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CHARACTERIZING TRIPLETS FOR MODULAR PSEUDOCOMPLEMENTED ORDERED SETS

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ABSTRACT. It was shown by T. Katriňák that every pseudocomplemented modular lattice is uniquely determined by the so called characterizing triplet. We prove that this method can be generalized also for modular pseudocomplemented ordered sets satisfying certain conditions on their ideals.

It was proved by T. Katriňák [7] that every pseudocomplemented distributive lattice $L = (L; \vee, \wedge, *, 0, 1)$ is determined uniquely up to isomorphism by the so called *characterizing triplet* $(\mathcal{B}(L), \mathcal{D}(L), \Phi_L)$, where $\mathcal{B}(L) = \{x \in L : x = x^{**}\}$, $\mathcal{D}(L) = \{y \in L : y^* = 0\}$ and Φ_L is a 0,1-homomorphism of L into the lattice $\mathcal{D}(L)$ of all filters of L. This means that two distributive pseudocomplemented lattices L_1, L_2 have isomorphic characterizing triplets if and only if $L_1 \cong L_2$. It was shown by T. Katriňák [5] that in addition for every triplet (B, D, Φ) such that B is a Boolean algebra, D is a distributive lattice and Φ is a suitable mapping there exists a construction of a distributive pseudocomplemented lattice L with $(B, D, \Phi) \cong (\mathcal{B}(L), \mathcal{D}(L), \Phi_L)$. This result was also generalized for modular pseudocomplemented lattices by T. Katriňák and P. Mederly [6]. We proceed to generalize this attempt for modular pseudocomplemented ordered sets. Our goal is to show that the characterizing triplet is much more an attribute of the property "to be pseudocomplemented" than the property "to be a lattice". Of course, we must add some conditions which are trivially valid for lattices but not for ordered sets in general.

Throughout the paper, every ordered set $A = (A, \leq)$ is assumed to have a least element 0, and it will be identified with its carrier set A. An ordered set A is said to be *pseudocomplemented* (briefly a *PC-set*) if for each $a \in A$ there exists $a^* \in A$ which is the greatest element of A with the property $\inf(a, a^*) = 0$ (see [3]). Of course, every PC-set has the greatest element 1 for which $0^* = 1$.

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Let A be an ordered set and $M \subseteq A$. We use the notations

$$L(M) = \{x \in A: \ x \le m \text{ for each } m \in M\},$$
$$U(M) = \{y \in A: \ m \le y \text{ for each } m \in M\}.$$

Further, if $M \subseteq S \subseteq A$ then $L_S(M) = L(M) \cap S$ and $U_S(M) = U(M) \cap S$. If M is finite, e.g. $M = \{a_1, \dots, a_n\}$, we will write briefly $L(a_1, \dots, a_n)$ or $U(a_1, \dots, a_n)$ instead of $L(M)$ or $U(M)$, respectively. If $B, C \subseteq A$, then $L(B,C)$ or $U(B,C)$ means $L(B \cup C)$ or $U(B \cup C)$. Finally, we will write briefly $LU(M)$ or $UL(M)$ instead of $L(U(M))$ or $U(L(M))$ to simplify the notation. It is immediate that if $L(a, a^*) = \{0\}$, then a, a^* have an infimum and $\inf(a, a^*) = 0$.

Recall some necessary concepts introduced in [1], [2] and [4]: an ordered set A is:

- *distributive* if $L(U(a,b),c) = LU(L(a,c), L(b,c))$,
- *modular* if $a \le c$ implies $L(U(a,b),c) = LU(a, L(b,c))$

for arbitrary a, b, c in A.

Further, A is *complemented* if for each $a \in A$ there exists $b \in A$ with $LU(a,b) = UL(a,b) = A$. The set A is called *boolean* if it is both distributive and complemented. Note that in a distributive set A, every $a \in A$ has at most one complement (see [1]).

Let A be a PC-set. An element $a \in A$ is called *boolean* whenever $a = a^{**}$. We use the notations

$$\mathcal{B}(A) = \{x \in A: \ x = x^{**}\},$$
$$\mathcal{D}(A) = \{y \in A: \ y^* = 0\}.$$

$\mathcal{D}(A)$ is called the set of *dense elements* of A.

The concept of an s-filter was introduced in [4]: a nonempty subset I of an ordered set A is an *s-filter* of A if $UL(M) \subseteq I$ for every finite subset $M \subseteq I$.

It is evident (see [4]) that the set $\mathrm{SFil}(A)$ of all s-filters of A forms a complete lattice with respect to set inclusion. Moreover, $\mathrm{SFil}(A)$ is isomorphic to the Dedekind-McNeille completion $N(A)$ of A whenever A is finite. For the reader's convenience, we denote by $\vee_F$ the join in $\mathrm{SFil}(A)$; of course, the meet coincides with set intersection. The following easy result was proved in [4]:

LEMMA 1. *Let $I, J \in \mathrm{SFil}(A)$. Then*

$$I \vee_F J = \bigcup \{UL(X): \ X \subseteq I \cup J, \ X \text{ finite}\}.$$

We can prove the following lemma.

LEMMA 2. *Let A be an ordered set. If the lattice* SFil(A) *is modular, then A is modular.*

Proof. Suppose $a, b, c \in A$ and $a \leq c$. Then $U(a) \supseteq U(c)$ and, by using of modularity of SFil(A), we have

$$U(a) \cap \big(U(b) \vee_F U(c)\big) = \big(U(a) \cap U(b)\big) \vee_F U(c) = U(a,b) \vee_F U(c) \subseteq UL\big(U(a,b), c\big).$$

Hence,

$$U\big(a, L(b,c)\big) = U(a) \cap UL(b,c) = U(a) \cap \big(U(b) \vee_F U(c)\big) \subseteq UL\big(U(a,b), c\big).$$

Conversely, $L(a) \subseteq L\big(U(a,b), c\big)$ and $L(b,c) \subseteq L\big(U(a,b), c\big)$ give $L(a) \cup L(b,c) \subseteq L\big(U(a,b), c\big)$, thus also

$$U\big(a, L(b,c)\big) \supseteq UL\big(U(a,b), c\big).$$

□

Remark 1. The converse of Lemma 2 does not hold in general. If X is an infinite set and A is a subset of the power set $\operatorname{Exp} X$ containing only at most one-element sets and complements of finite sets (i.e. $X \setminus Y$, where Y is finite), then A is distributive and hence modular but SFil(A) is not modular.

For a subset M of a PC-set A denote by $M^* = \{x^* : x \in M\}$, $U^*(M) = \{x^* : x \in U(M)\}$ and $L^*(M) = \{x^* : x \in L(M)\}$.

The following assertion was proved in [3; Proposition 2]:

LEMMA 3. *For each $a, b \in \mathcal{B}(A)$, A a PC-set,*

$$U(a,b) \cap \mathcal{B}(A) = L^*(a^*, b^*) \qquad \text{and} \qquad L(a,b) \cap \mathcal{B}(A) = U^*(a^*, b^*).$$

For the following result see [3; Lemma 1(viii)]:

LEMMA 4. *Let A be a PC-set and $B, C \subseteq A$. Then*

$$L(B,C) = \{0\} \implies L(B) \subseteq LL^*(C).$$

The following result is analogous to the lattice statement for PC-sets:

THEOREM 1. *Let A be a PC-set. Then*

(i) *$\mathcal{B}(A)$ is a boolean set with respect to the induced order;*
(ii) *$\mathcal{D}(A)$ is an s-filter of A with the greatest element 1;*
(iii) *the relation ϕ_A given by setting*

$$\langle x, y\rangle \in \phi_A \iff x^* = y^*$$

is an equivalence on A and all of its classes $[x]_{\phi_A}$ contain a greatest element $x^{**} \in \mathcal{B}(A)$.

Proof.

(i) To prove distributivity of $\mathcal{B}(A)$: let $x, y, z \in \mathcal{B}(A)$. Set $B = \{z\} \cup L^*(x^*, y^*)$ and $C = UU^*(x, L(y,z))$. Applying Lemma 4, we get $L(z, L^*(x^*, y^*)) \subseteq LL^*UU^*(x, L(y,z))$. Using the equalities $U^*(M) = L(M^*)$ and $L^*(M) = U(M^*)$ and Lemma 3, we have

$$\begin{aligned} L(z, U(x,y)) &= L(z, L^*(x^*, y^*)) \subseteq LL^*UU^*(x, L(y,z)) = LUU^*U^*(x, L(y,z)) \\ &= LULU(x, L(y,z)) = LU(x, L(y,z)) . \end{aligned}$$

We proceed to show that this inclusion implies distributivity of $\mathcal{B}(A)$. Suppose $j, m, x \in \mathcal{B}(A)$ and $L(j,x) \subseteq L(m)$, $U(m,x) \subseteq U(j)$. Then

$$\begin{aligned} L(j) &= L(j) \cap LU(m,x) = L(j, U(m,x)) \\ &\subseteq LU(m, L(j,x)) \subseteq LU(m, L(m)) \\ &= L(m) , \end{aligned}$$

whence $j \leq m$. By a result of M. Ernè [2] this proves distributivity of the lattice $N_0(\mathcal{B}(A))$, which is a sublattice of the Dedekind-McNeille completion $N(\mathcal{B}(A))$ of $\mathcal{B}(A)$ generated by the set $\{L(x) : x \in \mathcal{B}(A)\}$. It is easy to see that this implies distributivity of $\mathcal{B}(A)$. Prove that $\mathcal{B}(A)$ is complemented: for each $a \in \mathcal{B}(A)$ we have $L(a, a^*) = \{0\}$ and hence $UL(a, a^*) = \mathcal{B}(A)$. According to Lemma 3,

$$LU(a, a^*) = LL^*(a^*, a^{**}) = L(0^*) = L(1) = \mathcal{B}(A) ,$$

i.e. a^* is a complement of a.

(ii) Suppose $\mathcal{D}(A)$ is not an s-filter of A. Then there are $x_1, \dots, x_n \in \mathcal{D}(A)$ such that $z \in UL(x_1, \dots, x_n)$ and $z \notin \mathcal{D}(A)$, i.e. $z^* \neq 0$. Hence, there exists $g \in A$, $g \neq 0$ with $L(z,g) = 0$. Since $x_1^* = 0 = x_2^*$, we have $L(x_1, x_2) \neq \{0\}$ and $L(x_1, g) \neq \{0\} \neq L(x_2, g)$ (since $x_1^* \geq x_2 > 0$, $x_1^* \geq g > 0$ or $x_2^* \geq g > 0$ in the opposite case). Thus there are $g_1 \in L(x_1, x_2)$, $g_2 \in L(x_1, g)$ and $g_3 \in L(x_2, g)$ such that $g_1, g_2, g_3 \neq 0$. Further, $L(x_1, g_3) \neq \{0\}$ (since $x_1^* \geq g_3 > 0$ in the opposite case), thus there exists $0 \neq g_{12} \in L(x_1, g_3)$ and we have $g_{12} \leq x_1$, $g_{12} \leq g_3 \leq x_2$, i.e. $g_{12} \in L(x_1, x_2, g)$, see Fig. 1.

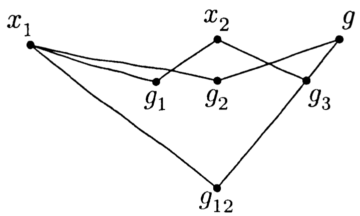

FIGURE 1.

Similarly, we can show the existence of $g_{123} \neq 0$ such that $g_{123} \in L(x_3, g_{12}) \subseteq L(x_1, x_2, x_3, g)$. We can proceed with this search up to the element x_n to prove the existence of $g_{12\dots n} \neq 0$ satisfying $g_{12\dots n} \in L(x_1, \dots, x_n, g)$. Since $z \in UL(x_1, \dots, x_n)$, we have

$$z \geq g_{12\dots n} \qquad \text{and} \qquad L(z, g) = \{0\} \supseteq L(g_{12\dots n}, g) = L(g_{12\dots n})$$

whence $g_{12\dots n} = 0$, a contradiction.

(iii) It is evident that ϕ_A is an equivalence on A. Let $x \in A$. Clearly $x^{**} \in \mathcal{B}(A)$ and $x \leq x^{**}$. Moreover, $y^{**} = x^{**}$ for each $\langle x, y\rangle \in \phi_A$, thus x^{**} is the greatest element of $[x]_{\phi_A}$. □

For the remainder of the paper, denote by D the set $\mathcal{D}(A)$ and by $\overset{\bullet}{\vee}$ the operation of join in $\mathrm{SFil}(D)$.

Definition. A PC-set A is *representable* if both the lattices $\mathrm{SFil}(A)$ and $\mathrm{SFil}(D)$ are modular and for each $x, y \in A$:

(a) $U_D(x)$ is an s-filter in D;
(b) *separation condition*: $x \neq y$ and $x^* = y^*$ imply $U_D(x) \neq U_D(y)$;
(c) $U_D(x^{**}) \overset{\bullet}{\vee} U_D(x^*) \supseteq U_D(x)$.

Remark 2. The aforementioned conditions (b), (c) are not exceptional for elements of $\mathcal{B}(A) \cup \mathcal{D}(A)$ for an arbitrary PC-set A. If $x, y \in \mathcal{B}(A)$, then $x^* = y^*$ implies $x = y$, thus (b) is satisfied trivially. If $x, y \in \mathcal{D}(A)$ and $U_D(x) = U_D(y)$, then $x = y$ again. If $x \in \mathcal{B}(A)$ and $y \in \mathcal{D}(A)$, then $y^* = 0$, thus $x^* = y^*$ gives $x^* = 0$, i.e. $x = 1$. Since $x \neq y$, we have $y \notin \{1\} = U_D(x)$ but $y \in U_D(y)$, i.e. $U_D(x) \neq U_D(y)$. For (c), if $x \in \mathcal{B}(A)$, then $x = x^{**} \in U_D(x^{**})$ and if $x \in \mathcal{D}(A)$, then $x^* = 0$ gives $U_D(x^*) = \mathcal{D}(A)$.

Example 1. The ordered set in Fig. 2 is a modular (even distributive) PC-set with $\mathcal{B}(A) = \{0, a, b, 1\}$, $\mathcal{D}(A) = \{y, c, d, 1\}$. Hence $\mathcal{D}(A)$ is not modular thus also $\mathrm{SFil}(D)$ is not modular. Moreover, $U(x) \cap \mathcal{D}(A) = \{c, d, 1\}$ is not an s-filter in $\mathcal{D}(A)$. Hence, A is not representable.

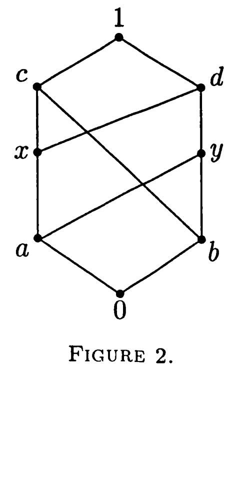

Figure 2.

EXAMPLE 2. The PC-set A in Fig. 3 is distributive and hence modular, $\mathcal{B}(A) = \{0, a, b, 1\}$, $\mathcal{D}(A) = \{c, d, 1\}$. Hence, SFil($D$) is the four element Boolean lattice $\{\{1\}, \{c, 1\}, \{d, 1\}, \{c, d, 1\}\}$ thus also SFil(D) is distributive. Since $A = \mathcal{B}(A) \cup \mathcal{D}(A)$, A is clearly representable.

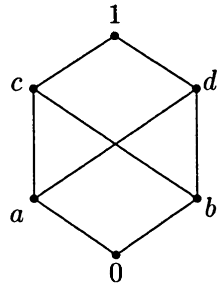

FIGURE 3.

EXAMPLE 3. We can easily verify that the PC-set A in Fig. 4 is representable. Moreover, $a, b \notin \mathcal{B}(A) \cup \mathcal{D}(A)$:

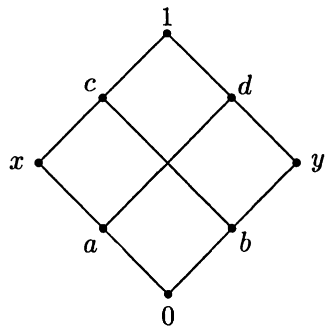

FIGURE 4.

Remark 3. If a PC-set A is modular lattice, then it is trivially a representable set. Of course, the lattices SFil(A) or SFil(D) coincide with the lattices of lattice filters on A or $\mathcal{D}(A)$, respectively, and hence they are modular. The separation condition holds trivially. Moreover, $U_D(x^{**}) \dot{\vee} U_D(x^*) = \mathcal{D}(A)$ thus also the condition (c) holds trivially.

LEMMA 5. *Let A be a PC-set and $x \in A$. Then*

(i) $U(x, x^*) \subseteq \mathcal{D}(A)$;
(ii) *if A is representable, then* $U(x, x^*) \in \mathrm{SFil}(D)$.

Proof. If $z \in U(x, x^*)$, then $z \geq x$, $z \geq x^*$ whence $z^* \leq x^*$ and $z^* \leq x^{**}$, i.e. $z^* \in L(x^*, x^{**}) = \{0\}$. Hence $z^* = 0$, proving that $U(x, x^*) \subseteq \mathcal{D}(A)$. Further, $U(x, x^*) = [U(x) \cap \mathcal{D}(A)] \cap [U(x^*) \cap \mathcal{D}(A)] = U_D(x) \cap U_D(x^*)$. If A is representable, then both $U_D(x)$ and $U_D(x^*)$ are s-filters in $\mathcal{D}(A)$, thus also $U(x, x^*) \in \mathrm{SFil}(D)$. □

DEFINITION. Let A be a representable set. The mapping Φ_A: $\mathcal{B}(A) \to \mathrm{SFil}(D)$ given by setting

$$\Phi_A(x) = U(x^*) \cap \mathcal{D}(A) = U_D(x^*)$$

is called the *characteristic mapping* on A. The triplet $(\mathcal{B}(A), \mathcal{D}(A), \Phi_A)$ is called the *characteristic triplet* on A.

LEMMA 6. *Let A be a representable set and $x \in A$. Then*

$$U_D(x) = \Phi_A(x^*) \overset{\bullet}{\vee} U(x, x^*).$$

Proof. By the definition of Φ_A and Lemma 5, both $\Phi_A(x^*)$ and $U(x, x^*)$ are s-filters in $\mathcal{D}(A)$. Further, applying modularity of $\mathrm{SFil}(D)$, we conclude

$$\Phi_A(x^*) \overset{\bullet}{\vee} U(x, x^*) = U_D(x^{**}) \overset{\bullet}{\vee} (U_D(x) \cap U_D(x^*)) = (U_D(x^{**}) \overset{\bullet}{\vee} U_D(x^*)) \cap U_D(x).$$

By the condition (c) of the definition, we have the assertion. □

For an ordered set B denote by B^d its dual set (i.e. ordered by the reverse order).

THEOREM 2. *Let A be a representable set. The mapping $h : A \to \mathcal{B}(A) \times \mathrm{SFil}(D)^d$ given by setting*

$$h(x) = \langle x^{**}, U_D(x) \rangle$$

is an order-preserving isomorphism of A onto $h(A)$.

Proof. By Lemmas 5 and 6, h is really a mapping of A into $\mathcal{B}(A) \times \mathrm{SFil}(D)^d$. Prove injectivity of h: let $x, y \in A$, $x \neq y$. By Theorem 1, x^{**} or y^{**} are the greatest elements of $[x]_{\phi_A}$ or $[y]_{\phi_A}$, respectively. If $x^{**} \neq y^{**}$, then $h(x) \neq h(y)$. If $x^{**} = y^{**}$, then $\langle x, y \rangle \in \phi_A$, i.e. $x^* = y^*$ and, by the separation condition, $U_D(x) \neq U_D(y)$, hence $h(x) \neq h(y)$ again.

Of course, h is an order preserving mapping since $x \leq y$ implies $x^{**} \leq y^{**}$ and $U_D(x) \supseteq U_D(y)$. It remains to show that also $h^{-1} : h(A) \to A$ is order preserving. Suppose $x^{**} \leq y^{**}$ and $U_D(x) \supseteq U_D(y)$. By using of Lemma 5(i) and modularity of $\mathrm{SFil}(A)$, we have

$$\begin{aligned} U(x^{**}) \vee_F \mathcal{D}(A) &\supseteq U(x^{**}) \vee_F U(x, x^*) = U(x^{**}) \vee_F (U(x) \cap U(x^*)) \\ &= (U(x^{**}) \vee_F U(x^*)) \cap U(x) \\ &= UL(x^{**}, x^*) \cap U(x) = U(x). \end{aligned}$$

Hence,

$$U(x^{**}) \vee_F \big(U(x) \cap \mathcal{D}(A)\big) = \big[U(x^{**}) \vee_F \mathcal{D}(A)\big] \cap U(x) = U(x)\,,$$

i.e

$$U(x) = U(x^{**}) \vee_F \big(U(x) \cap \mathcal{D}(A)\big) \supseteq U(y^{**}) \vee_F \big(U(y) \cap \mathcal{D}(A)\big) = U(y)$$

proving $x \leq y$. □

Having in mind the foregoing result, we can ask about conditions under which a representable set is reconstructable from its characterizing triplet. For this reason, we introduce the following concept:

DEFINITION. A representable set A is called *strongly representable* if for each $x \in A$

$$U_D(x^{**}) \overset{\bullet}{\vee} U_D(x^*) = \mathcal{D}(A)\,.$$

EXAMPLE 4. One can easily verify that the set A of Example 2 (see Fig. 3) is strongly representable. Also the set in Fig. 5 is strongly representable.

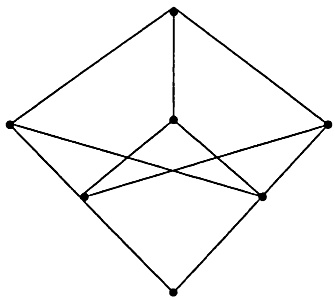

FIGURE 5.

Remark 4. It was shown in Remark 2 that a PC-set A with $A = \mathcal{B}(A) \cup \mathcal{D}(A)$ having modular lattices $\mathrm{SFil}(A)$ and $\mathrm{SFil}(D)$ is representable. However, for each $x \in \mathcal{B}(A) \cup \mathcal{D}(A)$ also $U_D(x^{**}) \overset{\bullet}{\vee} U_D(x^*) = \mathcal{D}(A)$, i.e. such a set A is also strongly representable. This is the case for the set A in Fig. 5.

It was proved in the proof of Theorem 1 that x^* is the complement of $x \in \mathcal{B}(A)$. We can prove the following theorem.

THEOREM 3. *Let A be a strongly representable set. Consider the subset $P_A \subseteq \mathcal{B}(A) \times \mathrm{SFil}(D)^d$ such that*

$$P_A = \Big\{ \langle y, \Phi_A(y^*) \overset{\bullet}{\vee} U_D(x) \rangle : \ y \in \mathcal{B}(A)\,, \ x \in \mathcal{D}(A) \Big\}\,. \qquad (*)$$

Then P_A is pseudocomplemented ordered set. More precisely,

(i) $\langle 0, \mathcal{D}(A)\rangle$ *and* $\langle 1, \{1\}\rangle$ *are the least and the greatest elements, respectively,*

(ii) $\langle a, \Phi_A(a^*) \overset{\bullet}{\vee} U_D(x)\rangle^* = \langle a^*, \Phi_A(a)\rangle$,

(iii) $\mathcal{B}(P_A) = \{\langle a, \Phi_A(a^*)\rangle : \ a \in \mathcal{B}(A)\}$ *and* $\mathcal{D}(P_A) = \{\langle 1, U_D(x)\rangle : \ x \in \mathcal{D}(A)\}$.

Proof. Since

$$\Phi_A(a) = \Phi_A(a^{**}) = \Phi_A(a^{**}) \overset{\bullet}{\vee} \big(U(1) \cap \mathcal{D}(A)\big),$$

we have $\langle a, \Phi_A(a^*)\rangle \in P_A$ for each $a \in \mathcal{B}(A)$. Of course, a^* is the greatest element of $\mathcal{B}(A)$ satisfying $L_{B(A)}(a, a^*) = \{0\}$. With respect to strong representability of A, we have

$$L_{P(A)}\Big(\big\langle a, \Phi_A(a^*) \overset{\bullet}{\vee} U_D(x)\big\rangle, \langle a^*, \Phi_A(a)\rangle\Big) = \langle 0, \mathcal{D}(A)\rangle.$$

Conversely, if

$$\big\langle b, \Phi_A(b^*) \overset{\bullet}{\vee} U_D(z)\big\rangle \in P_A$$

and

$$L_{P(A)}\Big(\big\langle a, \Phi_A(a^*) \overset{\bullet}{\vee} U_D(x)\big\rangle, \big\langle b, \Phi_A(b^*) \overset{\bullet}{\vee} U_D(z)\big\rangle\Big) = \langle 0, \mathcal{D}(A)\rangle,$$

then $L_{B(A)}(a, b) = 0$, i.e. $b \le a^*$, whence

$$\Phi_A(a) = U_D(a^*) \subseteq U_D(b) = \Phi_A(b^*) \subseteq \Phi_A(b^*) \overset{\bullet}{\vee} U_D(z).$$

Hence, $\langle a^*, \Phi_A(a)\rangle$ is the pseudocomplement of $\langle a, \Phi_A(a^*) \overset{\bullet}{\vee} U_D(x)\rangle$ in P_A. Further, if $\langle a^*, \Phi_A(a)\rangle = \langle 0, \mathcal{D}(A)\rangle$, then $a^* = 0$, thus $a = 1$ (because $a \in \mathcal{B}(A)$). Henceforth $\Phi_A(a) = U_D(a^*) = \mathcal{D}(A)$, i.e. $\mathcal{D}(P_A) = \{\langle 1, U_D(x)\rangle : x \in \mathcal{D}(A)\}$. The equality $\mathcal{B}(P_A) = \{\langle a, \Phi_A(a^*)\rangle : \ a \in \mathcal{B}(A)\}$ is almost evident. □

If we add two more conditions, we can show that the strongly representable sets are reconstructable by their characterizing triplets:

THEOREM 4. *Let A be a strongly representable set such that for each $x \in A$ and each $y \in \mathcal{D}(A)$ there exists $m \in A$ with $m^* = x^*$ satisfying*

$$U_D(x^{**}) \overset{\bullet}{\underset{\bullet}{\vee}} U_D(y) = U_D(m) \tag{$**$}$$

and for each $x \in A$ there exists $d \in \mathcal{D}(A)$ such that

$$U_D(x) = U_D(x^{**}) \overset{\bullet}{\vee} U_D(d). \tag{$***$}$$

Then A is order-isomorphic to the set P_A given by $(*)$ of Theorem 3.

Proof. In view of Theorem 2, it remains only to show that every element of P_A has the form $\langle m^{**}, U_D(m)\rangle$ for some $m \in A$ and that it is the whole $h(A)$. Let $p \in P_A$. By $(*)$, we have $p = \big\langle y, \Phi_A(y^*) \mathbin{\dot\vee} U_D(z)\big\rangle$ for some $y \in \mathcal{B}(A)$ and $z \in \mathcal{D}(A)$. Clearly, $\Phi_A(y^*) = U_D(y^{**})$. Applying the assumption $(**)$, there exists $m \in A$ such that

$$\Phi_A(y^*) \mathbin{\dot\vee} U_D(z) = U_D(y^{**}) \mathbin{\dot\vee} U_D(z) = U_D(m)$$

and $m^* = y^*$, i.e. $m^{**} = y^{**} = y$. Hence $p = \langle m^{**}, U_D(m)\rangle$. By Theorem 2, each element of $h(A)$ is of the form $\langle x^{**}, U_D(x)\rangle$ for some $x \in A$. By $(***)$, we have $\langle x^{**}, U_D(x)\rangle = \big\langle x^{**}, U_D(x^{**}) \mathbin{\dot\vee} U_D(d)\big\rangle$ for some $d \in \mathcal{D}(A)$. By the aforementioned fact, it is equal to $p \in P_A$ and hence $P_A = h(A)$. □

Remark 5. The assumptions $(**)$, $(***)$ of Theorem 4 are trivially satisfied in lattices. Of course, if A is a lattice and $x \in A$, then, by Lemma 6,

$$U_D(x) = U_D(x^{**}) \mathbin{\dot\vee} U_D(x, x^*) = U_D(x^{**}) \mathbin{\dot\vee} U_D(x \vee x^*)\,,$$

thus $d = x \vee x^*$ is the desired element of $(***)$.

Let A_1, A_2 be representable sets and

$$\big(\mathcal{B}(A_1), \mathcal{D}(A_1), \Phi_{A_1}\big)\,, \qquad \big(\mathcal{B}(A_2), \mathcal{D}(A_2), \Phi_{A_2}\big)$$

be their characterizing triplets. We say that these triplets are isomorphic if there exist lattice isomorphisms $i\colon \mathcal{B}(A_1) \to \mathcal{B}(A_2)$, $j\colon \mathcal{D}(A_1) \to \mathcal{D}(A_2)$ and the isomorphism $f_j\colon \operatorname{SFil}\big(\mathcal{D}(A_1)\big) \to \operatorname{SFil}\big(\mathcal{D}(A_2)\big)$ induced by j such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B}(A_1) \arrow[r, "\varphi_{A_1}"] \arrow[d, "i"'] & \operatorname{SFil}(\mathcal{D}(A_1)) \arrow[d, "f_j"] \\
\mathcal{B}(A_2) \arrow[r, "\varphi_{A_2}"'] & \operatorname{SFil}(\mathcal{D}(A_2))
\end{tikzcd}$$

We can formulate the corollary:

COROLLARY. *If A_1, A_2 are strongly representable sets with isomorphic characterizing triplets, then A_1 is order isomorphic to A_2. In other words, every strongly representable set is determined up to order-isomorphism by its characterizing triplet.*

Proof. We know by Theorem 4 that every strongly representable set can be reconstructed from its characterizing triplet as a set P_A of $(*)$ in Theorem 3. Consider the mapping $\psi\colon A_1 \to A_2$ given by

$$\psi\Big(\big\langle x^{**}, U_{D_1}(x^{**}) \mathbin{\dot\vee} U_{D_1}(m)\big\rangle\Big) = \Big\langle i(x^{**}), f_j\big(U_{D_1}(x^{**}) \mathbin{\dot\vee} U_{D_1}(m)\big)\Big\rangle\,.$$

Since the triplets are isomorphic, we have

$$U_{D_2}\big(i(x^{**})^*\big) = f_j\big(U_{D_1}(x^*)\big)$$

for each $x \in A_1$. Moreover, $i(x^{**})^* = i(x^*)$ since i is a boolean isomorphism, thus

$$f_j\big(U_{D_1}(x^{**})\big) = U_{D_2}\big(i(x^{**})\big).$$

Then also for each $m \in \mathcal{D}(A_1)$ we have

$$f_j\big(U_{D_1}(m)\big) = U_{D_2}\big(j(m)\big),$$

i.e.

$$f_j\big(U_{D_1}(x^{**}) \mathbin{\dot{\vee}} U_{D_1}(m)\big) = U_{D_2}\big(i(x^{**})\big) \mathbin{\dot{\vee}} U_{D_2}\big(j(m)\big).$$

Since both i, j are bijections, the foregoing equality yields that also ψ is a bijection. Isotony of ψ, ψ^{-1} is clear, hence ψ is an order isomorphism of A_1 onto A_2. □

EXAMPLE 5. Let A be the set of Example 2 (see Fig. 3). Then $\mathcal{B}(A) \times \mathrm{SFil}(D)^d$ is the 16-element Boolean lattice and the set P_A isomorphic to A is visualized by the solid circles in Fig. 6.

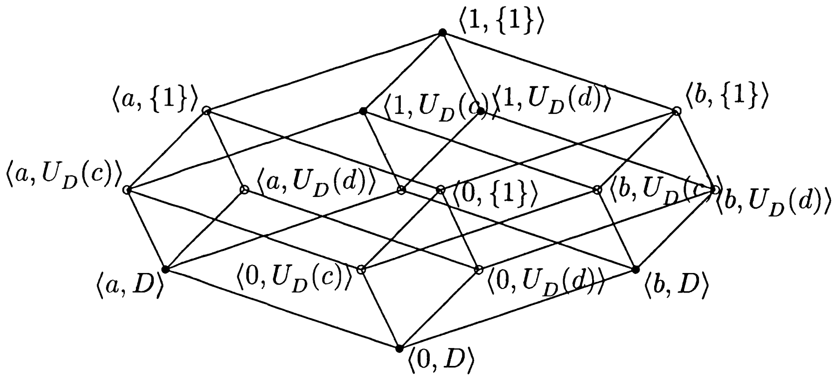

FIGURE 6.

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