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OSCILLATORY PROPERTIES OF SOLUTIONS OF A FOURTH-ORDER NONLINEAR DIFFERENTIAL EQUATION

VINCENT ŠOLTÉS

In [1] and [2] sufficient conditions were presented for the solutions of the equation

$$y^{(4)}+p(x)y''+q(x)y'+r(x)h(y)=f(x),$$

which satisfy an initial condition, to be oscillatory.

For $\varrho(x)\equiv 1$, the result of this paper do not follow from the results of [1], [2], and vice verba. The work extends to the set of theorems about the sufficient conditions for the oscillatory behaviour of the solutions. The same method is used in all the works.

The present paper presents sufficient conditions which ensure that the solutions of the equation

$$(1)\qquad [\varrho(x)y''']'+p(x)y''+q(x)y'+r(x)h(y)=f(x),$$

which satisfy a certain initial condition, are oscillatory. For $\varrho(x)\equiv 1$ the results are an extension of those of [1] and [2].

We shall assume throughout that $\varrho(x)>0$, $\varrho'(x)\geqq 0$, $\varrho''(x)\leqq 0$, $q(x)\geqq 0$, that $p(x)$, $q(x)$, $r(x)$, $f(x)$ and $h(y)$ are continuous for all $x\in\langle x_0,\infty)$, $y\in(-\infty,\infty)$ where $x_0\in(-\infty,\infty)$.

We shall consider the solutions of (1), which exist on $\langle x_0,\infty)$.

Let

$$F(x)=\varrho(x)y(x)y'''(x)-\varrho(x)y'(x)y''(x)+\tfrac{1}{2}\varrho'(x)y'^2(x)+\tfrac{1}{2}q(x)y^2(x)$$

$$F_1(x)=\varrho(x)y'(x)y'''(x)-\tfrac{1}{2}\varrho(x)y''^2(x)+\tfrac{1}{2}p(x)y'^2(x)+r(x)H(y(x))$$

$$H(y)=\int_0^y h(s)\,\mathrm{d}s$$

Lemma. *Suppose that* $q(x)\in C^1\langle x_0,\infty)$ *and that for all* $x\in\langle x_0,\infty)$ *and* $y\in(-\infty,\infty)$, $y\neq 0$

$$2\varrho(x)q'(x)+p^2(x)<0, \quad \operatorname{sgn} r(x)=\operatorname{sgn} h(y)y.$$

Then for any nonoscillatory solution of (1) *such that*

$$F(x_0)-\int_{x_0}^{\infty}\frac{f^2(x)\varrho(x)}{2q(x)\varrho(x)+p^2(x)}\,\mathrm{d}x=K_0\leqq 0 \tag{2}$$

exactly one of the following statements holds:

(i) $y(x)>0, \quad y'(x)>0, \quad y''(x)\geqq 0$ *for all* $x\geqq x_2\geqq x_0$
(ii) $y(x)<0, \quad y'(x)<0, \quad y''(x)\leqq 0$ *for all* $x\geqq x_2\geqq x_0$.

Proof. Let $y(x)$ be a nonoscillatory solution of (1) satisfying (2). Then there exists $x_1\geqq x_0$ such that $y(x)\neq 0$ for all $x\geqq x_1$. Multiply (1) by $y(x)$ and integrate from x_0 to $x\geqq x_0$, obtaining

$$F(x)+\int_{x_0}^{x}\varrho(t)y''^2(t)\,\mathrm{d}t+\int_{x_0}^{x}p(t)y''(t)y(t)\,\mathrm{d}t-\tfrac{1}{2}\int_{x_0}^{x}q'(t)y^2(t)\,\mathrm{d}t-$$
$$-\tfrac{1}{2}\int_{x_0}^{x}\varrho''(t)y'^2(t)\,\mathrm{d}t+\int_{x_0}^{x}r(t)h(y(t))y(t)\,\mathrm{d}t=F(x_0)+\int_{x_0}^{x}f(t)y(t)\,\mathrm{d}t. \tag{3}$$

Evidently for any real b, x, $a>0$ we have

$$ax^2+bx\geqq -\frac{b^2}{4a}. \tag{N}$$

Since $\varrho(x)>0$ and $2\varrho(x)q'(x)+p^2(x)<0$, we can use the last inequality to prove that

$$\varrho(x)y''^2(x)+p(x)y(x)y''(x)\geqq -\frac{p^2(x)}{4\varrho(x)}y^2(x),$$
$$-\tfrac{1}{4}\left[2q'(x)+\frac{p^2(x)}{\varrho(x)}\right]y^2(x)-f(x)y(x)\geqq\frac{f^2(x)\varrho(x)}{2q(x)\varrho(x)+p^2(x)}.$$

Using this, (3) yields

$$F(x)-\tfrac{1}{2}\int_{x_0}^{x}\varrho''(t)y'^2(t)\,\mathrm{d}t+\int_{x_0}^{x}r(t)h(y(t))y(t)\,\mathrm{d}t\leqq$$
$$\leqq F(x_0)-\int_{x_0}^{x}\frac{f^2(t)\varrho(t)}{2q(t)\varrho(t)+p^2(t)}\,\mathrm{d}t\leqq K_0\leqq 0 \tag{4}$$

for every $x\geqq x_0$. Omitting nonnegative terms on the left-hande side of the relation (4), we have for every $x\geqq x_1$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y''(x)}{y(x)}\right]\leqq -\tfrac{1}{2}\frac{q(x)}{\varrho(x)}. \tag{5}$$

Thus the function $\frac{y''(x)}{y(x)}$ is nonincreasing. This means that $y''(x)$, $y'(x)$ are monotonous in $\langle x_2, \infty)$, where $x_2 \geqq x_1$.

The following cases must be considered:

1. $y(x)>0, \quad y'(x)>0, \quad y''(x)\geqq 0$
2. $y(x)<0, \quad y'(x)<0, \quad y''(x)\leqq 0$
3. $y(x)>0, \quad y'(x)>0, \quad y''(x)<0$
4. $y(x)<0, \quad y'(x)<0, \quad y''(x)>0$
5. $y(x)>0, \quad y'(x)<0, \quad y''(x)>0$
6. $y(x)<0, \quad y'(x)>0, \quad y''(x)<0$
7. $y(x)>0, \quad y'(x)<0, \quad y''(x)\leqq 0$
8. $y(x)<0, \quad y'(x)>0, \quad y''(x)\geqq 0$

for every $x \geqq x_2 \geqq x_1$.

We shall prove the cases 3—8 are contradictory.

Suppose that case 3 holds. From (5) we have

$$\frac{y''(x)}{y(x_2)} \leqq \frac{y''(x)}{y(x)} \leqq \frac{y''(x_2)}{y(x_2)},$$

and therefore

$$y''(x) \leqq y''(x_2) < 0.$$

Integrating this from x_2 to $x \geqq x_2$, we obtain

$$y'(x) \leqq y'(x_2) + y''(x_2)\,(x - x_2),$$

which is a contradiction, since $y'(x)>0$.

Case 4 is disposed of analogously.

Suppose that case 5 holds. From (4) we can derive that there exists a positive constant A^2 such that

$$F(x) \leqq -A^2 < 0 \quad \text{for all} \quad x \geqq x_1 \geqq x_0$$

this means that

$$\varrho(x)y(x)y'''(x) < -A^2 \text{ for every } x \geqq x_2.$$

Since $y(x)$ decreases, we have

$$\varrho(x)y'''(x) < -\frac{A^2}{y(x_2)}$$

and integrating this from x_2 to $x \geqq x_2$ we obtain

$$\varrho(x)y''(x)-\varrho'(x)y'(x)+\int_{x_2}^{x}\varrho''(t)y'(t)\,dt<$$

$$<-\frac{A^2}{y(x_2)}(x-x_2)+\varrho(x_2)y''(x_2)-\varrho'(x_2)y'(x_2),$$

which again leads to a contradiction.

In the same way we show the impossibility of case 6.

If the case 7 or 8 hold, then there would exist $x_3 \geqq x_2$ such that $y(x_3)=0$ — again we have obtained a contradiction.

Theorem 1. *Suppose that the hypotheses of the Lemma hold and that, in addition, $h(y)$ is nondecreasing and $p(x)\geqq 0$ for every $y\in(-\infty,\infty)$ and $x\in\langle x_0,\infty)$, respectively.*

$$\text{If}\int_{x_0}^{\infty}\frac{dx}{\varrho(x)}=\int_{x_0}^{\infty}r(x)\,dx=\infty,\quad\left|\int_{x_0}^{\infty}f(x)\,dx\right|<\infty,$$

then any solution of (1) *satisfying* (2) *is oscillatory on* $\langle x_0,\infty)$.

Proof. Let $y(x)$ be a nonoscillatory solution of (1) which satisfies (2). According to our Lemma, there exists $x_1\geqq x_0$ such that either

1. $y(x)>0,\quad y'(x)>0,\quad y''(x)\geqq 0$ or
2. $y(x)<0,\quad y'(x)<0,\quad y''(x)\leqq 0$ for any $x\geqq x_1$.

Suppose that 1 is true. Then we have from (1)

$$[\varrho(x)y'''(x)]'\leqq f(x)-h(y)(x_1))r(x).$$

Integrating from x_1 to $x\geqq x_1$ we get

$$\varrho(x)y'''(x)\leqq\int_{x_1}^{x}f(t)\,dt-h(y(x_1))\int_{x_1}^{x}r(t)\,dt+\varrho(x_1)y'''(x_1).$$

Therefore there exists a positive constant B^2 such that for $x\geqq x_2\geqq x_1$ we have

$$y'''(x)\leqq-\frac{B^2}{\varrho(x)}$$

and thus $y''(x)\to-\infty$ as $x\to\infty$ — a contradiction.

Analogously we show that 2 cannot hold.

This completes the proof of Theorem 1.

Theorem 2. *Suppose that the hypotheses of our Lemma hold and that, in addition, for every $x\in\langle x_0,\infty)$*

$$p(x) \geqq 0, \quad \liminf_{y\to\infty} \frac{h(y)}{y} \geqq \varepsilon,$$

where ε *is a positive constant.*

$$\text{If } \int_{x_0}^{\infty} \frac{\mathrm{d}x}{\varrho(x)} = \int_{x_0}^{\infty} x r(x)\,\mathrm{d}x = \infty, \quad \left|\int_{x_0}^{\infty} f(x)\,\mathrm{d}x\right| < \infty,$$

then any solution of (1) *which satisfies* (2) *is oscillatory on* $\langle x_0, \infty)$.

Proof. Suppose that $y(x)$ is a nonoscillatory solution of (1) which satisfies (2). Suppose that, e.g.

$$y(x) > 0, \quad y'(x) > 0, \quad y''(x) \geqq 0 \quad \text{for all} \quad x \geqq x_1 \geqq x_0.$$

Since by hypothesis $\liminf_{y\to\infty} \frac{h(y)}{y} \geqq \varepsilon$, there exists a constant K such that for any $y \geqq K$

$$\frac{h(y)}{y} \geqq \frac{\varepsilon}{2}.$$

Since $y(x) \to \infty$ as $x \to \infty$, evidently there exists $x_2 \geqq x_1$ such that for all $x \geqq x_2$

$$\frac{h(y(x))}{y(x)} \geqq \frac{\varepsilon}{2}.$$

From (1) we have for every $x \geqq x_2$

$$[\varrho(x) y''']' \leqq f(x) - \frac{\varepsilon}{2} y(x) r(x).$$

Integraiting from x_2 to $x \geqq x_2$ we obtain

(6) $$\varrho(x) y'''(x) \leqq \varrho(x_2) y'''(x_2) + \int_{x_2}^{x} f(t)\,\mathrm{d}t - \frac{\varepsilon}{2} \int_{x_2}^{x} r(t) y(t)\,\mathrm{d}t.$$

In this case

$$y(x) - y(x_2) = \int_{x_2}^{x} y'(t)\,\mathrm{d}t \geqq y'(x_2)(x - x_2)$$

and thus

$$y(x) > y'(x_2)(x - x_2) \quad \text{for all} \quad x \geqq x_2.$$

The last inequality in conjuction with (6) shows that

$$\varrho(x) y'''(x) \leqq \varrho(x_2) y'''(x_2) + \left|\int_{x_2}^{x} f(t)\,\mathrm{d}t\right| - \frac{\varepsilon}{2} y'(x_2) \int_{x_2}^{x} (t - x_2) r(t)\,\mathrm{d}t,$$

which means that by the hypotheses of this Theorem there exists a positive constant C^2 such that for all $x \geqq x_3 \geqq x_2$

$$y'''(x) \leqq -\frac{C^2}{\varrho(x)}, \quad \text{whence}$$

$$y''(x) \leqq y''(x_3) - C^2 \int_{x_3}^{x} \frac{dt}{\varrho(t)},$$

which is again a contradiction, since $y''(x) \geqq 0$.

The proof for the case $y(x) < 0$, $y'(x) < 0$, $y''(x) \leqq 0$ is analogous.

This complete the proof of Theorem 2.

Theorem 3. *Suppose that the hypotheses of our Lemma hold. If*

$$\int_{x_0}^{\infty} \frac{q(x)}{\varrho(x)} \, dx = \infty,$$

then any solution of (1) *satisfying* (2) *is oscillatory on* $\langle x_0, \infty)$.

Proof. Let $y(x)$ be a nonoscillatory solution of (1) which satisfies (2). Integrating (5) from x_1 to $x \geqq x_1$ we get

$$\frac{y''(x)}{y(x)} \leqq \frac{y''(x_1)}{y(x_1)} - \frac{1}{2} \int_{x_1}^{x} \frac{q(t)}{\varrho(t)} \, dt,$$

whence $\dfrac{y''(x)}{y(x)} \to -\infty$ as $x \to \infty$; this is a contradiction according to our Lemma.

This completes the proof of Theorem 3.

Theorem 4. *Suppose that the hypotheses of our Lemma hold and that, in addition,* $p(x)$, $r(x) \in C^1 \langle x_0, \infty)$ *and for all* $x \in \langle x_0, \infty)$

$$2q(x) - p'(x) - |f(x)| \geqq 0, \quad r(x) \geqq 0, \quad r'(x) \leqq 0.$$

If

$$\int_{x_0}^{\infty} \frac{p(x)}{\varrho(x)} \, dx = +\infty,$$

then any solution of (1) *which satisfies the relations* (2) *and*

(7) $$F_1(x_0) + \frac{1}{2} \int_{x_0}^{\infty} |f(x)| \, dx = K_1 \leqq 0$$

is oscillatory on $\langle x_0, \infty)$.

Proof. Suppose that $y(x)$ is a nonoscillatory solution of (1) which satisfies (2) and (7). Multiply (1) by $y'(x)$ and integrate from x_0 to $x \geqq x_0$, obtaining

$$(8) \qquad F_1(x)+\frac{1}{2}\int_{x_0}^{x}[2q(t)-p'(t)]y'^2(t)\,dt-\int_{x_0}^{x}r'(t)H(y(t))\,dt=$$

$$=F_1(x_0)+\int_{x_0}^{x}f(t)y'(t)\,dt-\frac{1}{2}\int_{x_0}^{x}\varrho'(t)\,(y''(t))^2\,dt.$$

This leads easily to

$$F_1(x)+\tfrac{1}{2}\int_{x_0}^{x}[2q(t)-p'(t)-|f(t)|]y'^2(t)\,dt-$$

$$-\int_{x_0}^{x}r'(t)H(y(t))\,dt\leqq F_1(x_0)+\tfrac{1}{2}\int_{x_0}^{x}|f(t)|\,dt\leqq K_1\leqq 0.$$

Omitting nonnegative terms on the left-hande side of this relation, we have for all $x\geqq x_1\geqq x_0$

$$\frac{d}{dx}\left[\frac{y''(x)}{y'(x)}\right]\leqq-\tfrac{1}{2}\frac{p(x)}{\varrho(x)}$$

and therefore

$$\frac{y''(x)}{y'(x)}\leqq\frac{y''(x_1)}{y'(x_1)}-\tfrac{1}{2}\int_{x_1}^{x}\frac{p(t)}{\varrho(t)}\,dt.$$

Thus $\dfrac{y''(x)}{y'(x)}\to-\infty$ as $x\to\infty$ – a contradiction according to our Lemma.

This completes the proof of Theorem 4.

Theorem 5. *Suppose that the hypotheses of our Lemma hold and that, moreover* $p(x),\ r(x)\in C^1\langle x_0,\infty)$ *and*

$$2q(x)-p'(x)>0,\quad r(x)\geqq 0,\quad r'(x)\leqq 0$$

for all $x\in\langle x_0,\infty)$

If

$$\int_{x_0}^{\infty}\frac{p(x)}{\varrho(x)}\,dx=+\infty,$$

then any solution of (1) *which satisfies* (2) *and also*

$$(9) \qquad F_1(x_0)+\tfrac{1}{2}\int_{x_0}^{\infty}\frac{f^2(x)}{2q(x)-p'(x)}\,dx=K_2\leqq 0$$

is oscillatory on $\langle x_0,\infty)$.

Proof. Using (N), we see that

$$\tfrac{1}{2}[2q(x)-p'(x)]y'^2(x)-f(x)y'(x)\geqq-\frac{f^2(x)}{2[2q(x)-p'(x)]}.$$

Therefore (8) yields

$$F_1(x)-\int_{x_0}^{x} r'(t)H(y(t))\,\mathrm{d}t \leqq F_1(x_0)+\tfrac{1}{2}\int_{x_0}^{x} \frac{f^2(t)}{2q(t)-p'(t)}\,\mathrm{d}t \leqq 0$$

for all $x \geqq x_0$. The rest of the proof repeats that of Theorem 4.

In the sequel we shall assume that $f(x)\equiv 0$, thus considering the equation

$$[\varrho(x)y''']'+p(x)y''+q(x)y'+r(x)h(y)=0. \tag{10}$$

Theorem 6. *Suppose that the hypotheses of our Lemma hold and that in addition*

$$p(x)\geqq 0,\quad r(x)\geqq 0 \quad \textit{for all} \quad x\in\langle x_0,\infty).$$

If

$$\int_{x_0}^{\infty}\frac{\mathrm{d}x}{\varrho(x)}=\int_{x_0}^{\infty} xq(x)\,\mathrm{d}x=\infty,$$

then any solution of (10) *satisfying* (2) *is oscillatory on* $\langle x_0, \infty)$.

Proof. Let $y(x)$ be a nonoscillatory solution of (10) satisfying (2). According to our Lemma, there exists $x_1 \geqq x_0$ such that either

1. $y(x)>0,\quad y'(x)>0,\quad y''(x)\geqq 0$ or
2. $y(x)<0,\quad y'(x)<0,\quad y''(x)\leqq 0$ for all $x\geqq x_1$.

Consider the case 1. From (10) we see that for all $x \geqq x_1$

$$[\varrho(x)y''']'\leqq 0.$$

Suppose that there exists $x_2 \geqq x_1$ such that $y'''(x_2)<0$. Integrating the last relation from x_2 to $x\geqq x_2$, we obtain

$$\varrho(x)y'''(x)\leqq \varrho(x_2)y'''(x_2)$$

$$y'''(x)\leqq \varrho(x_2)y'''(x_2)\frac{1}{\varrho(x)} \quad \text{for all} \quad x\geqq x_2.$$

Since by hypotheses $\int_{x_0}^{\infty}\frac{\mathrm{d}x}{\varrho(x)}=\infty$, this means that $y''(x)\to-\infty$ as $x\to\infty$ – a contradiction. Thus $y'''(x)\geqq 0$ for every $x\geqq x_1$; double integration from x_1 to $x\geqq x_1$ shows that

$$y'(x)\geqq y''(x_1)\,(x-x_1) \quad \text{for all} \quad x\geqq x_1.$$

From (10) we have for every $x\geqq x_1$

$$[\varrho(x)y''']'\leqq -q(x)y'(x)\leqq -y''(x_1)\,(x-x_1)q(x)$$

and therefore

$$\varrho(x)y'''(x) \leqq \varrho(x_1)y'''(x_1) - y''(x_1)\int_{x_1}^{x} (t-x_1)q(t)\,dt.$$

Since $\int_{x_0}^{\infty} xq(x) = \infty$ by hypothesis, there exists a positive constant D^2 such that

$$y'''(x) \leqq -\frac{D^2}{\varrho(x)} \quad \text{for all} \quad x \geqq x_2 \geqq x_1,$$

which is a contradiction since $y'''(x) \geqq 0$.

The method of proof is analogous to that of case 2.

This completes the proof of Theorem 6.

The proof of the following Theorem would be analogous.

Theorem 7. *Suppose that the hypotheses of our Lemma hold and that in addition* $p(x) \in C^1\langle x_0, \infty)$ *and* $p(x) \geqq 0$, $r(x) \geqq 0$, $q(x) - p'(x) \geqq 0$ *for all* $x \in \langle x_0, \infty)$.

If

$$\int_{x_0}^{\infty} \frac{dx}{\varrho(x)} = \int_{x_0}^{\infty} x[q(x) - p'(x)]\,dx = \infty,$$

then any solution of (10) *which satisfies* (2) *is oscillatory on* $\langle x_0, \infty)$.

Remark. *Evidently any solution of* (10) *which has a double zero on* $\langle x_0, \infty)$ *satisfies at this point* (2) *as well as* (7) *and* (9).

Thus for equation (10) the requirement that a solution satisfy the initial condition (2), (7) or (9) may be substituted by the requirement that solution have a double zero in $\langle x_0, \infty)$.

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О КОЛЕБЛЕМОСТИ РЕШЕНИЙ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА

Винцент Шолтес

Резюме

В етой работе приведены теоремы, дающие достаточные условия для того, чтобы любое решение уравнения

$$[\varrho(x)y''']' + p(x)y'' + q(x)y' + r(x)h(y) = f(x),$$

удовлетворяющего начальному условию, колебалось в $\langle x_0), \infty)$.