Bohdan Zelinka

Polytopic locally linear graphs

Mathematica Slovaca, Vol. 38 (1988), No. 2, 99--103

Persistent URL: http://dml.cz/dmlcz/133017

Math. Slovaca 38, 1988, No. 2, 99—103

POLYTOPIC LOCALLY LINEAR GRAPHS

BOHDAN ZELINKA

Local properties of graphs were studied by many authors. Survey articles on this topic were written by J. Sedláček [2], [3].

At the Czechoslovak Conference on Graph Theory in the Raček Valley in May 1986, D. Fronček [1] has introduced locally linear graphs and presented his results on their fundamental properties. Here we shall study locally linear graphs which are polytopic. All considered graphs are finite without loops and multiple edges.

If G is an undirected graph and v is its vertex, then by $N_G(v)$ we denote the subgraph of G induced by the set of all vertices which are adjacent to v. If $N_G(v)$ is regular of degree 1 (i. e. if all connected components of $N_G(v)$ are isomorphic to K_2) for each vertex v of G, then G is called locally linear.

A polytopic graph is a graph isomorphic to the graph of a convex polyhedron (in the 3-dimensional Euclidean space), i. e. a planar 3-connected graph. Its faces are determined uniquely.

If in a polytopic locally linear graph G every triangle forms the boundary of a face, then G is said to have the property $\boldsymbol{P}$.

Theorem 1. *Let G be a polytopic locally linear graph. Then each edge of G belongs to at most one triangular face. If G has the property $\boldsymbol{P}$, then each edge of G belongs to exactly one triangular face.*

Proof. Each edge e of a locally linear graph belongs to exactly one triangle [1]. If this triangle is the boundary of a face, then e belongs to exactly one triangular face; in the opposite case to no triangular face. If G has the property $\boldsymbol{P}$, then only the first case can occur. □

Now we describe a certain construction.

Construction $\boldsymbol{C}$. *Let G_1, G_2 be two vertex-disjoint polytopic graphs, let F_1 (or F_2) be a triangular face of G_1 (or G_2 respectively.) We choose a bijective mapping φ of the set of vertices of F_1 onto the set of vertices of F_2. Then we identify each vertex x of F_1 with the vertex $\varphi(x)$ of F_2. The resulting graph will be denoted by G.*

Theorem 2. *Let G be a polytopic graph. Then the following two assertions are equivalent:*

(i) *G is locally linear and has not the property* $\boldsymbol{P}$.

(ii) *G is obtained from two polytopic locally linear graphs by Construction* $\boldsymbol{C}$.

Proof. (i) $\Rightarrow$ (ii). As G has not the property $\boldsymbol{P}$, it contains a triangle T which is not the boundary of any face; let u_1, u_2, u_3 be its vertices. Consider the representation of G by points and arcs in the plane. By G_1 (or G_2) denote the subgraph of G induced by the union of $\{u_1, u_2, u_3\}$ and the set of all vertices which lie outside (or inside respectively) of the triangle T. Evidently G is obtained from G_1 and G_2 by Construction $\boldsymbol{C}$ and the graphs G_1, G_2 are polytopic. It remains to prove that both G_1 and G_2 are locally linear. If v is a vertex of G_1, $v \notin \{u_1, u_2, u_3\}$, then evidently $N_{G_1}(v) = N_G(v)$ and thus it is regular of degree 1. Analogously if v is a vertex of G_2 and $v \notin \{u_1, u_2, u_3\}$. If $v \in \{u_1, u_2, u_3\}$, then $N_{G_1}(v)$ is the intersection of G_1 and $N_G(v)$. Let e be an edge of $N_G(v)$ with the end vertices v_1, v_2. If v_1 is in G_1 and $v_1 \notin \{u_1, u_2, u_3\}$, then also v_2 is in G_1; otherwise e should cross an edge of T. If $v_1 \in \{u_1, u_2, u_3\}$, then also $v_2 \in \{u_1, u_2, u_3\}$ and thus it is in G_1. Thus if one vertex of $N_G(v)$ is in G_1, then also the other vertex of the same connected component of $N_G(v)$ is in G_1. An analogous assertion holds for G_2. Hence $N_{G_1}(v)$ for each vertex v of G_1 and $N_{G_2}(v)$ for each vertex v of G_2 is regular of degree 1 and both G_1, G_2 are locally linear graphs.

(ii) $\Rightarrow$ (i). Let G be obtained from two polytopic locally linear graphs G_1 and G_2 by Construction $\boldsymbol{C}$. Let T be the triangle forming the identified boundaries of triangular faces of G_1 and G_2. The graph G can be represented in the plane in the following way. We represent G_1 (or G_2) so that its face with the boundary T is not outer (or is outer respectively). Then the whole representation of G_2 is inside the face of G_1 with the boundary T. Hence G is planar and evidently also polytopic. For a vertex v of G_1 (or G_2) not belonging to T we have $N_G(v) = N_{G_1}(v)$ (or $N_G(v) = N_{G_2}(v)$ respectively). For a vertex v of T the graph $N_G(v)$ is the union of $N_{G_1}(v)$ and $N_{G_2}(v)$. These graphs have a common connected component which is contained in T and no vertex of G_1 not in T can be adjacent to a vertex of G_2 not in T. hence G is locally linear. The triangle T is not the boundary of a face in G; hence G has not the property $\boldsymbol{P}$. $\square$

As all considered graphs are finite, any polytopic locally linear graph G which has not the property $\boldsymbol{P}$ can be obtained from a finite number of polytopic locally linear graphs with the property $\boldsymbol{P}$ by the repeated using od Construction $\boldsymbol{C}$. The minimum number of such graphs for a given graph G will be denoted by $s(G)$.

Theorem 3. *Let G be a polytopic locally linear graph. Then G is Eulerian.*

Proof. Evidently G is connected. Any regular graph of degree 1 has an even number of vertices; therefore $N_G(v)$ for each vertex v of G has an even number of vertices and thus v has an even degree. $\square$

Theorem 4. *Let G be a polytopic locally linear graph with n vertices and m edges. Then*

$$2n \leqq m \leqq 12(n-2)/5.$$

Proof. As G is polytopic, the degree of each vertex of G must be at least 3. As it is moreover Eulerian (Theorem 3), this degree must be even and thus at least 4. Hence the number of edges of G is at least $2n$. The other inequality will be proved by induction according to $s(G)$. If $s(G) = 1$, i.e. if G has the property $\boldsymbol{P}$, then each edge of G belongs to exactly one triangular face and to exactly one non-triangular face. Let f' (or f'') be the number of triangular (or non-triangular respectively) faces of G. Then $m = 3f'$ and $m \geqq 4f''$. For the total number f of faces of G we have $f = f' + f'' \leqq m/3 + m/4 = 7m/12$. From Euler's Formula we have $n = m - f + 2 \geqq m - 7m/12 + 2 = 5m/12 + 2$ and this yields $m \leqq 12(n-2)/5$. If $s(G) > 1$, then G is obtained from two graphs G_1, G_2 by Construction $\boldsymbol{C}$ and $s(G_1) < s(G)$, $s(G_2) < s(G)$; we may suppose according to the induction hypothesis that the assertion is true for G_1 and G_2. Let G_1 have n_1 vertices and m_1 edges, let G_2 have n_2 vertices and m_2 edges. The graphs G_1 and G_2 have a common triangle in G, thus $n = n_1 + n_2 - 3$, $m = m_1 + m_2 - 3$. According to the induction hypothesis $m_1 \leqq 12(n_1 - 2)/5$, $m_2 \leqq 12(n_2 - 2)/5$, hence $m = m_1 + m_2 - 3 \leqq 12(n_1 + n_2 - 4)/5 - 3 = 12(n-1)/5 - 3 < 12(n-2)/5$. □

Now we shall study the graphs at which one of the bounds is attained.

Theorem 5. *Let G be a polytopic locally linear graph with n vertices and m edges, let* $m = 2n$. *Then G is the line graph of a polytopic graph without triangles which is regular of degree* 3.

Proof. Take the graph G' whose vertices are triangular faces of G and in which two vertices are adjacent if and only if they have a common vertex (as faces of G). The graph G is regular of degree 4, therefore any triangular face of G has common vertices with exactly three others; hence G' is regular of degree 3. If we have drawn G in the plane, then we may draw G' in such a way that each vertex is drawn inside the corresponding triangular face of G and each edge is led over the vertex common to the faces of G corresponding to its end vertices; hence G' is polytopic. Evidently in G there are no three triangular faces with the property that any two of them have a vertex in common; otherwise there would be a fourth triangular face having an edge in common with all of them, which would contradict Theorem 3. This implies that G' is without triangles. Evidently G is the line graph of G'. □

The inverse assertion is evident.

Theorem 6. *Let G′ be a polytopic graph without triangles which is regular of degree* 3. *Then the line graph of G′ is a polytopic locally linear graph with n vertices and m edges, where* $m = 2n$. □

Theorem 7. *Let G be a polytopic locally linear graph with n vertices and m edges, let* $m = 12(n - 2)/5$. *Then there exists a positive integer k such that* $n = 5k + 2$, $m = 12k$ *and the number of faces of G is* $7k$.

Proof. Evidently, as m is maximal possible, all non-triangular faces of G are quadrangular. Let f' (or f'') be the number of triangular (or quadrangular respectively) faces of G. Then $m = 3f' = 4f''$ and $f'' = 3f'/4$. The total number of faces of G is $f = f' + f'' = 7f'/4$. As f must be an integer and the numbers 4 and 7 are comparatively prime, f must be divisible by 7 and thus $f = 7k$ for a positive integer k. Then $f' = 4k$ and $m = 12k$. From Euler's Formula we obtain $n = 5k + 2$. □

For each positive integer k by $\mathscr{L}(k)$ we denote the class of all polytopic locally linear graphs with $5k + 2$ vertices and $12k$ edges.

Theorem 8. *The class* $\mathscr{L}(1)$ *is empty and all classes* $\mathscr{L}(k)$ *for* $k \geqq 2$ *are non-empty.*

Proof. Suppose that $\mathscr{L}(1)$ is non-empty and let $G \in \mathscr{L}(1)$. Then G has 7 vertices and 12 edges, which contradicts the inequality $2n \leqq m$. Now we shall construct a graph $G \in \mathscr{L}(k)$ for $k \geqq 2$. Consider a circuit C of the length $3k$. We denote its vertices by $u_1, \ldots, u_k, u'_1, \ldots, u'_k, u''_1, \ldots, u''_k$ in such a way that the edges of G are $u_i u'_i$, $u'_i u''_i$, $u''_i u_{i+1}$ for $i = 1, \ldots, k$, the sum $i + 1$ being taken modulo k. To C we add new vertices $v_1, \ldots, v_k, v'_1, \ldots, v'_k$ and the edges $u_i v_i$, $u'_i v'_i$, $v_i v'_i$, $v'_i u''_i$, $v_i u''_{i-1}$ for $i = 1, \ldots, k$, the difference $i - 1$ being again taken modulo k. Finally we add two vertices x, y; the vertex x will be adjacent to the vertices $u_1, \ldots, u_k$, $u'_1, \ldots, u'_k$ and the vertex y to the vertices $v_1, \ldots, v_k$, $v'_1, \ldots, v'_k$. Evidently the number of vertices of the graph G thus obtained is $5k + 2$. The reader may verify it himself that G is polytopic and locally linear and has $12k$ edges. □

REFERENCES

[1] FRONČEK, D.: Locally linear graphs. (Slovak.) Lecture at the Czechoslovak Conference on Graph Theory in the Raček Valley in May 1986. (Unpublished.)

[2] SEDLÁČEK, J.: Local properties of graphs. (Czech.) Časop. pěst. mat. 106, 1981, 290—298.

[3] SEDLÁČEK, J.: On local properties of finite graphs. In: Graph Theory, Łagów 1981 (ed. by M. Borowiecki, J. W. Kennedy and M. M. Sysło). Springer Verlag Berlin—Heidelberg—New York—Tokyo 1983.

Received June 6, 1986

Katedra tváření a plastů
Vysoké školy strojní a textilní
Studentská 1292
461 17 Liberec

ПОЛИТОПИЧЕСКИЕ ЛОКАЛЬНО ЛИНЕЙНЫЕ ГРАФЫ

Bohdan Zelinka

Резюме

Локально линейным называетя граф, в котором окрестность любой вершины порождает подграф, который является регулярным степени 1. В статье исследованы локально линейные графы, которые являются политопическими. Найдены оценки для числа ребер и исследованы графы с экстремальными числами ребер.