is a n-point Hausdorff compactification of X that will be called as the compactification induced by the n-star $\{G_1, \dots, G_n\}$. Let us observe that in Z there exist n open sets $U_1, \dots, U_n$, whose closures are mutually disjoint, such that $z_i \in U_i$ for every $i \in \{1, \dots, n\}$. Hence, there exist n different numbers $a_1, \dots, a_n$ and $\overline{f} \in \mathcal{C}_+(Z)$ such that $\overline{f}(U_i) = \{a_i\}$ if $i \in \{1, \dots, n\}$. Let us denote $V_i = U_i \setminus \{z_i\}$ and $f = \overline{f} \circ \alpha$. We have that $\{V_1, \dots, V_n\}$ is a n-star of X such that there exist $f \in \mathcal{C}_+^*(X)$ and n different numbers $a_1, \dots, a_n$ such that $f(V_i) = \{a_i\}$ for every $i \in \{1, \dots, n\}$. In this case, the family $\{V_1, \dots, V_n\}$ will be called a separated n-star. It is easy to prove that the n-point compactification induced by $\{V_1, \dots, V_n\}$ is equivalent to the compactification induced by $\{G_1, \dots, G_n\}$. We can conclude that any n-point compactification is equivalent to the one induced by a separated n-star.

It should be mentioned that Magill's theorem ([6]) is still true if we change, in condition (b), the term n-star by the term separated n-star. For instance, if we consider in $\mathbb{R}$ the 2-star $\big\{(-\infty, 0), (0, +\infty)\big\}$, then a 2-point compactification is induced that is equivalent to the one induced by the 2-star $\big\{(-\infty, -1), (1, +\infty)\big\}$. The second 2-star is separated but the first is not separated.

THEOREM 5.1. *Let X be a locally compact Hausdorff space. Let $\{G_1,\dots,G_n\}$ be a separated n-star of X and let (Z,α) be the n-point compactification induced by $\{G_1,\dots,G_n\}$, where $Z = X\cup\{z_1,\dots,z_n\}$. Let $\mathcal{L}$ be the set of functions $f\in C^*_+(X)$ such that for every $i\in\{1,\dots,n\}$ there exists a compact set $K_i\subset G_i\cup M$, where $M = X\setminus\bigcup_{i=1}^{n} G_i$ is compact in X, such that f is constant in $G_i\setminus K_i$. Then $\mathcal{L}$ is a c-set of X and the compactifications $\big(P_*(\mathcal{L}),\varphi_*\big)$ and (Z,α) are equivalent.*

Proof. It is obvious that if $\{f,g\}\subset\mathcal{L}$, then $\{f\pm g, f\cdot g, f\vee g, f\wedge g\}\subset\mathcal{L}$. We will prove that $\{\mathrm{coz}(f) : f\in\mathcal{L}\}$ is an open basis of X. Let U be an open subset of X and let $x\in U$. Since X is locally compact, there exist an open set $V\subset X$ and a compact K in X such that $x\in V\subset K\subset U$. Let $f\in C^*_+(X)$ be such that $f(x)=1$ and $f(V^c)=\{0\}$. For every $i\in\{1,\dots,n\}$ we have that $G_i\cup M$ is compact and $\overline{G}_i\cap K\subset G_i\cup M$. Hence, $\overline{G}_i\cap K$ is a compact subset of X and $f(G_i\setminus\overline{G}_i\cap K)=\{0\}$. Therefore $f\in\mathcal{L}$, $x\in\mathrm{coz}(f)\subset U$ and we obtain the compactification $\big(P_*(\mathcal{L}),\varphi_*\big)$ of X. Since M is compact in X and $\mathrm{cl}\big(\varphi_*(M)\big)\subset\varphi_*(X)$, we have that

$$P(\mathcal{L})\setminus\varphi_*(X) = \big[\mathrm{cl}\big(\varphi_*(G_1)\big)\setminus\varphi_*(X)\big]\cup\dots\cup\big[\mathrm{cl}\big(\varphi_*(G_n)\big)\setminus\varphi_*(X)\big].$$

Let us prove that, for every $i\in\{1,\dots,n\}$, $\mathrm{cl}\big(\varphi_*(G_i)\big)\setminus\varphi_*(X)$ is an unitary set. If $\mathrm{cl}\big(\varphi_*(G_i)\big)\setminus\varphi_*(X)=\emptyset$, then we would have that $\mathrm{cl}\big(\varphi_*(G_i)\big)$ is compact in $\varphi_*(X)$, $\varphi_*(\overline{G}_i)$ is compact in $\varphi_*(X)$ and $\overline{G}_i$ is compact in X. Since $G_i\cup M$ is closed and $G_i\cup M\subset\overline{G}_i\cup M$, then we would have that $G_i\cup M$ is compact, which is false.

Let us suppose that $p\in\mathrm{cl}\big(\varphi_*(G_i)\big)\setminus\varphi_*(X)$ and let $f\in\mathcal{L}$ be such that $p\in o^*(f)$. Since $f\in\mathcal{L}$, there exists a compact K_i in X such that $K_i\subset G_i\cup M$ and there exists $a_i\in\mathbb{R}$ such that $f(G_i\setminus K_i)=\{a_i\}$. If $a_i=0$, then $f(G_i\setminus K_i)=\{0\}$ and $\big[\mathrm{cl}\big(\varphi_*(G_i)\big)\setminus\varphi_*(X)\big]\cap o^*(f)=\emptyset$. Therefore $a_i\neq 0$ and we can see that $\mathrm{cl}\big(\varphi_*(G_i)\big)\setminus\varphi_*(X)\subset o^*(f)$. This proves that $\mathrm{cl}\big(\varphi_*(G_i)\big)\setminus\varphi_*(X)$ is an unitary set, which will be denoted by p_i for every $i\in\{1,\dots,n\}$.

Let us prove that if $\{i,j\}\subset\{1,\dots,n\}$, $i\neq j$, then $p_i\neq p_j$. Since $\{G_1,\dots,G_n\}$ is a separated n-star there exists $g\in C^*_+(X)$ such that $g(G_i)=a_i$ for every $i\in\{1,\dots,n\}$, where $a_i\neq a_j$ if $i\neq j$; clearly $g\in\mathcal{L}$. Let us suppose that $a_i<a_j$. If $h=(g-a_i)\vee 0$, then $h\in\mathcal{L}$ and there exists $\overline{h}\in C_+\big(P_*(\mathcal{L})\big)$ such that $\overline{h}\circ\varphi_* = h$. It is clear that $\overline{h}(p_i)=0$ and $\overline{h}(p_j)\neq 0$. Therefore $h\in p_j\setminus p_i$.

Now, we prove that the compactifications $\big(P_*(\mathcal{L}),\varphi_*\big)$ and (Z,α) are equivalent. We consider the mapping $l\colon Z\to P_*(\mathcal{L})$ defined for every $x\in X$ by $l(x)=l\big(\alpha(x)\big)=\varphi_*(x)$ and, for $i\in\{1,\dots n\}$, by $l(z_i)=p_i$. We have that l is one to one and onto and that $l\circ\alpha=\varphi_*$. Let us check that l is continuous.

Let $i \in \{1, \dots, n\}$ and let $(x_\alpha)_{\alpha \in I}$ be a net in X that converges to z_i. Since $G_i \cup \{z_i\}$ is an open neighborhood of z_i in Z, let us suppose that $x_\alpha \in G_i$ for every $\alpha \in I$. Let $f \in \mathcal{L}$ be such that $p_i \in o^*(f)$. We have that there exist a compact set $K_i \subset G_i \cup M$ and $a_i \in \mathbb{R} \setminus \{0\}$ such that $f(G_i \setminus K_i) = \{a_i\}$. Since $(G_i \setminus K_i) \cup \{z_i\}$ is a neighborhood of z_i in Z, then there exists $\alpha_0 \in I$ such that $x_\alpha \in G_i \setminus K_i$ for every $\alpha \geq \alpha_0$, $\alpha \in I$.

Therefore $f(x_\alpha) = a_i$ and $l(x_\alpha) \in o^*(f)$ for every $\alpha \geq \alpha_0$. □

Remark 5.2. Let us observe that, under the hypotheses of Theorem 5.1, we have that $\mathcal{L} \cap P = \mathcal{L} \cap P_*$. Hence $\big(P(\mathcal{L}), \varphi\big)$ and $\big(P_*(\mathcal{L}), \varphi_*\big)$ are equivalent and (Z, α) is a Wallman-type compactification.

Let (Z, α) be a compactification of X. If $Z \setminus \alpha(X)$ is finite, then we have that $Z \setminus \alpha(X)$ is compact. If $Z \setminus \alpha(X)$ is compact, then X is locally compact. The spaces X such that the corresponding (Z, α) are compact and countable were characterized in [7]. Now, let us investigate when $Z \setminus \alpha(X)$ is homeomorphic to $\gamma\omega$, the Alexandroff compactification of ω.

Theorem 5.3. *Let X be a locally compact Hausdorff space and let (Z, α) be a compactification of X such that $Z \setminus \alpha(X)$ is homeomorphic to $\gamma\omega$. Then there exist two sequences, $(G_i)_{i \in \mathbb{N}}$ and $(B_i)_{\substack{i \in \mathbb{N} \\ i \geq 2}}$ of open sets of X such that:*

1) *For every $n \in \mathbb{N}$, $\{G_1, \dots, G_n, B_{n+1}\}$ is a separated $(n+1)$-star of X.*
2) $\bigcup_{k \geq n+1} G_k \subset B_{n+1}$.
3) *If $\{i, j\} \subset \mathbb{N}$ and $1 < i < j$, then $B_j \subset B_i$.*

Proof. We denote $Z \setminus \alpha(X) = \mathbb{N} \cup \{\infty\}$ and we define $f \colon Z \setminus \alpha(X) \to \mathbb{R}$ by $f(i) = 1/i$ for every $i \in \mathbb{N}$, and $f(\infty) = 0$. The continuous extension of f to Z will be also denoted by f. Let $(\varepsilon_i)_{i \in \mathbb{N}} \subset (0,1)$ be a sequence such that if $E_i = \left(\frac{1}{i} - \varepsilon_i, \frac{1}{i} + \varepsilon_i\right)$, then $\overline{E}_i \cap \overline{E}_j = \emptyset$ for $\{i,j\} \subset \mathbb{N}$ and $i \neq j$. We set $D_i = \left[0, \frac{1}{i} + \varepsilon_i\right)$. For every $i \in \mathbb{N}$, let $G_i = (f \circ \alpha)^{-1}(E_i)$ and let $B_{i+1} = (f \circ \alpha)^{-1}(D_{i+1})$. We will prove that $\{G_1, \dots, G_n, B_{n+1}\}$ is a separated $(n+1)$-star of X for every $n \in \mathbb{N}$. We have that,

$$Z \setminus [f^{-1}(E_1) \cup \dots \cup f^{-1}(E_n) \cup f^{-1}(D_{n+1})]$$

is compact in Z and is contained in $\alpha(X)$. Hence,

$$\alpha^{-1}\big[Z \setminus \big(f^{-1}(E_1) \cup \dots \cup f^{-1}(E_n) \cup f^{-1}(D_{n+1})\big)\big] = X \setminus (G_1 \cup \dots \cup G_n \cup B_{n+1})$$

is a compact set that will be denoted by M_{n+1}.

Let us prove that $M_{n+1} \cup G_j$ for $j \in \{1, \dots, n\}$, and that $M_{n+1} \cup B_{n+1}$ are not compact subsets of X. If $M_{n+1} \cup G_j$ is compact in X, then $Z \setminus \big(\alpha(M_{n+1}) \cup \alpha(G_j)\big)$ is open in Z and so $\big[Z \setminus \big(\alpha(M_{n+1}) \cup \alpha(G_j)\big)\big] \cap f^{-1}(E_j) = \{j\}$ is open, which is false. We also have that $\left[Z \setminus \left(\alpha(M_{n+1}) \cup \alpha(B_{n+1})\right)\right] \cap f^{-1}(D_{n+1}) = \{m \in \mathbb{N} : m \geq n+1\} \cup \{\infty\}$ and therefore, $M_{n+1} \cup B_{n+1}$ is not compact in X. It is clear how to prove (2) and (3). □

DEFINITION 5.4. Let X be a topological space. Any pair of sequences of open sets in X which satisfy the properties 1), 2), 3) in Theorem 5.3 will be called an *ω-star* of X.

THEOREM 5.5. *Let X be a locally compact space such that there exists an ω-star $(G_i)_{i\in\mathbb{N}}$, $(B_i)_{i\geq 2}$ of X. We denote $M_{i+1} = X \setminus G_1 \cup G_2 \cup \cdots \cup G_i \cup B_{i+1}$ for every $i \in \mathbb{N}$. Let $\mathcal{L}$ be the set of the functions $f \in \mathcal{C}_+^*(X)$ such that there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, $n \in \mathbb{N}$, then there exists a family of compact sets $K_1, \dots, K_{n+1}$ in X such that $K_i \subset G_i \cup M_{n+1}$, f is constant in $G_i \setminus K_i$ for every $i \in \{1, \dots, n\}$, $K_{n+1} \subset B_{n+1} \cup M_{n+1}$ and f is constant in $B_{n+1} \setminus K_{n+1}$. Then, $\mathcal{L}$ is a c-set of X and $P_*(\mathcal{L}) \setminus \varphi_*(X)$ is homeomorphic to $\gamma\omega$.*

Proof. It is clear that if $\{f, g\} \subset \mathcal{L}$, then $\{f \pm g, f \cdot g, f \vee g, f \wedge g\} \subset \mathcal{L}$. Let us prove that $\{\mathrm{coz}(f) : f \in \mathcal{L}\}$ is an open basis of X. Let U be an open set in X, and let $x \in U$. Since X is locally compact, there exist an open set V and a compact set K such that $x \in V \subset K \subset U$. Let $f \in \mathcal{C}_+^*(X)$ be such that $f(x) = 1$ and $f(V^c) = \{0\}$. For every $n \in \mathbb{N}$ and every $i \in \{1, \dots, n\}$ we have that $\overline{G}_i \cap K$ is a compact subset of $G_i \cup M_{n+1}$ and $f\left(G_i \setminus \overline{G}_i \cap K\right) = \{0\}$. We also have that $\overline{B}_{n+1} \cap K$ is a compact subset of $B_{n+1} \cup M_{n+1}$ and $f\left(B_{n+1} \setminus \overline{B}_{n+1} \cap K\right) = \{0\}$. Therefore $f \in \mathcal{L}$ and $x \in \mathrm{coz}(f) \subset U$.

Since $\mathcal{L}$ is a c-set of X, we obtain the compactification $\left(P_*(\mathcal{L}), \varphi_*\right)$ of X and, as in Theorem 5.1, we have that, for every $i \in \mathbb{N}$, $\mathrm{cl}\left(\varphi_*(G_i)\right) \setminus \varphi_*(X)$ is an unitary set, which will be denoted by $\{p_i\}$. Let us prove that if $\{i, j\} \subset \mathbb{N}$ and $i < j$, then $p_i \neq p_j$. We have that $\{G_1, \dots, G_i, \dots, G_j, B_{j+1}\}$ is a separated $(j+1)$-star of X and there exist $f \in \mathcal{C}_+^*(X)$ and $\{a_1, a_2, \dots, a_{j+1}\} \subset \mathbb{R}$ such that if $k \in \{1, \dots, j\}$, then $f(G_k) = \{a_k\}$ and $f(B_{j+1}) = \{a_{j+1}\}$. Since $\bigcup\limits_{k=j+1}^{n} G_k \subset B_{j+1}$ and $B_{n+1} \subset B_{j+1}$ for every $n > j$, we have that f is constant in G_i for $i \in \{1, \dots, n\}$, and f is constant in B_{n+1}. Therefore $f \in \mathcal{L}$.

Now, let us suppose that $a_i < a_j$. If $h = (f - a_i) \vee 0$, then $h \in \mathcal{L}$. Let $\overline{h} \in \mathcal{C}_+\left(P_*(\mathcal{L})\right)$ be such that $\overline{h} \circ \varphi_* = h$. We have that $\overline{h}(p_i) = 0$ and $\overline{h}(p_j) \neq 0$; therefore $h \in p_j \setminus p_i$.

Let us prove that $\{p_i : i \in \mathbb{N}\}$ is a discrete subset of $P_*(\mathcal{L})$. Let $i \in \{1 \dots, n\}$ and let us consider the $(i+1)$-star $\{G_1, \dots, G_i, B_{i+1}\}$. There exist $f \in \mathcal{C}^*(X)$ and $\{a_1, \dots, a_{i+1}\} \subset \mathbb{R}$ such that $f(G_1) = \{a_1\}$, $\dots$, $f(G_i) = \{a_i\}$, $f(B_{i+1}) = \{a_{i+1}\}$. We have that $f \in \mathcal{L}$. Let us suppose that $a_i < a_{i+1}$ and let $g = (f - a_i) \vee 0 \in \mathcal{L}$ and $\overline{g} \in \mathcal{C}_+\left(P_*(\mathcal{L})\right)$ be such that $\overline{g} \circ \varphi_* = g$. Then,

$\overline{g}(\mathrm{cl}(\varphi(G_i))) = \{0\}$ and $\overline{g}(\mathrm{cl}(\varphi_*(B_{i+1}))) = \{a_{i+1} - a_i\}$. The set

$$A = \{p \in P_*(\mathcal{L}) : \ \overline{g}(p) < (a_{i+1} - a_i)/2\} \setminus \{p_1, p_2, \dots, p_{i-1}\}$$

is open in $P_*(\mathcal{L})$ and $A \cap \{p_i : \ i \in \mathbb{N}\} = \{p_i\}$.

Let $K = P_*(\mathcal{L}) \setminus \varphi_*(X)$, then K is compact and we have that $K \setminus \{p_i : \ i \in \mathbb{N}\} \neq \emptyset$. We claim that $K \setminus \{p_i : \ i \in \mathbb{N}\}$ is an unitary set and K is homeomorphic to $\gamma\omega$. Let $\{p, q\} \subset K \setminus \{p_i : \ i \in \mathbb{N}\}$ and let $f \in \mathcal{L}$ be such that $p \in o^*(f)$. Let $n \in \mathbb{N}$ be such that there exists a family of compact sets $K_1, \dots, K_{n+1}$ such that $K_i \subset G_i \cup M_{n+1}$, f is constant in $G_i \setminus K_i$, $K_{n+1} \subset B_{n+1} \cup M_{n+1}$ and f is constant in $B_{n+1} \setminus K_{n+1}$ for $i \in \{1, \dots, n\}$. We have that

$$\begin{aligned} K &= P_*(\mathcal{L}) \setminus \varphi_*(X) \\ &= \left[\bigcup_{i=1}^{n} \Big(\mathrm{cl}(\varphi_*(G_i)) \setminus \varphi_*(X) \Big) \right] \cup \Big[\mathrm{cl}(\varphi_*(B_{n+1})) \setminus \varphi_*(X) \Big] \\ &= \{p_1, \dots, p_n\} \cup \Big[\mathrm{cl}(\varphi_*(B_{n+1})) \setminus \varphi_*(X) \Big]. \end{aligned}$$

Therefore $\{p, q\} \subset \mathrm{cl}(\varphi_*(B_{n+1})) \setminus \varphi_*(X)$, $\overline{f}(p) = \overline{f}(q)$ and $q \in o^*(f)$, which implies that $p = q$. □

Remark 5.6.

a) Let X be a locally compact space and let (Z, α) be a compactification of X such that $Z \setminus \alpha(X)$ is homeomorphic to $\gamma\omega$ and let $(G_i)_{i \in \mathbb{N}}$, $(B_i)_{i \geq 2}$ be the ω-star obtained in the proof of Theorem 5.3. Let $\mathcal{L}$ be the subset of $\mathcal{C}_+^*(X)$ defined in Theorem 5.5. It is not difficult to prove that $(P_*(\mathcal{L}), \varphi_*)$ and (Z, α) are equivalent.

b) If $h \in \mathcal{L}$ is invertible in $\mathcal{C}(X)$, then h is also invertible in $\mathcal{C}^*(X)$ and consequently, $(P(\mathcal{L}), \varphi)$ and $(P_*(\mathcal{L}), \varphi_*)$ are equivalent.

c) If (Z, α) is a compactification of X such that $Z \setminus \alpha(X)$ is homeomorphic to $\gamma\omega$, then (Z, α) is a Wallman-type compactification.

6. Zero-dimensional compactifications

Let us recall that a topological space X is called 0-*dimensional* if X has a base of clopen (open and closed) subsets. In this section, we will study the 0-dimensional compactifications of a completely regular Hausdorff space X.

We denote by $S\mathcal{C}_+^*(X)$ the subset of $\mathcal{C}_+^*(X)$ of finite-valued functions. If $f \in S\mathcal{C}_+^*(X)$, then $f = a_1\chi_{A_1} + \cdots + a_n\chi_{A_n}$, where $\{a_1, \dots, a_n\} \subset \mathbb{R}^+$ and $\{A_1, \dots, A_n\}$ is a partition consisting of disjoint clopen subsets of X. It is well known ([8]) that if X has a 0-dimensional compactification, then X is 0-dimensional.

THEOREM 6.1.

a) *The following statements are equivalent:*

1) X *is* 0*-dimensional.*
2) $SC^*_+(X)$ *is a c-set of* X.
3) *There exists a c-set* $\mathcal{L}$ *of* X *such that* $\mathcal{L} \subset SC^*_+(X)$.

b) *If* $\mathcal{L} \subset SC^*_+(X)$ *is a c-set of* X, *then* $(P_*(\mathcal{L}), \varphi_*)$ *and* $(P(\mathcal{L}), \varphi)$ *are equivalent* 0*-dimensional compactifications of* X.

Proof.

a) $1 \implies 2$: It is sufficient to show that the family of cozero-sets of functions in $SC^*_+(X)$ is an open basis of X. Let U be an open set in X and $x \in U$. There exists a clopen set $A \subset X$ such that $x \in A \subset U$ and $x \in \operatorname{coz}(\chi_A) \subset U$.

It is obvious that $2 \implies 3$.

$3 \implies 1$: Let U be an open set in X and let $x \in U$. There exists $f \in \mathcal{L}$ such that $x \in \operatorname{coz}(f) \subset U$. We have that $f = a_1\chi_{A_1} + \cdots + a_n\chi_{A_n}$, where $\{a_1, \ldots, a_n\} \subset \mathbb{R}$ and $\{A_1, \ldots, A_n\}$ is a partition of clopen sets of X. Let us assume that $f(x) = a_1$ and let $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$, be such that $[a_1 - \varepsilon, a_1 + \varepsilon] \cap \{0, a_2, \ldots, a_n\} = \emptyset$. If $Q = f^{-1}([a_1 - \varepsilon, a_1 + \varepsilon])$, then Q is a clopen set of X and $x \in Q \subset U$.

b) It is obvious that $\mathcal{L} \cap P = \mathcal{L} \cap P_*$. □

THEOREM 6.2. *Let* $\mathcal{L}$ *be a c-set of* X *such that* $(P_*(\mathcal{L}), \varphi_*)$ *is* 0*-dimensional. Then:*

1) *For every clopen set* A *of* $P_*(\mathcal{L})$ *there exists* $g \in \mathcal{L}$ *such that:*
   a) $A = o^*(g)$.
   b) *There exists* $a > 0$, $a \in \mathbb{R}$, *such that* $\overline{g}(p) \geq a$ *for every* $p \in o^*(g)$, *where* $\overline{g} \in C_+(P_*(\mathcal{L}))$ *is such that* $\overline{g} \circ \varphi_* = g$.
   c) $C = \operatorname{coz}(g)$ *is a clopen of* X *and* $\chi_C \in \mathcal{L}$. *Moreover,* $A = \operatorname{coz}(\overline{\chi_C})$, *where* $\overline{\chi_C} \in C_+(P_*(\mathcal{L}))$ *satisfies* $\overline{\chi_C} \circ \varphi_* = \chi_C$.
2) *Let* $\mathcal{L}'$ *be the set of finite-valued functions* $f \in \mathcal{L}$. *Then,* $\mathcal{L}'$ *is a c-set of* X *and the compactifications* $(P_*(\mathcal{L}), \varphi_*)$ *and* $(P_*(\mathcal{L}'), \varphi'_*)$ *of* X *are equivalent.*
3) *Let* $\mathcal{F}$ *be the set of clopen subsets* $A \subset X$ *such that* $\chi_A \in \mathcal{L}$. *Then* $\mathcal{F}$ *is a Boolean algebra on* X *and is an open basis of* X. *Therefore it is a Wallman base on* X. *If* $(w(\mathcal{F}), w)$ *is the Wallman-Shanin compactification of* X, *then* $(w(\mathcal{F}), w)$ *and* $(P_*(\mathcal{L}), \varphi_*)$ *are equivalent.*

Proof.

1) Let A be a clopen set in $P_*(\mathcal{L})$. Since A is compact, there exists a family $\{f_1, \ldots, f_n\} \subset \mathcal{L}$ such that $A = o^*(f_1) \cup \cdots \cup o^*(f_n)$. Let $g = f_1 + \cdots + f_n$. Then $g \in \mathcal{L}$ and $A = o^*(g)$. For b), assume that there exists $p_n \in o^*(g)$ such

that $0 \leq \overline{g}(p_n) \leq 1/n$ for every $n \in \mathbb{N}$. If q is a cluster point of $\{p_n : n \in \mathbb{N}\}$, we have that $\overline{g}(q) = 0$. On the other hand $q \in g^*$ and $\overline{g}(q) \neq 0$, because $o^*(g)$ is a clopen set in $P(\mathcal{L})$. For c), let us observe that $C = \operatorname{coz}(g)$ and $B = z(g)$ are clopen sets of X, because $\operatorname{coz}(g) = g^{-1}([a, +\infty))$. Let $h = (a - g) \vee 0$; we have that $h = a\chi_B$ and therefore $\chi_B \in \mathcal{L}$. Since $\chi_C = (1 - \chi_B) \vee 0$, we also have that $\chi_C \in \mathcal{L}$. It is easy to prove that if $\overline{\chi}_C \in \mathcal{C}_+(P_*(\mathcal{L}))$, then $A = \operatorname{coz}(\overline{\chi}_C)$, where $\overline{\chi}_C \circ \varphi_* = \chi_C$.

2) Let us prove that $\{\operatorname{coz}(g) : g \in \mathcal{L}'\}$ is an open basis of X. Let G be an open set in X and let $x \in G$. There exist an open set H in $P_*(\mathcal{L})$ and a clopen set A in $P_*(\mathcal{L})$ such that $\varphi(G) = H \cap \varphi(X)$ and $\varphi(x) \in A \subset H$. By 1c), we have that $A = \operatorname{coz}(\overline{\chi}_C)$, where $\chi_C \in \mathcal{L}$ and $\overline{\chi}_C \circ \varphi_* = \chi_C$. Therefore $\chi_C \in \mathcal{L}'$ and $x \in \operatorname{coz}(\chi_C) \subset G$. Hence $\mathcal{L}'$ is a c-set of X. Let $(P_*(\mathcal{L}'), \varphi'_*)$ be the corresponding compactification of X. If $g \in \mathcal{L}'$, then $g \in \mathcal{L}$ and there exists $\overline{g} \in \mathcal{C}_+(P_*(\mathcal{L}))$ such that $\overline{g} \circ \varphi_* = g$. Moreover, the family $\{\operatorname{coz}(\overline{g})\}_{g \in \mathcal{L}'}$, where $\overline{g} \circ \varphi_* = g$ for every $g \in \mathcal{L}'$, is an open basis of $P_*(\mathcal{L})$, because if H is open in $P_*(\mathcal{L})$ and $q \in H$, then, by 1c), we have that $q \in \operatorname{coz}(\overline{\chi}_A) \subset H$, where $\chi_A \in \mathcal{L}'$ and $\overline{\chi}_A \circ \varphi_* = \chi_A$. Hence, by Remark 4.5, we have the required equivalence.

3) It may be seen, as in the proof of 2), that $\mathcal{F}$ is an open basis of X. Let us prove that $\mathcal{F}$ is a Boolean algebra. If $A \in \mathcal{F}$, then $\chi_{A^c} = (1 - \chi_A) \vee 0 \in \mathcal{L}$ and therefore $A^c \in \mathcal{F}$. If $\{A, B\} \subset \mathcal{F}$, then $\chi_A + \chi_B \in \mathcal{L}$ and $\frac{1}{2}\chi_{A \cap B} = ((\chi_A + \chi_B) - \frac{3}{2}) \vee 0 \in \mathcal{L}$ therefore $A \cap B \in \mathcal{F}$. Finally, if $\{A, B\} \subset \mathcal{F}$ and $A \cap B = \emptyset$, then $\chi_{A \cup B} = \chi_A + \chi_B \in \mathcal{L}$ and $A \cup B \in \mathcal{F}$; if $A \cap B \neq \emptyset$, then we have that $A \cup B \in \mathcal{F}$ because $A \setminus B, B \setminus A, A \cap B \in \mathcal{F}$. Hence $\mathcal{F}$ is a Wallman base on X. Let $(w(\mathcal{F}), w)$ be the Wallman-Shanin compactification of X. Let us observe that $w(\mathcal{F})$ is the Stone space of the Boolean algebra $\mathcal{F}$ ([8]).

For every $p \in P_*(\mathcal{L})$, we define $H(p) = \{A \in \mathcal{F} : \chi_A \in p\}$. We claim that $H(p)$ is a maximal filter of $\mathcal{F}$. Let $\{A, B\} \subset H(p)$, then $\chi_{A \cap B} = \chi_A \cdot \chi_B \in p$ and $A \cap B \in H(p)$. Let $\{A, B\} \subset \mathcal{F}$ be such that $A \subset B$ and $A \in H(p)$. Then we have that $\chi_A + \chi_{A^c}$ is invertible in $\mathcal{C}^*(X)$ and $\chi_{A^c} \notin p$. Therefore $\chi_B \in p$ and $B \in H(p)$. Moreover, if $A \in \mathcal{F}$ and $A \notin H(p)$, then $\chi_A \notin p$ and $\chi_A + \chi_{A^c} = 1 \in p$. Hence $\chi_{A^c} \in p$ and $A^c \in H(p)$.

Let $H\colon P_*(\mathcal{L}) \to w(\mathcal{F})$ be the mapping defined by $H(p) = \{A \in \mathcal{F} : \chi_A \in p\}$ for every $p \in P_*(\mathcal{L})$. This mapping is one to one because if $\{p, q\} \subset P_*(\mathcal{L})$ and $p \neq q$, then there exists $f \in \mathcal{L}$ such that $f \in p \setminus q$ and there exists $A \in \mathcal{F}$ such that $p \in \operatorname{coz}(\overline{\chi}_A) \subset o^*(f)$, where $\overline{\chi}_A \in \mathcal{C}_+(P_*(\mathcal{L}))$ and $\overline{\chi}_A \circ \varphi_* = \chi_A$. Therefore $A \in H(p) \setminus H(q)$.

If $A \in \mathcal{F}$, then $\operatorname{coz}(\overline{\chi}_A) = o^*(\chi_A)$ and $\{o^*(\chi_A) : A \in \mathcal{F}\}$ is a closed base of $P_*(\mathcal{L})$. For every $A \in \mathcal{F}$ we have that $H(o^*(\chi_A)) = H(\{p \in P_*(\mathcal{L}) : \chi_A \in p\})$ $= \{H(p) : A \in H(p)\} = A^*$ and $\{A^* : A \in \mathcal{F}\}$ is a closed basis of $w(\mathcal{F})$. $\square$

Remark 6.3. Let $H^{-1}\colon w(\mathcal{F}) \to P_*(\mathcal{L})$ be the inverse of H. For every $F \in w(\mathcal{F})$, $H^{-1}(F)$ is the set of functions $f \in \mathcal{L}$ such that there exists some $A \in \mathcal{F}$ such that $\inf\{f(x):\ x \in A\} > 0$.

COROLLARY 6.4.

a) *Let $\mathcal{L}$ be a c-set of X. Then $(P_*(\mathcal{L}), \varphi_*)$ is 0-dimensional if and only if $(P_*(\mathcal{L}), \varphi_*)$ and $(P_*(\mathcal{L}'), \varphi'_*)$ are equivalent, where $\mathcal{L}'$ is the set of the finite-valued functions of $\mathcal{L}$.*

b) *Let (Z, α) be a Hausdorff compactification of X. Then (Z, α) is 0-dimensional if and only if (Z, α) and $(P_*(\mathcal{L}), \varphi_*)$ are equivalent, where $\mathcal{L}$ is the set of finite-valued functions $f \in \mathcal{C}_+(X)$ such that $\overline{f} \circ \alpha = f$, $\overline{f} \in \mathcal{C}(Z)$.*

c) *If (Z, α) is a 0-dimensional compactification of X, then (Z, α) is a Wallman-type compactification.*

Let us recall that a space X is called *strongly 0-dimensional* if for every $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(X)$ such that $z(f) \cap z(g) = \emptyset$ there exists a clopen set A in X such that $z(f) \subset A$ and $z(g) \subset A^c$. It is easy to see that X is strongly 0-dimensional if and only if for every $\{f, g\} \subset \mathcal{C}^*_+(X)$ such that $f+g$ is invertible in $\mathcal{C}^*(X)$ there exists a clopen set A in X such that $z(f) \subset A$ and $z(g) \subset A^c$. It is well known ([8]) that βX is 0-dimensional if and only if X is strongly 0-dimensional; we will obtain a stronger version of this results by generalizing the concept of strongly 0-dimensional space.

DEFINITION 6.5. Let $\mathcal{L}$ be a c-set of X, we will say that X is *$\mathcal{L}$ 0-dimensional* if for every $\{f, g\} \subset \mathcal{L}$ such that $f+g$ is invertible in $\mathcal{C}^*(X)$ there exists a clopen A of X such that $z(f) \subset A$, $z(g) \subset A^c$ and $\chi_A \in \mathcal{L}$.

It is easy to prove that if X is $\mathcal{L}$ 0-dimensional, then X is 0-dimensional.

THEOREM 6.6. *Let $\mathcal{L}$ be a c-set of X. Then $P_*(\mathcal{L})$ is 0-dimensional if and only if X is $\mathcal{L}$ 0-dimensional.*

Proof. If $P_*(\mathcal{L})$ is 0-dimensional, $\{f, g\} \subset \mathcal{L}$ and $f+g$ is invertible in $\mathcal{C}^*(X)$, then we have that $\overline{f}+\overline{g}$ is invertible in $\mathcal{C}(P_*(\mathcal{L}))$, where $\overline{f} \in \mathcal{C}_+(P_*(\mathcal{L}))$ and $\overline{g} \in \mathcal{C}_+(P_*(\mathcal{L}))$ are such that $\overline{f} \circ \varphi_* = f$, $\overline{g} \circ \varphi_* = g$ and $z(\overline{f}) \cap z(\overline{g}) = \emptyset$. There exists a clopen set in $P_*(\mathcal{L})$, $\operatorname{coz}(\overline{\chi}_A)$, where $\chi_A \in \mathcal{L}$ and $\overline{\chi}_A \circ \varphi_* = \chi_A$, such that $z(\overline{f}) \subset \operatorname{coz}(\overline{\chi}_A)$ and $z(\overline{g}) \subset z(\overline{\chi}_A)$. Therefore $z(f) \subset A$, $z(g) \subset A^c$ and $\chi_A \in \mathcal{L}$.

Now, let us assume that X is $\mathcal{L}$ 0-dimensional. We claim that $P_*(\mathcal{L})$ is 0-dimensional. Let $p \in P_*(\mathcal{L})$ and let $f \in \mathcal{L}$ be such that $p \in o^*(f)$. We can suppose that $\overline{f}(p) = b > 0$, where $b \in \mathbb{R}$, $\overline{f} \in \mathcal{C}_+(P_*(\mathcal{L}))$, $\overline{f} \circ \varphi_* = f$. Let $f_1 = \left(f - \frac{b}{6}\right) \vee 0$ and $f_2 = \left(\frac{5b}{6} - f\right) \vee 0$. We have that $\{f_1, f_2\} \subset \mathcal{L}$ and that $f_1 + f_2$ is invertible in $\mathcal{C}^*(X)$. Hence, there exists a clopen set $A \subset X$ such that $z(f_1) \subset A$, $z(f_2) \subset A^c$ and $\chi_A \in \mathcal{L}$. We have that $\operatorname{coz}(\overline{\chi}_{A^c})$ is a clopen in $P_*(\mathcal{L})$, where $\overline{\chi}_{A^c} \in \mathcal{C}_+(P_*(\mathcal{L}))$, $\overline{\chi}_{A^c} \circ \varphi_* = \chi_{A^c}$. It is easy to check that $p \in \mathrm{coz}(\overline{\chi}_{A^c}) \subset o^*(f)$. □

7. Appendix

In this section, we will introduce the concept of multiplicative $P\mathcal{C}_+(X)$-filter or, briefly, multiplicative filter.

By using Zorn's lemma, it is easy to check that if q is a multiplicative filter, then there exists a maximal multiplicative filter p such that $q \subset p$. Now, we will give a characterization for the maximal multiplicative filters.

THEOREM 7.1. *Let p be a multiplicative filter, then p is maximal if and only if the existence of a function $g \in \mathcal{C}_+(X)$ such that $f \cdot g \neq 0$ for every $f \in p$ implies that $g \in p$.*

Proof. Sufficiency is obvious. Let $g \in \mathcal{C}_+(X)$ be such that $f \cdot g \neq 0$ for every $f \in p$. We have that $q = \{f \cdot g^n : f \in p, \ n \in \mathbb{N}\}$ is a multiplicative filter and $g \in q$. Therefore $p = q$ and $g \in p$. □

DEFINITION 7.2. Let $(P(\mathcal{C}_+(X)), \varphi)$ be the Stone-Čech compactification of X. For every $p \in P(\mathcal{C}_+(X))$ we denote $S^p = \{f \in \mathcal{C}_+(X) : p \in \mathrm{cl}(\varphi(\mathrm{coz}(f)))\}$.

If $x \in X$, we denote $S_x = \{f \in \mathcal{C}_+(X) : x \in \overline{\mathrm{coz}(f)}\}$.

Remark 7.3.

a) Let $p \in P(\mathcal{C}_+(X))$. Then we have:

1) If $\{f, g\} \subset \mathcal{C}_+(X)$ and $f \cdot g \in S^p$, then $\{f, g\} \subset S^p$.
2) $p \subset S^p$.
3) $f \in S^p$ if and only if for every $g \in p$ we have that $f \cdot g \neq 0$.
4) If $f \in S^p$ and $g \in p$, then $f \cdot g \in S^p$.
5) p is the unique minimal filter contained in S^p.
6) If q is a multiplicative filter and $p \subset q$, then $q \subset S^p$.
7) If $f \in S^p \setminus p$, then there exists a multiplicative filter q such that $f \in q$ and $p \subset q \subset S^p$.

b) Let us recall that a point x of a topological space X is a p-*point* if and only if, for each function $f \in \mathcal{C}_+(X)$, the condition $f(x) = 0$ implies that there exists a neighborhood V_x of x such that $f(V_x) = \{0\}$. It is plain to prove that x is a p-point of X if and only if $p_x = S_x$. A topological space X is called a P-*space* if and only if every point is a p-point. We also have that X is a P-space if and only if every cozero-set is a clopen in X. The following statements are equivalent:

1) X is a P-space.

2) Every multiplicative filter is minimal.
3) Every multiplicative filter is a maximal multiplicative filter.
4) Every multiplicative filter q satisfies that if $\{f, g\} \subset \mathcal{C}_+(X)$ and $f \cdot g \in q$, then $\{f, g\} \subset q$.
5) $S^p = p$ for each minimal filter p.

A completely regular space X is a F-space if and only if two disjoint cozero-sets are completely separated.

THEOREM 7.4. *X is a F-space if and only if S^p is multiplicative for every $p \in P(\mathcal{C}_+(X))$.*

P r o o f. If X is a F-space and S^p is not multiplicative, where $p \in P(\mathcal{C}_+(X))$, then there exist $f \in S^p$ and $g \in S^p$ such that $p \notin \mathrm{cl}(\varphi(\mathrm{coz}(f \cdot g)))$. Therefore there exists $\overline{h} \in \mathcal{C}_+(P(\mathcal{C}_+(X)))$ such that $\overline{h}(p) \neq 0$ and $\overline{h}(\mathrm{cl}(\varphi(\mathrm{coz}(f \cdot g)))) = \{0\}$. We have that $h = \overline{h} \circ \varphi \in p$ and $h \cdot f \cdot g = 0$. Since $\mathrm{coz}(f \cdot h) \cap \mathrm{coz}(g \cdot h) = \emptyset$, we have that there exists $l \in \mathcal{C}_+(X)$ such that $l(\mathrm{coz}(f \cdot h)) = \{1\}$ and $l(\mathrm{coz}(g \cdot h)) = \{0\}$. Since $f \in S^p$ and $h \in p$, then $f \cdot h \in S^p$ and $p \in \mathrm{cl}(\varphi(\mathrm{coz}(f \cdot h)))$. Similarly, we have that $p \in \mathrm{cl}(\varphi(\mathrm{coz}(g \cdot h)))$. If $\overline{l} \in \mathcal{C}_+(P(\mathcal{C}_+(X)))$ is such that $\overline{l} \circ \varphi = l$, then $\overline{l}(p) = 1$ and $\overline{l}(p) = 0$, which is a contradiction.

Conversely, let $f \in \mathcal{C}_+(X)$ and $g \in \mathcal{C}_+(X)$ be such that $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$. Since the hypotheses implies that $\mathrm{cl}(\varphi(\mathrm{coz}(f))) \cap \mathrm{cl}(\varphi(\mathrm{coz}(g))) = \emptyset$, the Theorem is a consequence of Uryson's lemma. □

THEOREM 7.5. *X is an F-space if and only if for every multiplicative filter q there exists $p \in P(\mathcal{C}_+(X))$ such that $q = S^p$.*

P r o o f. The necessity is almost obvious. We shall verify only the sufficiency. Let $p \in P(\mathcal{C}_+(X))$ and let q be a maximal multiplicative filter such that $p \subset q$. There exists, by hypothesis, $p' \in P(\mathcal{C}_+(X))$ such that $q = S^{p'}$. Since $p \subset S^p$ and $p' \subset S^{p'}$, we have that $p = p'$ and S^p is multiplicative. □

It is well known ([4]), that X is a F-space if and only if βX is a F-space. Now we will use our model of βX, the space $P(\mathcal{C}_+(X))$, and our techniques to prove this result.

THEOREM 7.6. *X is a F-space if and only if $P(\mathcal{C}_+(X))$ is a F-space.*

P r o o f.

Necessity.
Let $\{\overline{f}, \overline{g}\} \subset \mathcal{C}_+(P(\mathcal{C}_+(X)))$ be such that $\mathrm{coz}(\overline{f}) \cap \mathrm{coz}(\overline{g}) = \emptyset$. Since X is a F-space there exists $h \in \mathcal{C}_+(X)$ such that $h(\mathrm{coz}(f)) = \{1\}$ and $h(\mathrm{coz}(g)) = \{0\}$, where $f = \overline{f} \circ \varphi$ and $g = \overline{g} \circ \varphi$. Let $\overline{h} \in \mathcal{C}_+(P(\mathcal{C}_+(X)))$, where $\overline{h} \circ \varphi = h$.

Clearly $\overline{h}(\mathrm{cl}(\varphi(\mathrm{coz}(f)))) = \{1\}$ and we have $\overline{h}(\mathrm{coz}(\overline{f})) = \{1\}$. Similarly $\overline{h}(\mathrm{coz}(\overline{g})) = \{0\}$.

Sufficiency.

Let $\{f, g\} \subset \mathcal{C}_+^*(X)$ and let $\{\overline{f}, \overline{g}\} \subset \mathcal{C}_+(P(\mathcal{C}_+(X)))$, where $\overline{f} \circ \varphi = f$ and $\overline{g} \circ \varphi = g$. It is easy to prove that $\mathrm{coz}(\overline{f}) \cap \mathrm{coz}(\overline{g}) = \emptyset$. By hypothesis, there exists $\overline{h} \in \mathcal{C}_+(P(\mathcal{C}_+(X)))$ such that $\overline{h}(\mathrm{coz}(\overline{f})) = \{1\}$ and $\overline{h}(\mathrm{coz}(\overline{g})) = \{0\}$. If $h = \overline{h} \circ \varphi$, we have that $h(\mathrm{coz}(f)) = \{1\}$ and $h(\mathrm{coz}(g)) = \{0\}$. □

Remark 7.7. In this paper, our main interest has been in obtaining compactifications, of a Hausdorff completely regular space X, of type $(P(\mathcal{L}), \varphi)$ and $(P_*(\mathcal{L}), \varphi_*)$, where $\mathcal{L} \subset \mathcal{C}_+(X)$. For some remarkable compactifications, we have found the corresponding family $\mathcal{L}$ of functions. For other compactifications, we propose the characterization of the corresponding families of functions. Moreover, we suggest the manipulation of the initial sets, A, $\mathcal{L}$ and P to get interesting compact sets.

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