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Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday

ON ISOMORPHISMS OF INNER PRODUCT SPACES

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ABSTRACT. In this paper, we show that if S_1 and S_2 are two separable, real inner product spaces such that $P(S_1)$ is algebraically isomorphic to $P(S_2)$, where $P(S)$ denotes the modular lattice of finite and cofinite dimensional subspaces of an inner product space S, then S_1 and S_2 are isomorphic as inner product spaces. The proof makes use of Gleason's theorem. We also remark that, as a consequence of this, if for two separable, real inner product spaces S_1, and S_2, the respective complete lattices of strongly closed subspaces are isomorphic, then S_1 and S_2 are unitarily equivalent. In particular, if we just restrict ourselves to complete inner product spaces, we obtain the classical Wigner's theorem ([WIGNER, E. P.: *Group Theory and its Applications to Quantum Mechanics of Atomic Spectra*, Acad. Press. Inc., New York, 1959]).

1. Introduction

For an inner product space S, let $P(S)$ (see [3]) denote the family of finite and cofinite dimensional subspaces of S.[1] The idea is to show that if S_1 and S_2 are two separable real inner product spaces such that $P(S_1)$ is *isomorphic* to $P(S_2)$, then S_1 and S_2 are isomorphic as inner product spaces.

We say that $P(S_1)$ is *isomorphic* to $P(S_2)$ when there exists a bijective mapping $\psi\colon P(S_1) \to P(S_2)$ such that:

(1) $\psi(S_1) = S_2$;

(2) $\psi(A^{\perp_{S_1}}) = (\psi(A))^{\perp_{S_2}}$ for all $A \in P(S_1)$;

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[1] A subspace A of S is cofinite dimensional if there exists a finite dimensional subspace M of S such that $A = M^{\perp}$.

(3) $\psi(A \vee B) = \psi(A) \vee \psi(B)$, whenever $A, B \in P(S_1)$ and $A \subset B^{\perp_{S_1}}$;
(4) ψ^{-1} satisfies (1), (2) and (3).

2. Preliminary results

We first prove that $P(S)$ is an orthomodular lattice.

PROPOSITION 2.1. *$P(S)$ is an orthomodular lattice with the largest and smallest elements being S and $\{0\}$ respectively.*

Proof. First we show that $P(S)$ is a lattice. If A and B are either both finite or cofinite dimensional, then obviously we have $A \vee B = A + B$. If A is finite and B is cofinite dimensional, then, by noting that

$$(A+B)^{\perp} = A^{\perp} \cap B^{\perp} \subset B^{\perp},$$

it follows that $A + B$ is cofinite dimensional. (The other case is the same).

We now show that $P(S)$ is orthomodular. Let $A \subset B$ be elements of $P(S)$. We certainly have that $A \oplus (B \cap A^{\perp}) \subset B$. Moreover, since $A \subset B$, we have $B = B \cap (A \oplus A^{\perp}) \subset (B \cap A) \oplus (B \cap A^{\perp}) = A \oplus (B \cap A^{\perp})$. □

In [6], the family of complete-cocomplete subspaces of an inner product space, denoted by $C(S)$, was defined and investigated. It was shown that the structure of $C(S)$ can be very different for different separable inner product spaces. It is evident that $P(S)$ is a suborthomodular lattice of $C(S)$, and using an argument similar to that used in [1], one can easily show that $P(S)$ admits no σ-additive states.

LEMMA 2.2. *Let $A \in P(S_1)$, $\dim A = n < \infty$, then $\dim \psi(A) = n$.*

Proof. Let $\{e_i : i \le n\}$ be an ONB for A. Then

$$\psi(A) = \psi\Big(\bigvee_{i \le n} [e_i]\Big) = \bigvee_{i \le n} \psi([e_i]).$$

Since for $i \neq j$ we have $\psi([e_i]) \perp \psi([e_j])$, it follows that $\dim A \le \dim \psi(A)$. On the other hand, let $\{f_i : i \in I\}$ be a MONS in $\psi(A)$. Then $\psi(A) = \bigvee_{i \in I} [f_i]$. Let $I_0 \subset I$ such that $|I_0| = n$. Then

$$A = \psi^{-1}(\psi(A)) = \psi^{-1}\Big(\bigvee_{i \in I_0} [f_i] \vee \bigvee_{i \in I \setminus I_0} [f_i]\Big) = \bigvee_{i \in I_0} \psi^{-1}([f_i]) \vee \psi^{-1}\Big(\bigvee_{i \in I \setminus I_0} [f_i]\Big)$$

and therefore $\dim A \ge \dim \psi(A)$. □

As a consequence of Lemma 2.2, we have that atoms in $P(S_1)$ are mapped onto the atoms of $P(S_2)$. Since S_1 is separable, we can always find an orthonormal basis $\{e_i\}$ of $\overline{S_1}$ in S_1, see [3], [5]. For every $i \in \mathbb{N}$, let f_i be a unit vector in S_2 such that $\psi([e_i]) = [f_i]$.

For every atom in $P(S_1)$, choose a representative vector — i.e., a unit vector in S_1 which spans the atom. For the atom $[e_i]$, the representative is chosen to be e_i, and to make the proof of Lemma 2.9 free of unnecessary awkward notation, we also take the representative of the following atoms to be as follows:

$$[e_i + e_j] \to y_{ij} = \frac{1}{\sqrt{2}}(e_i + e_j) \qquad i,j \in \mathbb{N};$$

$$[e_k + e_{k+1} + \cdots + e_l] \to z_{kl} = \frac{1}{\sqrt{l-k+1}} \sum_{i=k}^{l} e_i \qquad l > k \in \mathbb{N}.$$

Denote by $\mathfrak{S}_1^+$ the union of $\{0\}$ and the collection of all these unit vectors. For every $x \in \mathfrak{S}_1^+$ $(x \neq 0)$, let $\hat{x}$ be a unit vector in $\psi([x])$. (To simplify the notation we set $\hat{e}_i$ to be equal to f_i). The union of $\{0\}$ and the collection of all these unit vectors in S_2 is denoted by $\mathfrak{S}_2^+$.

Moreover, for every $i \in \mathbb{N}$ let $A_i = \operatorname{span}\{e_i, e_{i+1}, e_{i+2}\}$. It is then not difficult to see that $\psi(A_i) = \operatorname{span}\{f_i, f_{i+1}, f_{i+2}\}$.

Consider the Gleason state s_{e_i} on $P(S_1)$ defined by

$$s_{e_i}(M) = \langle P_M e_i, e_i \rangle .$$

This state induces a state $\hat{s}_{e_i}$ on $P(S_2)$ as follows:

$$\hat{s}_{e_i}(N) = s_{e_i}\big(\psi^{-1}(N)\big) . \tag{2.1}$$

One can easily verify that $\hat{s}_{e_i}\big(\psi(A_j)\big) = 1$ if and only if $i \in \{j, j+1, j+2\}$. Moreover, for every $i \in \mathbb{N}$, the restriction of $\hat{s}_{e_i}$ on $L\big(\psi(A_i)\big)$ defines a state on $L\big(\psi(A_i)\big)$.

The cornerstone of quantum logic theory on $L(H)$ (the complete orthomodular poset of closed subspaces of a Hilbert space) is Gleason's theorem ([3], [4], [7]). This states that:

If H is a separable Hilbert space, $\dim H \geq 3$, then for every state s on $L(H)$, there exists an orthonormal sequence of vectors $\{x_i\} \subset H$ such that

$$s(M) = \sum_{i \in \mathbb{N}} s([x_i]) \langle P_M x_i, x_i \rangle , \qquad M \in L(H) ,$$

where P_M denotes the orthoprojection of H onto M.

This fundamental and highly non-trivial result is of crucial importance for the probabilistic theory of $L(H)$ and has many generalization and applications (see, for example [3]).

We shall need the following proposition ([2]).

LEMMA 2.4. *Let S be any inner product space, and suppose that s_1, s_2 are two (finitely-additive) states on $P(S)$ such that:*

(i) $s_1(M) = s_2(M) = 1$ *for some $M \subset S$, M finite dimensional;*
(ii) $s_1(K) = s_2(K)$ *for all $K \subset M$.*

Then $s_1(L) = s_2(L)$ holds for all $L \in P(S)$.

Proof. It suffices to show that $s_1([x]) = s_2([x])$ holds for all $x \in S$. Let $x \in S$, $\|x\| = 1$, be arbitrary. If $x \in M$, result follows by hypothesis. Suppose that $x \notin M$. Let N be a finite dimensional subspace of S, of dimension at least equal to 3, including M and x. We certainly have that $s_1|_N$ and $s_2|_N$ are states on $L(N)$, and therefore, by Gleason's theorem, there exist finite orthonormal sequences $\{e_i :\ i \le n\}$ and $\{f_i :\ i \le n\}$ $(n = \dim N)$ in N such that

$$s_1|_N(K) = s_1(K) = \sum_{i \le n} s([e_i])\langle P_K e_i, e_i\rangle,$$
$$s_2|_N(K) = s_2(K) = \sum_{i \le n} s([f_i])\langle P_K f_i, f_i\rangle$$

for all $K \subset N$.

Let $z \in M^{\perp_N}$. Then

$$0 = s_1([z]) = \sum_{i \le n} s([e_i])\langle P_{[z]} e_i, e_i\rangle,$$
$$0 = s_2([z]) = \sum_{i \le n} s([f_i])\langle P_{[z]} f_i, f_i\rangle.$$

This implies that

$$z \in \operatorname{span}\{e_i :\ i \le n\}^{\perp_N},$$
$$z \in \operatorname{span}\{f_i :\ i \le n\}^{\perp_N}.$$

Hence, $\{e_i :\ i \le n\} \subset M$ and $\{f_i :\ i \le n\} \subset M$.

But

$$x = P_M x + P_{M^\perp} x = x_M + x_{M^\perp},$$

and therefore,

$$\begin{aligned} s_1([x]) &= \sum_{i<n} s([e_i])\langle P_{[x]}e_i, e_i\rangle \\ &= \sum_{i<n} s([e_i])\left|\langle x, e_i\rangle\right|^2 \\ &= \sum_{i<n} s([e_i])\left|\langle x_M, e_i\rangle\right|^2 \\ &= \sum_{i<n} s([e_i])\|x_M\|^2\left|\left\langle \frac{x_M}{\|x_M\|}, e_i\right\rangle\right|^2 \\ &= \|x_M\|^2\sum_{i<n} s([e_i])\langle P_{[x_M]}e_i, e_i\rangle \\ &= \|x_M\|^2 s_1([x_M]) . \end{aligned}$$

Similarly, $s_2([x]) = \|x_M\|^2 s_2([x_M])$. Then,

$$\begin{aligned} s_1([x]) &= \|x_M\|^2 s_1([x_M]) \\ &= \|x_M\|^2 s_2([x_M]) \qquad \text{(by hypothesis)} \\ &= s_2([x]) . \end{aligned}$$

This completes the proof. □

COROLLARY 2.5. *If s is a state on $P(S)$ that lives on an atom (i.e. there exists a unit vector $u \in S$ such that $s([u]) = 1$), then s is determined by*

$$s(N) = \langle P_N u, u\rangle .$$

COROLLARY 2.6. *The state $\hat{s}_{e_i}$ defined in equation (2.1) satisfies:*

$$\hat{s}_{e_i}(N) = \langle P_N f_i, f_i\rangle \tag{2.2}$$

for all $N \in P(S_2)$.

LEMMA 2.7. *Let $0 \neq x = \sum_{i\in\mathbb{N}} \alpha_i e_i \in \mathfrak{S}_1^+$. Then for every $i \in \mathbb{N}$, we have:*

$$\langle \hat{x}, f_i\rangle = \pm\alpha_i .$$

Proof. This follows from the following equalities:

$$|\alpha_i|^2 = s_{e_i}([x]) = \hat{s}_{e_i}([\hat{x}]) = \langle \hat{x}, f_i\rangle^2 .$$

□

DEFINITION 2.1. For any unit vector $x \in \mathfrak{S}_1^+$ and $i \in \mathbb{N}$ satisfying $\langle x, e_i \rangle \neq 0$, define

$$\beta(x,i) = \frac{\langle \hat{x}, f_i \rangle}{\langle x, e_i \rangle} \, (= \pm 1) .$$

When $\langle x, e_i \rangle = 0$, we set $\beta(x,i) = 1$.

LEMMA 2.8. *For any unit vector $x \in \mathfrak{S}_1^+$, the vector $\hat{x}$ can be expressed in terms of the f_i's as follows:*

$$\hat{x} = \sum_{i \in \mathbb{N}} \beta(x,i) \alpha_i f_i .$$

Proof. First we observe that

$$\begin{aligned} \hat{x} &= \hat{x}_{\psi(A_n)} + \hat{x}_{(\psi(A_n))^\perp} \\ &= \sum_{i \le n} \langle \hat{x}, f_i \rangle f_i + \hat{x}_{(\psi(A_n))^\perp} \\ &= \sum_{i \le n} \beta(x,i) \alpha_i f_i + \hat{x}_{(\psi(A_n))^\perp} . \end{aligned}$$

Therefore

$$\begin{aligned} \left\| \hat{x}_{(\psi(A_n))^\perp} \right\|^2 &= \|\hat{x}\|^2 - \sum_{i \le n} |\alpha_i|^2 \\ &= 1 - \sum_{i \le n} |\alpha_i|^2 \to 0 \qquad \text{as} \quad n \to \infty . \end{aligned}$$

□

LEMMA 2.9. *Let $x \in \mathfrak{S}_1^+$. If $\langle x, e_i \rangle \neq 0$ and $\langle x, e_j \rangle \neq 0$, then*

$$\frac{\beta(x,i)}{\beta(x,j)} = \frac{\beta(y_{ij},i)}{\beta(y_{ij},j)} .$$

Proof. Recall that

$$y_{ij} = \frac{1}{\sqrt{2}} e_i + \frac{1}{\sqrt{2}} e_j \in \mathfrak{S}_1^+ .$$

It is not difficult to see that

$$\hat{y}_{ij} = \frac{1}{\sqrt{2}} \beta(y_{ij},i) f_i + \frac{1}{\sqrt{2}} \beta(y_{ij},j) f_j .$$

We have

$$\begin{aligned}\left(\frac{\alpha_i}{\sqrt{2}}+\frac{\alpha_j}{\sqrt{2}}\right)^2 &= |\langle y_{ij}, x\rangle|^2 \\ &= s_{y_{ij}}([x]) = \hat{s}_{y_{ij}}([\hat{x}]) = |\langle \hat{y}_{ij}, \hat{x}\rangle|^2 \\ &= \left(\frac{\beta(x,i)\beta(y_{ij},i)\alpha_i}{\sqrt{2}}+\frac{\beta(x,j)\beta(y_{ij},j)\alpha_j}{\sqrt{2}}\right)^2 .\end{aligned}$$

Since the field is real, it follows that

$$\beta(x,i)\beta(y_{ij},i) = \beta(x,j)\beta(y_{ij},j),$$

and therefore,

$$\frac{\beta(x,i)}{\beta(x,j)} = \frac{\beta(y_{ij},i)}{\beta(y_{ij},j)}.$$

□

3. Main result

Let $0 \neq x \in \mathfrak{S}_1^+$ be arbitrary and let k be the smallest natural number satisfying $\langle x, e_k\rangle \neq 0$. For any $j \in \mathbb{N}$ satisfying $\langle x, e_j\rangle \neq 0$, by Lemma 2.9, we have

$$\frac{\beta(x,j)}{\beta(x,k)} = \frac{\beta(y_{kj},j)}{\beta(y_{kj},k)}.$$

This implies that

$$\beta(x,j) = \beta(x,k)\frac{\beta(z_{1j},j)}{\beta(z_{1j},k)}.$$

But since, from Lemma 2.9,

$$\frac{\beta(z_{1j},k)}{\beta(z_{1j},1)} = \frac{\beta(z_{1k},k)}{\beta(z_{1k},1)},$$

we have that

$$\beta(x,j) = \beta(x,k)\frac{\beta(z_{1k},1)}{\beta(z_{1k},k)}\frac{\beta(z_{1j},j)}{\beta(z_{1j},1)}.$$

For any $j \in \mathbb{N}$, define:

$$\gamma_j = \frac{\beta(z_{1j},j)}{\beta(z_{1j},1)}.$$

Thus, we have that

$$\hat{x} = \beta(x,k)\frac{\beta(z_{1k},1)}{\beta(z_{1k},k)}\sum_{i\in\mathbb{N}}\gamma_i\alpha_i f_i .$$

So if we define $U\colon \mathfrak{S}_1^+ \to S_2$ by

$$U(x) = \begin{cases} U\Big(\sum\limits_{i\in\mathbb{N}} \alpha_i e_i\Big) = \sum\limits_{i\in\mathbb{N}} \gamma_i \alpha_i f_i & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,, \end{cases} \tag{3.1}$$

we certainly have that U is well defined on $\mathfrak{S}_1^+$ and moreover, it is an injection into S_2. We now prove the claim put in the abstract.

THEOREM 3.1. *Let S_1 and S_2 be two separable real inner product spaces. Then, $P(S_1)$ is isomorphic to $P(S_2)$ if and only if S_1 and S_2 are isomorphic as inner product spaces.*

Proof. If S_1 is isomorphic to S_2, then we obviously have that $P(S_1)$ is isomorphic to $P(S_2)$. Suppose that $P(S_1)$ is isomorphic to $P(S_2)$ as understood in the beginning of this note. We show that there exists a bijective operator T from S_1 onto S_2 such that $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in S_1$.

Define $T\colon S_1 \to S_2$ by:

$$\begin{aligned} T(v) &= T(\lambda x) \qquad \text{for some unique } x \in \mathfrak{S}_1^+ \\ &= \lambda U(x) \end{aligned}$$

where U is as defined in equation (3.1). Clearly T is a bijection between S_1 and S_2. We show that T is linear. From the same definition, it is clear that for every $\rho \in \mathbb{R}$, $T(\rho v) = \rho T(v)$. Let $v, w \in S_1$. Put $\delta_i = \langle v, e_i\rangle$ and $\theta_i = \langle w, e_i\rangle$. Then

$$\begin{aligned} T(v+w) &= T\Big(\sum_{i\in\mathbb{N}} (\delta_i + \theta_i) e_i\Big) \\ &= T\left((\kappa\|v+w\|) \left[\frac{\kappa}{\|v+w\|} \sum_{i\in\mathbb{N}} (\delta_i + \theta_i) e_i \right] \right), \end{aligned}$$

where $\kappa = \pm 1$ so that $\left[\frac{\kappa}{\|v+w\|} \sum\limits_{i\in\mathbb{N}} (\delta_i + \theta_i) e_i\right] \in \mathfrak{S}_1^+$. Then we have

$$\begin{aligned} T(v+w) &= \kappa\|v+w\| U\left(\sum_{i\in\mathbb{N}} \left(\frac{\kappa\delta_i}{\|v+w\|} + \frac{\kappa\theta_i}{\|v+w\|} \right) e_i \right) \\ &= \kappa\|v+w\| \sum_{i\in\mathbb{N}} \left(\frac{\kappa\gamma_i\delta_i}{\|v+w\|} + \frac{\kappa\gamma_i\theta_i}{\|v+w\|} \right) f_i \\ &= \sum_{i\in\mathbb{N}} \gamma_i\delta_i f_i + \sum_{i\in\mathbb{N}} \gamma_i\theta_i f_i \\ &= T(v) + T(w)\,. \end{aligned}$$

This completes the proof. □

Let $F(S)$ denote the complete lattice of strongly closed subspaces of S and $E(S)$ the orthomodular poset of splitting subspaces of S. We recall that

$$P(S) \subset C(S) \subset E(S) \subset F(S).$$

COROLLARY 3.2. *The following statements are equivalent:*

(1) S_1 *is isomorphic to* S_2 *(as inner product spaces);*
(2) $P(S_1)$ *is isomorphic to* $P(S_2)$ *(as orthomodular lattices);*
(3) $C(S_1)$ *is isomorphic to* $C(S_2)$ *(as orthomodular posets);*
(4) $E(S_1)$ *is isomorphic to* $E(S_2)$ *(as orthomodular posets);*
(5) $F(S_1)$ *is isomorphic to* $F(S_2)$ *(as complete lattices).*

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