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ON ζ-CONVERGENCE OF SEQUENCES

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ABSTRACT. A new number theoretical method of summability is defined, which turns out to be equivalent to the Cesaro method for bounded sequences. As a corollary, one gets for example the following theorem, which contains the prime number theorem:

For any bounded arithmetical function f the condition $\sum\limits_{n<x} f(n) = o(x)$ implies $\sum\limits_{n<x} (\mu * f)(n) = o(x)$, where $*$ denotes the Dirichlet convolution, and μ is the Möbius function.

Let (a_n) be a sequence of complex numbers. We shall say that (a_n) is ζ*-convergent* to $a \in \mathbb{C}$ if and only if the functional Dirichlet series

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s} \right) \cdot \zeta(s)^{-1}$$

is convergent to a for $s = 1$. This means that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{d|n} a_d \mu(n/d) \right)$$

is convergent to a.

We shall prove the following theorem giving more than the regularity of this summation method.

THEOREM 1. *If (a_n) is a bounded sequence of complex numbers, then the following two conditions are equivalent:*

(a) *(a_n) is Cesaro-convergent to a,*

$$\sum_{n \le x} a_n = ax + o(x) .$$

(b) *(a_n) is ζ-convergent to a.*

As simple corollaries we obtain:

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THEOREM 2. *Let (a_n) be a bounded Cesaro-convergent sequence of complex numbers, and let (b_n) be defined as follows*

$$b_n = \sum_{d_1 d_2 = n} \mu(d_1) a_{d_2} \,. \tag{1}$$

Then (b_n) is Cesaro-convergent to 0.

THEOREM 3. *Let f be a complex valued function of a real variable which is periodic with period 2π and Riemann integrable on $[0, 2\pi]$. Then the following "arithmetical formula for the integral" holds*

$$\frac{1}{2\pi} \int_0^{2\pi} f(x)\, \mathrm{d}x = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cdot \sum_{n=1}^{\infty} \frac{f(n)}{n} \,.$$

Proof of Theorem 1.

(a) $\Longrightarrow$ (b) :

Consider the sequence

$$a'_n := a_n - a \,.$$

It is bounded and

$$\sum_{n \le x} a'_n = 0 \cdot x + o(x) \,.$$

By the formal equality,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cdot \sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cdot \sum_{n=1}^{\infty} \frac{a'_n}{n} + (a + 0 + 0 + 0 + \ldots) \,,$$

it suffices to prove the theorem in the case $a = 0$. As in [3; p. 685], we start with the identity

$$\begin{aligned} &B(x) - G(\sqrt{x}\,) A(\sqrt{x}\,) \\ &= \sum_{k \le \sqrt{x}} \frac{\mu(k)}{k} \Big(A\Big(\frac{x}{k}\Big) - A(\sqrt{x}\,) \Big) + \sum_{m \le \sqrt{x}} \frac{a_m}{m} \Big(G\Big(\frac{x}{m}\Big) - G(\sqrt{x}\,) \Big) \,, \end{aligned} \tag{2}$$

where

$$A(y) = \sum_{n \le y} \frac{a_n}{n} \,, \qquad G(y) = \sum_{n \le y} \frac{\mu(n)}{n} \,, \qquad B(y) = \sum_{n \le y} \frac{b_n}{n}$$

and

$$\sum_{n=1}^{\infty} \frac{b_n}{n} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cdot \sum_{n=1}^{\infty} \frac{a_n}{n} \,.$$

We have to prove that

$$\lim_{x\to\infty} B(x) = 0\,.$$

By the identity (2), it suffices to prove the following three statements

$$\lim_{x\to\infty} G(\sqrt{x}\,)A(\sqrt{x}\,) = 0\,, \tag{3}$$

$$\lim_{x\to\infty} \sum_{m\le\sqrt{x}} \frac{a_m}{m}\Big(G\Big(\frac{x}{m}\Big) - G(\sqrt{x}\,)\Big) = 0\,, \tag{4}$$

$$\lim_{x\to\infty} \sum_{k\le\sqrt{x}} \frac{\mu(k)}{k}\Big(A\Big(\frac{x}{k}\Big) - A(\sqrt{x}\,)\Big) = 0\,. \tag{5}$$

Proof of (3):

The sequence (a_n) is bounded, and therefore

$$A(x) = O\Big(\sum_{n\le x}\frac{1}{n}\Big) = O(\log x)\,.$$

Landau proved in 1903 ([3; p. 570]) that for any $q > 0$

$$G(x) = O(\log^{-q} x)\,, \tag{6}$$

and the proof of (3) is finished.

Proof of (4):

Because of $m \le \sqrt{x}$ and (6), we obtain

$$G\Big(\frac{x}{m}\Big) - G(\sqrt{x}\,) = o\big(\log^{-1}\sqrt{x}\,\big)\,.$$

Hence

$$\sum_{m\le\sqrt{x}} \frac{a_m}{m}\Big(G\Big(\frac{x}{m}\Big) - G(\sqrt{x}\,)\Big) = O\big(\log\sqrt{x}\,\big)\cdot o\big(\log^{-1}\sqrt{x}\,\big) = o(1)\,.$$

Proof of (5):

Partial summation gives

$$\sum_{k\le\sqrt{x}} \frac{\mu(k)}{k}\Big(A\Big(\frac{x}{k}\Big) - A(\sqrt{x}\,)\Big)$$

$$= G([\sqrt{x}\,])\cdot\Big(A\Big(\frac{x}{[\sqrt{x}\,]}\Big) - A(\sqrt{x}\,)\Big) + \sum_{k+1\le\sqrt{x}} G(k)\Big(A\Big(\frac{x}{k}\Big) - A\Big(\frac{x}{k+1}\Big)\Big)\,.$$

Hence, we only have to prove that

$$\lim_{x\to\infty} \sum_{k+1\le\sqrt{x}} G(k)\Big(A\Big(\frac{x}{k}\Big) - A\Big(\frac{x}{k+1}\Big)\Big) = 0\,.$$

From

$$\left|A\left(\frac{x}{k}\right) - A\left(\frac{x}{k+1}\right)\right| = O\left(\sum_{\frac{x}{k+1}<n\le\frac{x}{k}} \frac{1}{n}\right) = O\left(\log\left(1+\frac{1}{k}\right) + O\left(\frac{k+1}{x}\right)\right)$$

and $k+1 \le \sqrt{x}$, we get

$$\left|A\left(\frac{x}{k}\right) - A\left(\frac{x}{k+1}\right)\right| = O\left(\frac{1}{k}\right).$$

Fix any $\varepsilon > 0$. Because of

$$G(k)\cdot\left(A\left(\frac{x}{k}\right) - A\left(\frac{x}{k+1}\right)\right) = O(|G(k)|\cdot k^{-1})$$

and (6), there exists $K \in \mathbb{N}$ such that

$$\left|\sum_{k=K+1}^{\sqrt{x}-1} G(k)\cdot\left(A\left(\frac{x}{k}\right) - A\left(\frac{x}{k+1}\right)\right)\right| < \varepsilon$$

independently of x. Now we only need to prove that

$$\lim_{x\to\infty}\sum_{k=1}^{K} G(k)\cdot\left(A\left(\frac{x}{k}\right) - A\left(\frac{x}{k+1}\right)\right) = 0.$$

This is a consequence of the fact that for a fixed k

$$\lim_{x\to\infty}\left(A\left(\frac{x}{k}\right) - A\left(\frac{x}{k+1}\right)\right) = 0,$$

which can be shown as follows. Let $a(x) = \sum_{n\le x} a_n$. Partial summation gives

$$\begin{aligned} &A\left(\frac{x}{k}\right) - A\left(\frac{x}{k+1}\right)\\ &= a\left(\left[\frac{x}{k}\right]\right)\left[\frac{x}{k}\right]^{-1} - a\left(\left[\frac{x}{k+1}\right]\right)\left(\left[\frac{x}{k+1}\right]+1\right)^{-1} + \sum_{\frac{x}{k+1}<r<\frac{x-k}{k}} \frac{a(r)}{r(r+1)}. \end{aligned}$$

By the assumption

$$\lim_{x\to\infty} a\left(\left[\frac{x}{k}\right]\right)\left[\frac{x}{k}\right]^{-1} = \lim_{x\to\infty} a\left(\left[\frac{x}{k+1}\right]\right)\left(\left[\frac{x}{k+1}\right]+1\right)^{-1} = 0,$$

for a given $\varepsilon > 0$ and sufficiently large x

$$\left|\frac{a(r)}{r}\right| < \varepsilon \qquad \text{for } r = \left[\frac{x}{k+1}\right]+1, \dots, \left[\frac{x}{k}\right]-1.$$

Hence

$$\Bigg| \sum_{\frac{x}{k+1}<r<\frac{x-k}{k}} \frac{a(r)}{r(r+1)} \Bigg| < \varepsilon \sum_{\frac{x}{k+1}<r<\frac{x-k}{k}} \frac{1}{r+1}$$
$$= \varepsilon \left(\log\left(1+\frac{1}{k}\right) + O\left(\frac{k+1}{x}\right)\right) = \varepsilon O(1)\,,$$

which completes the proof of (5) and the implication (a) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (a) : (Drmota)

Let (b_n) be defined by (1). By the identity

$$\sum_{n=1}^{\infty} \frac{b_n x^n}{1-x^n} = \sum_{n=1}^{\infty} a_n x^n \qquad \text{for} \quad |x|<1$$

and the regularity of Lambert's method, we get

$$\lim_{x\to 1-} (1-x) \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{b_n}{n} = a\,.$$

The assertion now follows from the tauberian theorem of Hardy-Littlewood-Karamata. □

P r o o f o f T h e o r e m 2. This is an immediately consequence of Theorem 1 and the fact:

if $\sum_{n=1}^{\infty} \frac{b_n}{n}$ *is convergent, then* $\sum_{n\le x} b_n = o(x)$. □

P r o o f o f T h e o r e m 3. The sequence of all natural numbers (n) is uniformly distributed mod 2π, and therefore

$$\sum_{n\le x} f(n) = x \cdot \frac{1}{2\pi} \int_0^{2\pi} f(t)\, \mathrm{d}t + o(x)$$

([2; Theorem 1.1]). The conclusion now follows from Theorem 1. □

Remark 1. Theorem 1 resembles I n g h a m's tauberian theorem ([1; Theorem 2]), which can be reformulated as follows:

THEOREM 4. (I n g h a m) *Let* (a_n) *be a sequence of real (complex) numbers which is Cesaro-convergent to* a, *and let* (b_n) *be defined by* (1). *If there exists* $K>0$ *such that* $b_n > -K$ *(respectively* $|\, b_n \,| < K$*), then* $\sum_{n=1}^{\infty} \frac{b_n}{n}$ *converges to* a.

Remark 2. Theorem 3 motivates the following definition.

The sequence (x_n) of real numbers from the interval $[0,1)$ is *arithmetically uniformly distributed* mod 1 if and only if for any subinterval $[a,b)$ the following series converges to $b-a$

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} \mu(d) c_{[a,b)}(x_{\frac{n}{d}}), \tag{7}$$

where $c_{[a,b)}$ is the characteristic function of $[a,b)$. From Theorem 1, it follows that this definition is equivalent to the classical one. Despite this, one can define the *arithmetical discrepancy* of a finite sequence $x_1,\dots,x_n$ as follows

$$D_n(x_1,\dots,x_n) := \sup_{[a,b)\subseteq[0,1)} |s_n - (b-a)|,$$

where s_n is the nth partial sum of the series (7).

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