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THE PROJECTIVE PROPERTIES OF THE EXTREME PATH DERIVATIVES

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ABSTRACT. The paper studies the projective and measurable properties that the extreme path derivatives as well as the multifunction of all path derived numbers must have under certain measurable and projective conditions about the system of paths.

I. Introduction

One approach to get a unified method of the study of a number of generalized derivatives is based on the concept of path system differentiation [2]. Namely a collection $E = \{E(x):\ x \in \mathbb{R}\}$ ($\mathbb{R}$ – real line) is a system of paths if each set $E(x) \subset \mathbb{R}$ has x as a point of accumulation. It can be considered as a multifunction $E\colon x \mapsto E(x)$. If $f\colon \mathbb{R} \to \mathbb{R}$ is a function, then the upper and lower E-derivatives of f at x are defined as follows:

$$\overline{f}'_E(x) = \limsup_{\substack{y \to x \\ y \in E(x)}} \frac{f(x) - f(y)}{x - y} \quad \text{and} \quad \underline{f}'_E(x) = \liminf_{\substack{y \to x \\ y \in E(x)}} \frac{f(x) - f(y)}{x - y}\,.$$

If $\underline{f}'_E(x) = \overline{f}'_E(x)$, their common value is called the E-derivative of f at x $\big(f'_E(x)\big)$. By $E_f\colon \mathbb{R} \to \mathbb{R}^*$ ($\mathbb{R}^*$ – the extended real line with the topology of two-point compactification) we denote the following non-empty and compact-valued multifunction defined as

$$E_f(x) = \left\{ y \in \mathbb{R}^*:\ \text{there is a sequence } \{x_n\}_{n=1}^{\infty} \text{ in } E(x) \setminus \{x\} \right.$$
$$\left. \text{so that } \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = y \right\}.$$

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There are cases, where the behaviour of E_f is very bad. For example, there is a continuous function f such that, given any function g, a system of paths E can be found so that $f'_E = g$ [4].

The aim of this paper is the investigation of measurability properties that the multifunction E_f and $\overline{f}'_E$, $\underline{f}'_E$ must have under certain measurability conditions concerning the system of paths E. We shall namely study projective and measurability properties of E_f under the following assumptions about E:

(a) Gr (E) (graph of E) $= \bigcup\limits_{l=1}^{\infty} \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{j=1}^{\infty} A_{j,k,l} \times B_{j,k,l}$, where $A_{j,k,l}$ is of projective class n and $B_{j,k,l} \subset \mathbb{R}$ (Theorem 13),
(b) Graph of E is of projective class n (Theorem 12),
(c) Graph of E belongs to a given σ-algebra (Theorem 17, Corollary 24),
(d) E is measurable with respect to a given σ-algebra (Theorem 22, Corollary 23).

This paper can be considered as a continuation of [8], where a semi-Borel classification of E_f was given and where measurability properties of E_f were studied under stronger conditions about the graph of E than it is in this paper. We hope that this paper will give a comprehensive and deeper information concerning the measurability of generalized derivatives. We note that the motivation for our concept came from paper [1] by Alikhani-Koopaei. His method of considering E as a multifunction seems to be a convenient tool for investigating some problems connected with path derivatives.

2. Notation and preliminary results

The set of all positive integers is denoted by $\mathbb{N}$. Let $F\colon \mathbb{R} \to \mathbb{R}^*$ be a multifunction, $f\colon \mathbb{R} \to \mathbb{R}$ be a function, $T \subset \mathbb{R}$, $A \subset \mathbb{R} \times \mathbb{R}$.

We set

$$f_0\big((x,y)\big) = \frac{f(x) - f(y)}{x - y}, \qquad (x,y) \in \mathbb{R} \times \mathbb{R}, \quad x \neq y,$$

$$F^-(T) = \big\{x \in \mathbb{R}\colon\ F(x) \cap T \neq \emptyset\big\},$$

$$F^+(T) = \big\{x \in \mathbb{R}\colon\ F(x) \subset T\big\},$$

$$\mathrm{Gr}\,(F) = \big\{(x,y)\colon\ y \in F(x)\big\} \quad \text{(graph of } F\text{)},$$

$$\mathrm{pr}\,(A) = \{x\colon\ \exists y \in \mathbb{R} \quad \text{s.t.} \quad (x,y) \in A\}\,.$$

If a single valued function $f\colon \mathbb{R} \to \mathbb{R}^*$ is given, then under the natural interpretation of $f(x)$ as a one point set we have $f^+(T) = f^-(T) = f^{-1}(T)$.

LEMMA 1.

(a) $F^-(T) = \mathbb{R} \setminus F^+(\mathbb{R} \setminus T)$.

(b) $F^-\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} F^-(A_n)$.

(c) $F^+\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \bigcap_{n=1}^{\infty} F^+(A_n)$.

(d) *If K is a closed set in $\mathbb{R}^*$, F is compact-valued and $G_n = \{x: d(x,K) < \frac{1}{n}\}$ (d is a metric for $\mathbb{R}^*$), then $F^-(K) = \bigcap_{n=1}^{\infty} F^-(G_n)$. As special cases we have $F^-(\langle -\infty, b\rangle) = \bigcap_{n=1}^{\infty} F^-(\langle -\infty, b+\frac{1}{n}))$, $F^-(\langle a, \infty\rangle) = \bigcap_{n=1}^{\infty} F^-((a-\frac{1}{n}, \infty\rangle)$, $F^-(\langle a, b\rangle) = \bigcap_{n=1}^{\infty} F^-((a-\frac{1}{n}, b+\frac{1}{n}))$ for any $a, b \in \mathbb{R}$, $a < b$.*

The trivial proof of Lemma 1 is omitted.

Moreover, we use the following notation for families of subsets of $\mathbb{R}$:

A_α (M_α) – the family of all subsets of the Borel additive (multiplicative) class α.
P_n – the family of all projective subsets of class n, $n = 0, 1, 2, \dots$.
L – the family of all Lebesgue measurable subsets.
Br – the family of all subsets having the Baire property.

LEMMA 2. (*see* [5, pp. 361–362]). *Let $A, B \in P_n$ and let $S \subset \mathbb{R} \times \mathbb{R}$ be a set of projective class n, $n = 0, 1, 2, \dots$. Let $P_{n-1} = P_0$ for $n = 0$.*

Then

(a) *$A \times B$ is of projective class n,*
(b) *$\mathrm{pr}(S) \in P_n$ for n odd and $\mathrm{pr}(S) \in P_{n+1}$ for n even,*
(c) *$\mathbb{R} \setminus A \in P_{n-1}$ for n even and $\mathbb{R} \setminus A \in P_{n+1}$ for n odd,*
(d) *$P_{2n} \subset P_{2n+2} \cap P_{2n+1}$,*
(e) *$P_{2n+1} \subset P_{2n+3}$,*
(f) *$P_{2n+1} \subset P_{2n+4}$,*
(g) *P_n is closed relatively to countable unions and intersections and if $0 \neq n \neq 2$, then P_n is closed with respect to operation $\mathcal{A}$* [5, p. 375],
(h) *The assertions (c)–(g) hold for the family of all planar projective subsets of class n.*

If E is a system of paths, then for $j = 1, 2, 3, \dots$ we define multifunctions

E^j, E^j_+, E^j_- as follows:

$$\begin{aligned} E^j(x) &= E(x) \cap (x - \tfrac{1}{j}, x + \tfrac{1}{j}), \\ E^j_+(x) &= E(x) \cap (x, x + \tfrac{1}{j}), \\ E^j_-(x) &= E(x) \cap (x - \tfrac{1}{j}, x). \end{aligned}$$

The following lemma is the essence for investigating the measurable properties of E_f.

LEMMA 3. *Let E be an arbitrary system of paths and f be a function. If K is a closed set in $\mathbb{R}^*$, then*

$$E_f^-(K) = \bigcap_{j=1}^{\infty} \operatorname{pr}\left(f_0^{-1}(V^j) \cap \operatorname{Gr}(E^j)\right),$$

where $V^j = \{x\colon\ d(x,K) < \frac{1}{j}\}$ (d is a metric for $\mathbb{R}^*$), $j = 1, 2, \dots$.

The proof is trivial and hence omitted.

3. The Borel classification of E_f

DEFINITION 4. *A multifunction $F\colon \mathbb{R} \to \mathbb{R}^*$ is said to be of lower (upper) class α, if $F^-(G) \in A_\alpha$ ($F^+(G) \in A_\alpha$) for each open $G \subset \mathbb{R}^*$. A function $f\colon \mathbb{R} \to \mathbb{R}^*$ is said to be of lower (upper) class α, if $f^{-1}\left((a,\infty\rangle\right) \in A_\alpha$ ($f^{-1}\left(\langle-\infty,a)\right) \in A_\alpha$) for all $a \in \mathbb{R}$.*

THEOREM 5. *Let f be a function of class α. If $\operatorname{Gr}(E) = \bigcup_{n=1}^{\infty} A_n \times B_n$, $A_n \in A_\alpha$, $B_n \subset \mathbb{R}$, then E_f is a multifunction of upper class $\alpha + 1$ and consequently $\overline{f}'_E$ ($\underline{f}'_E$) is a function of upper (lower) class $\alpha + 1$.*

Proof. By [8, Lemma 2.6], $f_0^{-1}\left((a,b)\right) \cap \operatorname{Gr}(E^j) = \left[f_0^{-1}\left((a,b)\right) \cap \operatorname{Gr}(E^j_+)\right] \cup \left[f_0^{-1}\left((a,b)\right) \cap \operatorname{Gr}(E^j_-)\right] = \left[S_a \cap T_b \cap \operatorname{Gr}(E^j_+)\right] \cup \left[S_b \cap T_a \cap \operatorname{Gr}(E^j_-)\right]$, where $T_a = \{(x,y)\colon\ f(x) - ax < f(y) - ay\}$ and $S_a = \{(x,y)\colon\ f(x) - ax > f(y) - ay\}$ (see [8, Lemma 2.5]).

Hence the set $f_0^{-1}(V^j) \cap \operatorname{Gr}(E^j)$ can be expressed as $\bigcup_{k=1}^{\infty} X^j_k \times Y^j_k$, $X^j_k \in A_\alpha$, $Y^j_k \subset \mathbb{R}$, $j = 1, 2, 3 \dots$ (see also [8, Lemma 2.5]).

By Lemma 3, $E_f^-(K) = \bigcap_{j=1}^{\infty}\Big(\bigcup_{k=1}^{\infty} X_k^j\Big) \in M_{\alpha+1}$ for any closed $K \subset \mathbb{R}^*$, hence $E_f^+(G) = \mathbb{R} \setminus E_f^-(\mathbb{R} \setminus G) \in A_{\alpha+1}$ for any open $G \subset \mathbb{R}^*$. The equalities $\overline{f}'^{-1}_E(\langle a,\infty\rangle) = E_f^-(\langle a,\infty\rangle)$ and $\underline{f}'^{-1}_E(\langle -\infty,a\rangle) = E_f^-(\langle -\infty,a\rangle)$ finish the proof.

THEOREM 6. *If f is Baire 1 and $\mathrm{Gr}(E)$ is an F_σ-set, then E_f is a multifunction of upper class 2 and consequently $\overline{f}'_E$ ($\underline{f}'_E$) is a function of upper (lower) class 2.*

Proof. This follows directly from Lemma 3, since if $f_0^{-1}(V^j) \cap \mathrm{Gr}(E^j)$ is an F_σ-set, then $E_f^-(K)$ is an $F_{\sigma\delta}$-set. Hence $E_f^+(G)$ is a $G_{\delta\sigma}$-set for any open $G \subset \mathbb{R}^*$.

THEOREM 7. *Let E be a lower semi-continuous system of paths (i.e. $E^-(G)$ is open for any open set $G \subset \mathbb{R}$) and let f be a continuous function. Then E_f is a multifunction of upper class 1 and $\overline{f}'_E$ ($\underline{f}'_E$) is a function of upper (lower) class 1.*

Proof. We shall show that $A^j = \mathrm{pr}\big(f_0^{-1}(V^j) \cap \mathrm{Gr}(E^j)\big)$ is open for any $j = 1,2,3,\ldots$. Let $x_0 \in A^j$. Then there is $y_0 \in \mathbb{R}$ s.t. $(x_0,y_0) \in f_0^{-1}(V^j)$ and $y_0 \in E^j(x_0)$. Since f_0 is continuous, there are open intervals I, J such that $(x_0,y_0) \in I \times J \subset f_0^{-1}(V^j)$. Since E^j is lower semi-continuous, there is an open set $G \subset I$ with $x_0 \in G$ such that $E^j(x) \cap J \neq \emptyset$ for any $x \in G$. Thus for any $x \in G$ there is $y_x \in E^j(x) \cap J$. Since $(x,y_x) \in \mathrm{Gr}(E^j) \cap f_0^{-1}(V^j)$, $x \in A^j$ for any $x \in G$. Hence A^j is open. By Lemma 3, $E_f^-(K) \in M_1$ for any closed $K \subset \mathbb{R}^*$, hence E_f is a multifunction of upper class 1. The classification of the extreme E-derivatives is similar to that in the proof of Theorem 5.

Remark 8. Note that the assertions of this section are stronger than those of paper [8], where we consider only the inverse image of intervals under E_f. Further results as well as open problems concerning the Borel classification of E_f can be found in [9].

4. The projective properties of E_f

DEFINITION 9. *A multifunction $F\colon \mathbb{R} \to \mathbb{R}^*$ is a lower (upper) projective multifunction of class n with respect to the upper inverse image (briefly $F \in UP_n^-$ ($F \in UP_n^+$)) if $F^+(\langle a,\infty\rangle) \in P_n$ ($F^+(\langle -\infty,a\rangle) \in P_n$) for all $a \in \mathbb{R}$. F is a lower (upper) projective multifunction of class n with respect to the lower inverse image (briefly $F \in LP_n^-$ ($F \in LP_n^+$)) if $F^-(\langle a,\infty\rangle) \in P_n$*

($F^-(\langle -\infty, a))\in P_n$) for all $a \in \mathbb{R}$. For single-valued functions we use the notation P_n^- ($= UP_n^- = LP_n^-$), P_n^+ ($= UP_n^+ = LP_n^+$).

LEMMA 10. *If F is a compact-valued multifunction, then for $n = 1, 2, 3, \ldots$ we have*

$$\text{(a)}\quad UP_{2n}^- = LP_{2n-1}^+ , \qquad \text{(b)}\quad UP_{2n}^+ = LP_{2n-1}^- ,$$

$$\text{(c)}\quad UP_{2n-1}^- = LP_{2n}^+ , \qquad \text{(d)}\quad UP_{2n-1}^+ = LP_{2n}^- .$$

Proof.

(a) If $F \in UP_{2n}^-$, then

$$F^-(\langle -\infty, a)) = \bigcup_{i=1}^{\infty} F^-(\langle -\infty, a - \tfrac{1}{i}\rangle) = \mathbb{R} \setminus \bigcap_{i=1}^{\infty} F^+((a - \tfrac{1}{i}, \infty\rangle) \in P_{2n-1}$$

(see Lemmas 1b and 2c, g).

If $F \in LP_{2n-1}^+$, then

$$F^+((a, \infty\rangle) = \mathbb{R} \setminus F^-(\langle -\infty, a\rangle) = \mathbb{R} \setminus \bigcap_{i=1}^{\infty} F^-(\langle -\infty, a + \tfrac{1}{i})) \in P_{2n}$$

(see Lemmas 1d and 2c, g).

(b) If $F \in UP_{2n}^+$, then

$$F^-((a, \infty\rangle) = \bigcup_{i=1}^{\infty} F^-(\langle a + \tfrac{1}{i}, \infty\rangle) = \bigcup_{i=1}^{\infty} \Big(\mathbb{R} \setminus F^+(\langle -\infty, a + \tfrac{1}{i}))\Big) \in P_{2n-1}$$

(see Lemmas 1b and 2c, g).

If $F \in LP_{2n-1}^-$, then

$$F^+(\langle -\infty, a)) = F^+\Big(\bigcup_{i=1}^{\infty} \langle -\infty, a - \tfrac{1}{i}\rangle\Big) = \mathbb{R} \setminus F^-\Big(\bigcap_{i=1}^{\infty} (a - \tfrac{1}{i}, \infty\rangle\Big)$$

$$= \mathbb{R} \setminus \bigcap_{i=1}^{\infty} F^-((a - \tfrac{1}{i}, \infty\rangle) \in P_{2n}$$

(see Lemmas 1d and 2c, g).

(c) If $F \in UP_{2n-1}^-$, then

$$F^-(\langle -\infty, a)) = \bigcup_{i=1}^{\infty} F^-(\langle -\infty, a - \tfrac{1}{i}\rangle) = \mathbb{R} \setminus \bigcap_{i=1}^{\infty} F^+((a - \tfrac{1}{i}, \infty\rangle) \in P_{2n}$$

(see Lemmas 1b and 2c, g).

If $F \in LP^{+}_{2n}$, then

$$F^{+}(\langle a,\infty\rangle) = \mathbb{R} \setminus F^{-}(\langle -\infty, a\rangle) = \mathbb{R} \setminus \bigcap_{i=1}^{\infty} F^{-}(\langle -\infty, a+\tfrac{1}{i}\rangle) \in P_{2n-1}$$

(see Lemmas 1d and 2c, g).

(d) If $F \in UP^{+}_{2n-1}$, then

$$F^{-}(\langle a,\infty\rangle) = \bigcup_{n=1}^{\infty} F^{-}(\langle a+\tfrac{1}{n},\infty\rangle) = \bigcup_{i=1}^{\infty}\Big(\mathbb{R} \setminus F^{+}(\langle -\infty, a+\tfrac{1}{i}\rangle)\Big) \in P_{2n}$$

(see Lemmas 1b and 2c, g).

If $F \in LP^{-}_{2n}$, then

$$F^{+}(\langle -\infty, a\rangle) = F^{+}\Big(\bigcup_{i=1}^{\infty}\langle -\infty, a-\tfrac{1}{i}\rangle\Big) = \mathbb{R} \setminus F^{-}\Big(\bigcap_{i=1}^{\infty}\langle a-\tfrac{1}{i},\infty\rangle\Big)$$

$$= \mathbb{R} \setminus \bigcap_{i=1}^{\infty} F^{-}(\langle a-\tfrac{1}{i},\infty\rangle) \in P_{2n-1}$$

(see Lemmas 1d and 2c, g).

If $\mathrm{Gr}(E) = \bigcup_{i=1}^{\infty} A_i \times B_i$, where $A_i \in P_n$, $B_i \in \mathbb{R}$, then the first information about the projective classification of E_f is given by [8, Theorem 3.3]. It is clear from the following tables (for $n = 0$, let $P_{n-1} = P_0$).

n-even

f	E	E_f
P^{+}_{n}	$+$	UP^{+}_{n-1}
P^{-}_{n}	$-$	UP^{+}_{n-1}
P^{+}_{n}	$-$	UP^{-}_{n-1}
P^{-}_{n}	$+$	UP^{-}_{n-1}

n-odd

f	E	E_f
P^{+}_{n}	$+$	UP^{+}_{n+1}
P^{-}_{n}	$-$	UP^{+}_{n+1}
P^{+}_{n}	$-$	UP^{-}_{n+1}
P^{-}_{n}	$+$	UP^{-}_{n+1}

The sign $+$ $(-)$ corresponds to a right-(left-)sided system of paths. For example, if $f \in P^{+}_{n}$ and E is left sided, then $E_f \in UP^{-}_{n-1}$ for n even and $E_f \in UP^{-}_{n+1}$ for n odd.

By Lemma 10, the preceding tables can be joined by the following table which holds for E with a more general graph as we shall see in Theorem 13 below.

f	E	E_f
P_n^+	$+$	LP_n^-
P_n^-	$-$	LP_n^-
P_n^+	$-$	LP_n^+
P_n^-	$+$	LP_n^+

EXAMPLE A. *Let n be odd. There is a right sided system E in the lower Borel class 1 and a function f in P_n^- such that $f'_E \in P_n^+ \setminus \bigcup_{k=0}^{n-1} P_k^+$.*

EXAMPLE B. *Let $n > 0$ be even. There is a right-sided system of paths E in the lower Borel class 1 and a function f in P_n^+ such that $f'_E \in P_{n-1}^+ \setminus \bigcup_{k=0}^{n-2} P_k^+$.*

Proof A. Let $Z \in P_n \setminus \bigcup_{k=0}^{n-1} P_k$ [5, p. 368] and let Z_r be a set of all right cluster points of Z. Let $E(x) = Z \cap (x, \infty)$ for $x \in Z_r$ and $E(x) = (x, \infty)$ for $x \notin Z_r$. Since $Z_r \in A_1 \cap M_1$ and $\mathrm{Gr}(E) = \big[(Z_r \times Z) \cup (\mathbb{R} \setminus Z_r) \times \mathbb{R}\big] \cap H$, where $H = \big\{(x,y)\colon\ x < y\big\}$, $\mathrm{Gr}(E)$ can be expressed as $\mathrm{Gr}(E) = \bigcup_{n=1}^{\infty} C_n \times B_n$, where $C_n \in A_1 \cap M_1$. Hence $E^-(S) \in \bigcup C_{n_i}$ for any $S \subset \mathbb{R}$ (sum is taken over all C_{n_i} s.t. $B_{n_i} \cap S \neq \emptyset$). Thus E is in the lower Borel class 1.

Denote the characteristic function of Z by f. Then

$$f'_E(x) = \begin{cases} -\infty & \text{for } x \in Z \cap (\mathbb{R} \setminus Z_r), \\ 0 & \text{for } x \in Z \cap Z_r, \\ 0 & \text{for } x \in (\mathbb{R} \setminus Z) \cap (\mathbb{R} \setminus Z_r), \\ \infty & \text{for } x \in (\mathbb{R} \setminus Z) \cap Z_r. \end{cases}$$

Since

$$f^{-1}(a, \infty) = \begin{cases} \emptyset \in P_0 & \text{if } a \geqq 1, \\ Z \in P_n & \text{if } 0 \leqq a < 1, \\ \mathbb{R} \in P_0 & \text{if } a < 0. \end{cases} \qquad f \in P_n^-,$$

It is clear that $Z \cap (\mathbb{R} \setminus Z_r)$ is a countable set, hence $Z \cap (\mathbb{R} \setminus Z_r) \in P_0$. Moreover $\mathbb{R} \setminus Z_r \in P_0$.

Since

$$f_E'^{-1}\big((-\infty, a)\big) = \begin{cases} Z \cap (\mathbb{R} \setminus Z_r) \in P_0 & \text{if } \ a \leqq 0, \\ Z \cup (\mathbb{R} \setminus Z_r) \in P_n & \text{if } \ a > 0 \end{cases}$$

and $f_E'^{-1}\big((-\infty, \infty)\big) = \mathbb{R} \in P_0$, $f_E' \in P_n^+ \setminus \bigcup\limits_{k=0}^{n-1} P_k^+$.

Proof B. By Example A, there is a right-sided system of paths E in the lower Borel class 1 and a function f in $P_{n-1}^- = P_n^+$ such that

$$f_E' \in P_{n-1}^+ \setminus \bigcup_{k=0}^{n-2} P_k^+ .$$

Lemma 11. *For any $a \in \mathbb{R}$ and $j = 1, 2, 3 \dots$ we have*

(a) *If $f \in P_n^+$ and E is right-sided, then*

$$f_0^{-1}\big((a - \tfrac{1}{j}, \infty)\big) \cap \operatorname{Gr}(E^j) = \left[\bigcup_{i=1}^{\infty} (A_j^i \times B_j^i)\right] \cap \operatorname{Gr}(E_+^j), \quad A_j^i,\ \mathbb{R} \setminus B_j^i \in P_n .$$

(b) *If $f \in P_n^-$ and E is left-sided, then*

$$f_0^{-1}\big((a - \tfrac{1}{j}, \infty)\big) \cap \operatorname{Gr}(E^j) = \left[\bigcup_{i=1}^{\infty} (A_j^i \times B_j^i)\right] \cap \operatorname{Gr}(E_-^j), \quad A_j^i,\ \mathbb{R} \setminus B_j^i \in P_n .$$

(c) *If $f \in P_n^+$ and E is left-sided, then*

$$f_0^{-1}\big((-\infty, a + \tfrac{1}{j})\big) \cap \operatorname{Gr}(E^j) = \left[\bigcup_{i=1}^{\infty} (A_j^i \times B_j^i)\right] \cap \operatorname{Gr}(E_-^j), \quad A_j^i,\ \mathbb{R} \setminus B_j^i \in P_n .$$

(d) *If $f \in P_n^-$ and E is right-sided, then*

$$f_0^{-1}\big((-\infty, a + \tfrac{1}{j})\big) \cap \operatorname{Gr}(E^j) = \left[\bigcup_{i=1}^{\infty} (A_j^i \times B_j^i)\right] \cap \operatorname{Gr}(E_+^j), \quad A_j^i,\ \mathbb{R} \setminus B_j^i \in P_n .$$

This lemma follows directly from [8, Lemmas 2.5 and 2.6].

THEOREM 12. *Let E be a system of paths with the graph of projective class n. Then E_f has properties given in the following tables:*

n-even

f	E	E_f
P_n^+	$+$	LP_{n+1}^-
P_n^-	$-$	LP_{n+1}^-
P_n^+	$-$	LP_{n+1}^+
P_n^-	$+$	LP_{n+1}^+

n-odd

f	E	E_f
P_n^+	$+$	LP_{n+2}^-
P_n^-	$-$	LP_{n+2}^-
P_n^+	$-$	LP_{n+2}^+
P_n^-	$+$	LP_{n+2}^+

P r o o f. Note that if X^i, $\mathbb{R}\backslash Y^i \in P_n$ and S is of the projective class n, then $\mathrm{pr}\Big[\Big(\bigcup\limits_{i=1}^{\infty} X^i\times Y^i\Big)\cap S\Big] = \bigcup\limits_{i=1}^{\infty}(X^i\cap \mathrm{pr}(S\cap\mathbb{R}\times Y^i))$. Thus $\mathrm{pr}\Big[\Big(\bigcup\limits_{i=1}^{\infty} X^i\times Y^i\Big)\cap S\Big]$ belongs to P_{n+1} (P_{n+2}) for n even (odd), by Lemma 2. Let $a\in\mathbb{R}$. By Lemmas 3 and 11 we have

(a) If $f\in P_n^+$ and E is right-sided, then $E_f^+(\langle-\infty,a)) = \mathbb{R}\setminus E_f^-(\langle a,\infty\rangle) = \mathbb{R}\setminus\bigcap\limits_{j=1}^{\infty}\mathrm{pr}\left[f_0^{-1}((a-\frac{1}{j},\infty))\cap\mathrm{Gr}(E^j)\right]$ belongs to P_{n+2} (P_{n+3}) for n even (odd).

(b) If $f\in P_n^-$ and E is left-sided, then $E_f^+(\langle-\infty,a)) = \mathbb{R}\setminus E_f^-(\langle a,\infty\rangle) = \mathbb{R}\setminus\bigcap\limits_{j=1}^{\infty}\mathrm{pr}\left[f_0^{-1}((a-\frac{1}{j},\infty))\cap\mathrm{Gr}(E^j)\right]$ belongs to P_{n+2} (P_{n+3}) for n even (odd).

(c) If $f\in P_n^+$ and E is left-sided, then $E_f^+((a,\infty\rangle) = \mathbb{R}\setminus E_f^-(\langle-\infty,a\rangle) = \mathbb{R}\setminus\bigcap\limits_{j=1}^{\infty}\mathrm{pr}\left[f_0^{-1}((-\infty,a+\frac{1}{j}))\cap\mathrm{Gr}(E^j)\right]$ belongs to P_{n+2} (P_{n+3}) for n even (odd).

(d) If $f\in P_n^-$ and E is right-sided, then $E_f^+((a,\infty\rangle) = \mathbb{R}\setminus E_f^-((-\infty,a\rangle) = \mathbb{R}\setminus\bigcap\limits_{j=1}^{\infty}\mathrm{pr}\left[(f_0^{-1}((-\infty,a+\frac{1}{j}))\cap\mathrm{Gr}(E^j)\right]$ belongs to P_{n+2} (P_{n+3}) for n even (odd). Lemma 10 finishes the proof.

THEOREM 13. *Let* $\mathrm{Gr}(E) = \bigcup\limits_{l=1}^{\infty}\bigcap\limits_{k=1}^{\infty}\bigcup\limits_{r=1}^{\infty} Z_{r,k,l}\times T_{r,k,l}$, *where* $Z_{r,k,l}\in P_n$,

$T_{r,k,l} \subset \mathbb{R}$, $0 \neq n \neq 2$. *Then* E_f *has properties given in the following table:*

f	E	E_f
P_n^+	$+$	LP_n^-
P_n^-	$-$	LP_n^-
P_n^+	$-$	LP_n^+
P_n^-	$+$	LP_n^+

Proof. We shall show that if $S = \bigcup_{l=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{r=1}^{\infty} X_{r,k,l} \times Y_{r,k,l}$, where $X_{r,k,l} \in P_n$ $(0 \neq n \neq 2)$ and $Y_{r,k,l} \subset \mathbb{R}$, then $\operatorname{pr}(S) \in P_n$.

$$\operatorname{pr}(S) = \bigcup_{l=1}^{\infty} \operatorname{pr}\Big(\bigcap_{k=1}^{\infty} \bigcup_{r=1}^{\infty} X_{r,k,l} \times Y_{r,k,l} \Big) = \bigcup_{l=1}^{\infty} \operatorname{pr}\Big(\bigcup_{r_k} \bigcap_{k=1}^{\infty} X_{r_k,k,l} \times Y_{r_k,k,l} \Big)$$

$$= \bigcup_{l=1}^{\infty} \bigcup_{r_k} \operatorname{pr}\Big[\Big(\bigcap_{k=1}^{\infty} X_{r_k,k,l} \Big) \times \Big(\bigcap_{k=1}^{\infty} Y_{r_k,k,l} \Big) \Big] = \emptyset \qquad \text{if} \quad \bigcap_{k=1}^{\infty} Y_{r_k,k,l} = \emptyset$$

and

$$\operatorname{pr}(S) = \bigcup_{l=1}^{\infty} \bigcup_{r_k} \bigcap_{k=1}^{\infty} X_{r_k,k,l} \in P_n \qquad \text{otherwise}$$

since P_n is closed with respect to operation $\mathcal{A}$ for $0 \neq n \neq 2$ (see [5, p. 375]). $\bigcup_{r_k}$ is the sum taken over all the subsequences of $\mathbb{N}$.

The proof of the assertions is similar to that of Theorem 12.

(1) Suppose that $f \in P_n^+$ and E is right-sided. Then

$$E_f^+(\langle -\infty, a)) = \mathbb{R} \setminus E_f^-(\langle a, \infty\rangle) = \mathbb{R} \setminus \bigcap_{j=1}^{\infty} \operatorname{pr}\Big[f_0^{-1}((a - \tfrac{1}{j}, \infty)) \cap \operatorname{Gr}(E^j) \Big]$$

by Lemma 3.

By Lemma 11a, $f_0^{-1}((a - \frac{1}{j}, \infty)) \cap \operatorname{Gr}(E^j) = \bigcup_{l=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{r=1}^{\infty} X_{r,k,l} \times Y_{r,k,l}$. Hence $E_f^+(\langle -\infty, a))$ belongs to P_{n-1} for n even and to P_{n+1} for n odd.

Using Lemmas 3 and 11b, c, d the proof of the remaining cases is exactly like that of the cases b, c, d of the proof of Theorem 12.

Namely

(2) If $f \in P_n^-$ and E is left sided, then $E_f^+(\langle -\infty, a))$ belongs to P_{n-1} for n even and to P_{n+1} for n odd.

(3) If $f \in P_n^+$ and E is left sided, then $E_f^+((a, \infty\rangle)$ belongs to P_{n-1} for n even and to P_{n+1} for n odd.

(4) If $f \in P_n^-$ and E is right sided, then $E_f^+((a, \infty\rangle)$ belongs to P_{n-1} for n even and to P_{n+1} for n odd.

Lemma 10 finishes the proof.

We turn now to the discussion of the measurability properties of E_f with respect to a given σ-algebra.

DEFINITION 14. *Given a family A of subsets of $\mathbb{R}$, we say that a multifunction $F\colon \mathbb{R} \to \mathbb{R}^*$ is an upper (lower) A-measurable if $F^+(G) \in A$ ($F^-(G) \in A$) for each G open in $\mathbb{R}^*$.*

Remark 15. If A is a σ-algebra and F is compact-valued, then the condition of being upper and lower A-measurable are equivalent (see Lemma 1). It is then motivated to use the denomination A-measurable, e.g. Borel measurable, Lebesgue measurable, Baire measurable etc.

DEFINITION 16. *The σ-algebra generated by P_{2n} ($n = 0, 1, 2, \dots$) will be denoted $S_{2n}(\mathbb{R})$. Since P_{2n} and P_{2n-1} are closed relatively to countable unions and intersections, $S_{2n}(\mathbb{R}) \subset P_{2n+1} \cap P_{2n+2}$. The σ-algebra generated by projective sets of class $2n$ in $\mathbb{R} \times \mathbb{R}$ will be denoted $S_{2n}(\mathbb{R} \times \mathbb{R})$. As above any set from $S_{2n}(\mathbb{R} \times \mathbb{R})$ is of projective class $2n+1$ and $2n+2$ (for $n = 1$ see [6]).*

THEOREM 17. *If f is $S_{2n}(\mathbb{R})$-measurable and $\mathrm{Gr}(E) \in S_{2n}(\mathbb{R} \times \mathbb{R})$, then $E_f^-(K) \in P_{2n+1}$ for any closed set $K \subset \mathbb{R}^*$ and consequently E_f is $S_{2(n+1)}$-measurable.*

Proof. By Lemma 3, $E_f^-(K) = \bigcap_{j=1}^{\infty} \mathrm{pr}\left(f_0^{-1}(V^j) \cap \mathrm{Gr}(E^j)\right)$. It is clear that $f_0^{-1}(V^j) \cap \mathrm{Gr}(E^j) \in S_{2n}(\mathbb{R} \times \mathbb{R})$. Since any set belonging to $S_{2n}(\mathbb{R} \times \mathbb{R})$ is of projective class $2n+1$, $E_f^-(K) \in P_{2n+1}$, by Lemma 2b. Since $E_f^-(G) = \bigcup_{j=1}^{\infty} E_f^-(K_j)$ (K_j closed, $\bigcup_{j=1}^{\infty} K_j = G$), E_f is $S_{2(n+1)}$-measurable.

A. Alikhani-Koopaei in [1] quotes the following fact due to Laczkovich. There is a Baire 2 function and a continuous system of path E such that

$\overline{f}'_E$ is not Borel measurable. The following corollary says that it is an analytic function.

COROLLARY 18. *If f is Borel measurable and $\mathrm{Gr}(E)$ is a Borel set, then $E_f^-(K)$ is an analytic set for any $K \subset \mathbb{R}^*$ and consequently E_f is Lebesgue and Baire measurable* [7, p. 424].

LEMMA 19. *If E is a closed valued system of paths, then*

$$\mathrm{Gr}(E) = \bigcap_{j=1}^{\infty} (\mathbb{R} \times (\mathbb{R} \setminus U_j) \cup E^-(U_j) \times \mathbb{R})$$

where $U_1, U_2, U_3 \dots$ is an open base of $\mathbb{R}$.

From Lemma 19 follows:

Remark 20. If E is a lower P_n-measurable system of paths with closed values ($0 \neq n \neq 2$), then Theorem 13 holds.

LEMMA 21. *Let A be a σ-algebra closed with respect to operation $\mathcal{A}$ (see* [5, p. 4]). *If $M = \bigcup_{l=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{r=1}^{\infty} X_{l,k,r} \times Y_{l,k,r}$, $X_{l,k,r} \in A$, $Y_{l,k,r} \subset \mathbb{R}$, then $\mathrm{pr}(M) \in A$.*

The proof is exactly like the first part of the proof of Theorem 13.

THEOREM 22. *Let E have closed values. If f and E are A-measurable where A is a σ-algebra closed with respect to operation $\mathcal{A}$, then $\overline{f}'_E$, $\underline{f}'_E$, E_f are A-measurable.*

The proof follows from Lemmas 3, 19, 21.

COROLLARY 23. *Let E have closed values. If f and E are Lebesgue (Baire) measurable, then $\overline{f}'_E$, $\underline{f}'_E$, E_f are Lebesgue (Baire) measurable.*

This is just a special case of Theorem 22, since L and Br are closed with respect to operation $\mathcal{A}$ [7, p. 403].

COROLLARY 24. *Let f be Lebesgue measurable with closed values. If $\mathrm{Gr}(E)$ belongs to a σ-algebra generated by $\{A \times B\colon\ A, B \in L\}$, then $\overline{f}'_E$, $\underline{f}'_E$, E_f are Lebesgue measurable.*

Proof. By [3, Theorem 3.4], E is Lebesgue measurable.

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