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Local Versions of some Congruence Properties in Single Algebras *

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Abstract

We investigate some local versions of congruence permutability, regularity, uniformity and modularity. The results are applied to several examples including implication algebras, orthomodular lattices and relative pseudocomplemented lattices.

Key words: Congruence permutability, congruence regularity, congruence uniformity, congruence modularity.

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Congruence permutability, regularity, uniformity and modularity are well studied concepts in universal algebra. For the convenience of the reader we refer to [4]. We introduce and study some local versions of these notions.

In the following let $\mathcal{A} = (A, F)$ be an arbitrary but fixed algebra and a, b arbitrary but fixed elements of A.

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Definition 1 For every positive integer n and every $i \in \{1, \ldots, n\}$ let C_{ni} denote the set of all n-ary functions on A which are compatible with all congruences on $\mathcal{A}$ with respect to the i-th variable, i.e. C_{ni} consists of all functions $f : A^n \to A$ satisfying the following condition: If $a_1, \ldots, a_n, \bar{a}_i \in A$, $\theta \in \mathrm{Con}(\mathcal{A})$ and $a_i \theta \bar{a}_i$ then

$$f(a_1, \ldots, a_i, \ldots, a_n) \theta f(a_1, \ldots, \bar{a}_i, \ldots, a_n).$$

Moreover, put $C_n := C_{n1} \cap \ldots \cap C_{nn}$ the set of all compatible n-ary functions on $\mathcal{A}$ for all positive integers n.

Definition 2 $\mathcal{A}$ is called (a, b)-*permutable* if for all $\theta, \phi \in \mathrm{Con}(\mathcal{A})$ the assertions $a(\theta \circ \phi)b$ and $a(\phi \circ \theta)b$ are equivalent. $\mathcal{A}$ is called (a, b)-*regular* if for all $\theta, \phi \in \mathrm{Con}(\mathcal{A})$, $[a]\theta = [a]\phi$ implies $[b]\theta = [b]\phi$. $\mathcal{A}$ is called (a, b)-*uniform* if $|[a]\theta| = |[b]\theta|$ for all $\theta \in \mathrm{Con}(\mathcal{A})$.

Remark 1 The following properties follow directly from Defintion 2:

- $\mathcal{A}$ is (a, b)-permutable if and only if $\mathcal{A}$ is (b, a)-permutable.
- $\mathcal{A}$ is permutable if and only if it is (c, d)-permutable for all $c, d \in A$.
- $\mathcal{A}$ is regular if and only if it is (c, d)-regular for all $c, d \in A$.
- $\mathcal{A}$ is (a, b)-uniform if and only if $\mathcal{A}$ is (b, a)-uniform.
- $\mathcal{A}$ is uniform if and only if it is (c, d)-uniform for all $c, d \in A$.

Theorem 1 *(i) If there exists an $f \in C_1$ with $f(b) = a$ and $f(a) = b$ then $\mathcal{A}$ is (a, b)-permutable.*

(ii) If there exist $f, g \in C_1$ with $f(b) = a$ and $g(f(x)) = x$ for all $x \in A$ then $\mathcal{A}$ is (a, b)-regular.

(iii) If there exist $f, g \in C_1$ such that $f(b) = a$ and $f(g(x)) = g(f(x)) = x$ for all $x \in A$ then $\mathcal{A}$ is (a, b)-uniform.

Proof Let $\theta, \phi \in \mathrm{Con}(\mathcal{A})$.

(i) If $a(\theta \circ \phi)b$ then there exists an element $c \in A$ with $a\theta c\phi b$ and hence $a = f(b)\phi f(c)\theta f(a) = b$ showing $a(\phi \circ \theta)b$, i.e. $a(\theta \circ \phi)b$ implies $a(\phi \circ \theta)b$. The converse implication follows by symmetry.

(ii) Assume $[a]\theta = [a]\phi$. If $c \in [b]\theta$ then $f(c) \in [f(b)]\theta = [a]\theta = [a]\phi$ and hence $c = g(f(c)) \in [g(a)]\phi = [g(f(b))]\phi = [b]\phi$ showing $[b]\theta \subseteq [b]\phi$. The converse inclusion follows by symmetry.

(iii) If $c \in [a]\theta$ then $g(c) \in [g(a)]\theta = [g(f(b))]\theta = [b]\theta$. If $d \in [b]\theta$ then $f(d) \in [f(b)]\theta = [a]\theta$. Moreover, $f(g(x)) = g(f(x)) = x$ for all $x \in A$. Hence $g|_{[a]\theta}$ and $f|_{[b]\theta}$ are mutually inverse bijections between $[a]\theta$ and $[b]\theta$ proving $|[a]\theta| = |[b]\theta|$. □

Example 1 An implication algebra (cf. [1]) is a groupoid $(A, \cdot)$ satisfying the identities

$$(xy)x = x, \quad (xy)y = (yx)x, \quad x(yz) = y(xz).$$

This implies $xx = yy$, i.e. xx is a constant denoted by 1 (if $A \neq \emptyset$ which we will assume). Moreover, $1x = (xx)x = x$ and $x1 = (1x)1 = 1$. With the partial order

$$x \leq y \text{ if and only if } xy = 1$$

$(A, \leq)$ is a $\vee$-semilattice with $x \vee y = (xy)y$ in which every interval $[c, 1]$ is a Boolean algebra. The element xy coincides with the complement of $x \vee y$ in the interval $[y, 1]$.

An implication algebra is (a, b)-permutable if and only if a and b have a common lower bound, i.e. if and only if there exists an interval $[c, 1]$ with $a, b \in [c, 1]$: Firstly suppose that such an element c exists. Let $+_c$ denote the symmetric difference in $[c, 1]$. $+_c$ can be represented as a polynomial function and thus $x +_c y$ makes sense for all $x, y \in A$ and is in C_2. Consequently $f(x) = x +_c (a +_c b)$ is in C_1 and obviously satisfies condition (i) of Theorem 1.

On the other hand, suppose a and b do not have a common lower bound. Let θ and ϕ be the principal congruences generated by $(a, 1)$ and $(b, 1)$, respectively. It can be verified easily that $(x, y) \in \theta$ if and only if $x \wedge y$ exists in A and $1 +_{x \wedge y} (x +_{x \wedge y} y) \geq a \vee (x \wedge y)$. Similarly ϕ can be characterized.

Obviously $(a, b) \in \theta \circ \phi$. Assume $(a, b) \in \phi \circ \theta$, i.e. there is $d \in A$ such that $(a, d) \in \phi$ and $(d, b) \in \theta$. $(a, d) \in \phi$ implies $(a, a \vee d) \in \phi$ which means $1 +_a (a +_a (a \vee d)) \geq b \vee a$ by the above characterization of ϕ. This implies $a \vee d \leq 1 +_a (a \vee b)$ and hence $(a \vee b) \wedge (a \vee d) = a$. $(d, b) \in \theta$ implies the existence of $b \wedge d$ and we infer $a \vee (b \wedge d) \leq (a \vee b) \wedge (a \vee d) = a$, hence $b \wedge d \leq a$. This is a contradiction to the assumption that a and b do not have a common lower bound.

One might suspect that (a, b)-regularity and (a, b)-uniformity can be characterized by the same condition as (a, b)-permutability. This is not the case: We consider the implication algebra $\mathcal{A}$ with $A = \{1, a, b, c, d\}$ consisting of the two Boolean subalgebras $\{1, a, b, c\}$ with $c \leq a, b \leq 1$ and $\{1, d\}$.

One can check easily that $\theta = \{a, c\}^2 \cup \{1, b, d\}^2$ and $\phi = \{a, c\}^2 \cup \{1, b\}^2 \cup \{d\}^2$ are congruences of $\mathcal{A}$. We have $c = a \wedge b$, $[a]\theta = [a]\phi$ but $[b]\theta \neq [b]\phi$, thus $\mathcal{A}$ is not (a, b)-regular. Moreover, $|[a]\theta| = 2$ and $|[b]\theta| = 3$, hence $\mathcal{A}$ is not (a, b)-uniform.

Example 2 Let $\mathcal{A}$ denote the algebra (A, s_1, s_2) with $A = \{a, b, c, d\}$ and unary operations s_1, s_2 defined as follows:

	a	b	c	d
s_1	d	c	c	d
s_2	b	a	d	c

$\mathcal{A}$ has exactly 3 non-trivial congruences, namely

$$\begin{aligned} \theta &= \{a\}^2 \cup \{b\}^2 \cup \{c, d\}^2, \\ \phi &= \{a, d\}^2 \cup \{b, c\}^2 \text{ and} \\ \psi &= \{a, b\}^2 \cup \{c, d\}^2. \end{aligned}$$

It follows

$$\begin{aligned}\theta\circ\phi &= \theta\cup\phi\cup\{(c,a),(d,b)\},\\ \phi\circ\theta &= \theta\cup\phi\cup\{(a,c),(b,d)\},\\ \theta\circ\psi &= \psi\circ\theta=\psi,\\ \phi\circ\psi &= \psi\circ\phi=A^2.\end{aligned}$$

$\mathcal{A}$ is (c,d)-permutable: For $f := s_1\circ s_2$ it holds $f(c)=d$ and $f(d)=c$. Since $(b,d)\in(\phi\circ\theta)\setminus(\theta\circ\phi)$, $\mathcal{A}$ is not (b,d)-permutable.

$\mathcal{A}$ is (a,b)-regular: For $f=g:=s_2$ it holds $f(b)=a$ and $g(f(x))=x$ for all $x\in A$. Since $[a]\theta=[a]\omega$ (where ω denotes the least congruence on $\mathcal{A}$) and $[d]\theta\neq[d]\omega$, $\mathcal{A}$ is not (a,d)-regular.

$\mathcal{A}$ is (a,b)-uniform: In fact, for $f=g:=s_2$ it holds $f(b)=a$ and $f(g(x))=g(f(x))=x$ for all $x\in A$. Since $|[a]\theta|\neq|[d]\theta|$, $\mathcal{A}$ is not (a,d)-uniform.

Corollary 1 *(i) If there exists $f\in C_{32}$ with $f(x,x,y)=f(y,x,x)=y$ for all $x,y\in A$ then $\mathcal{A}$ is permutable.*

(ii) If there exist $f,g\in C_{32}$ with $f(x,x,y)=y$ and $g(x,f(x,y,z),z)=y$ for all $x,y,z\in A$ then $\mathcal{A}$ is regular.

(iii) If there exist $f,g\in C_{32}$ with $f(x,x,y)=y$ and $f(x,g(x,y,z),z)=g(x,f(x,y,z),z)=y$ for all $x,y,z\in A$ then $\mathcal{A}$ is uniform.

Proof (i) Put $f_{cd}(x):=f(c,x,d)$ for all $c,d,x\in A$. Then $f_{cd}\in C_1$, $f_{cd}(c)=d$ and $f_{cd}(d)=c$ for all $c,d\in A$. According to Theorem 1, $\mathcal{A}$ is (c,d)-permutable for all $c,d\in A$ and hence permutable.

(ii) Put $f_{cd}(x):=f(d,x,c)$ and $g_{cd}(x):=g(d,x,c)$ for all $c,d,x\in A$. Then $f_{cd},g_{cd}\in C_1$, $f_{cd}(d)=c$ and $g_{cd}(f_{cd}(x))=g(d,f(d,x,c),c)=x$ for all $c,d,x\in A$. Hence $\mathcal{A}$ is (c,d)-regular for all $c,d\in A$ according to Theorem 1 and therefore regular.

(iii) With the same notation as in the proof of (ii) we now have $f_{cd}(d)=c$, $f_{cd}(g_{cd}(x))=f(d,g(d,x,c),c)=x$ and $g_{cd}(f_{cd}(x))=g(d,s(d,x,c),c)=x$ for all $c,d,x\in A$. By Theorem 1 $\mathcal{A}$ is (c,d)-uniform for all $c,d\in A$ and hence uniform. □

Example 3 Let $\mathcal{L}=(L,\vee,\wedge,',0,1)$ be an orthomodular lattice. For $x,y\in L$ we define

$$x+y:=(x\vee(y\wedge x'))\wedge(x'\vee y').$$

Then it can be proved with standard methods:

$$x+0=0+x=x,\quad x+x=0,\quad (x+y)+y=x.$$

Let $f(x,y,z):=(x+y)+z$, then we have $f(x,x,y)=(x+x)+y=0+y=y$ and $f(y,x,x)=(y+x)+x=y$. Therefore $\mathcal{L}$ is permutable according to Corollary 1.

Now let $f(x,y,z):=(y+x)+z$ and $g(x,y,z):=(y+z)+x$. Then we have for all $x,y,z\in L$:

$$\begin{aligned}&f(x,x,y)=(x+x)+y=0+y=y,\\ &f(x,g(x,y,z),z)=(((y+z)+x)+x)+z=(y+z)+z=y,\\ &g(x,f(x,y,z),z)=(((y+x)+z)+z)+x=(y+x)+x=y.\end{aligned}$$

By Corollary 1 $\mathcal{L}$ is both regular and uniform.

In the following let 0 be a fixed element of A. Recall that $\mathcal{A}$ is called

- permutable at 0 (cf. [2], [4], [6]) if $[0](\theta\circ\phi) = [0](\phi\circ\theta)$ for all $\theta, \phi \in \mathrm{Con}(\mathcal{A})$,
- weakly regular, (cf. [4], [5], [7]) if $\theta, \phi \in \mathrm{Con}(\mathcal{A})$ and $[0]\theta = [0]\phi$ imply $\theta = \phi$,
- locally regular (cf. [3], [4]) if $a \in A$, $\theta, \phi \in \mathrm{Con}(\mathcal{A})$ and $[a]\theta = [a]\phi$ imply $[0]\theta = [0]\phi$.

Corollary 2 *(i) If there exists $f \in C_{22}$ with $f(x, 0) = x$ and $f(x, x) = 0$ for all $x \in A$ then $\mathcal{A}$ is permutable at* 0.

(ii) If there exist $f, g \in C_{22}$ with $f(x, x) = 0$ and $g(x, f(x, y)) = y$ for all $x, y \in A$ then $\mathcal{A}$ is weakly regular.

(iii) If there exist $f, g \in C_{22}$ with $f(x, 0) = x$ and $g(x, f(x, y)) = y$ for all $x, y \in A$ then $\mathcal{A}$ is locally regular.

Proof It is easy to see that $\mathcal{A}$ is permutable at 0 if and only if $\mathcal{A}$ is $(c, 0)$-permutable for all $c \in A$, that $\mathcal{A}$ is weakly regular if and only if $\mathcal{A}$ is $(0, c)$-regular for all $c \in A$ and that $\mathcal{A}$ is locally regular if and only if $\mathcal{A}$ is $(c, 0)$-regular for all $c \in A$. Applying Theorem 1 to $f_c(x) := f(c, x)$ and $g_c(x) := g(c, x)$ the assertions follow immediately. $\square$

Definition 3 $\mathcal{A}$ is called (a, b)-*semiuniform* if $|[a]\theta| \le |[b]\theta|$ for all $\theta \in \mathrm{Con}(\mathcal{A})$. $\mathcal{A}$ is called 0-*semiuniform* if $\mathcal{A}$ is $(c, 0)$-semiuniform for all $c \in A$.

Theorem 2 *(i) If there exist $f, g \in C_1$ with $f(a) = b$ and $g(f(x)) = x$ for all $x \in A$ then $\mathcal{A}$ is (a, b)-semiuniform.*

(ii) If there exist $f, g \in C_{22}$ with $f(x, x) = 0$ and $g(x, f(x, y)) = y$ for all $x, y \in A$ then $\mathcal{A}$ is 0-semiuniform.

Proof (i) Let $\theta \in \mathrm{Con}(\mathcal{A})$. If $c \in [a]\theta$ then $f(c) \in [f(a)]\theta = [b]\theta$. If $d, e \in [a]\theta$ and $f(d) = f(e)$ then $d = g(f(d)) = g(f(e)) = e$. Hence $f|_{[a]\theta}$ is an injective mapping from $[a]\theta$ to $[b]\theta$ proving $|[a]\theta| \le |[b]\theta|$.

(ii) Put $f_c(x) := f(c, x)$ and $g_c(x) := g(c, x)$ for all $c, x \in A$. Then $f_c, g_c \in C_1$, $f_c(c) = 0$ and $g_c(f_c(x)) = x$ for all $c, x \in A$. According to (i) $\mathcal{A}$ is $(c, 0)$-uniform for all $c \in A$, i.e. $\mathcal{A}$ is 0-semiuniform. $\square$

Example 4 Every finite relatively pseudocomplemented lattice $\mathcal{L} = (L, \vee, \wedge, *, 0, 1)$ is 1-semiuniform: Let $\theta \in \mathrm{Con}(\mathcal{L})$. Since L is finite the class $[c]\theta$ contains the greatest element $\bar{c}$. Consider the function $\varphi_c(x) := \bar{c} * x$. For $x \in [c]\theta$ we have $\bar{c} * x\theta\bar{c} * \bar{c} = 1$, i.e. $\varphi_c(x) \in [1]\theta$. Suppose $x, y \in [c]\theta$ and $\varphi_c(x) = \varphi_c(y)$. Then

$$x = \bar{c} \wedge (\bar{c} * x) = \bar{c} \wedge \varphi_c(x) = \bar{c} \wedge \varphi_c(y) = \bar{c} \wedge (\bar{c} * y) = y.$$

This shows that φ_c is an injection from $[c]\theta$ into $[1]\theta$, i.e. $\mathcal{L}$ is $(c, 1)$-semiuniform for all $c \in L$.

Example 5 Every finite implication algebra $\mathcal{A} = (A, \cdot)$ is 1-semiuniform: Let $\theta \in \mathrm{Con}(\mathcal{A})$ and $c \in A$. Since A is finite, the class $[c]\theta$ has a greatest element $\bar{c}$. We consider $\varphi_c(x) := \bar{c}x$. Then for $x \in [c]\theta$ we have

$$(\bar{c}x)\theta\bar{c}\bar{c} = 1,$$

hence $\varphi_c(x) \in [1]\theta$. Suppose $\varphi_c(x) = \varphi_c(y)$ for $x, y \in [c]\theta$. We prove $\bar{c}x \wedge \bar{c} = x$: Since $x \in [c]\theta$ we have $x \leq \bar{c}$ and $x(\bar{c}x) = \bar{c}(xx) = 1$ implies $x \leq \bar{c}x$. Now suppose $z \leq \bar{c}x$ and $z \leq \bar{c}$, i.e. $z(\bar{c}x) = 1$ and $z\bar{c} = 1$. We have to show that $z \leq x$:

$$\begin{aligned} zx &= (z(\bar{c}x))(zx) = (\bar{c}(zx))(zx) = ((zx)\bar{c})\bar{c} = ((zx)((z\bar{c})\bar{c}))\bar{c} \\ &= ((zx)((\bar{c}z)z))\bar{c} = ((\bar{c}z)((zx)z))\bar{c} = ((\bar{c}z)z)\bar{c} = ((z\bar{c})\bar{c})\bar{c} = \bar{c}\bar{c} = 1. \end{aligned}$$

This proves $\bar{c}x \wedge \bar{c} = x$ and analogously we obtain $\bar{c}y \wedge \bar{c} = y$, thus we infer

$$x = (\bar{c}x) \wedge \bar{c} = (\bar{c}y) \wedge \bar{c} = y.$$

Consequently φ_c is an injection of $[c]\theta$ into $[1]\theta$, whence $|[c]\theta| \leq |[1]\theta|$. Thus $\mathcal{A}$ is 1-semiuniform.

Definition 4 Let $n > 1$. $\mathcal{A}$ is called *n-(a, b)-permutable* if $(a, b) \in \theta \circ \phi \circ \theta \circ \ldots$ (n factors) is equivalent to $(a, b) \in \phi \circ \theta \circ \phi \circ \ldots$ (n factors) for all $\theta, \phi \in \mathrm{Con}(\mathcal{A})$.

Theorem 3 *(i) If there exist functions $f_1 \in C_{31} \cap C_{33}$ and $f_2 \in C_{32} \cap C_{33}$ satisfying*

$$f_1(a, x, x) = a, \;\; f_1(x, x, b) = f_2(x, b, b), \;\; f_2(x, x, b) = b,$$

for all $x \in A$ then $\mathcal{A}$ is 3-(a, b)-permutable.

(ii) If there exists $f \in C_4$ satisfying

$$f(x, x, x, a) = a, \;\; f(x, x, x, b) = b, \;\; f(x, x, b, b) = f(b, x, b, x)$$

for all $x \in A$ then $\mathcal{A}$ is 3-(a, b)-permutable.

Proof (i) Let $\theta, \phi \in \mathrm{Con}(\mathcal{A})$ and $(a, b) \in \theta \circ \phi \circ \theta$. Then there are elements $c, d \in A$ with $a\theta c\phi d\theta b$. We infer

$$a = f_1(a, c, c)\phi f_1(a, c, d)\theta f_1(c, c, b) = f_2(c, b, b)\theta f_2(c, d, b)\phi f_2(c, c, b) = b,$$

whence $(a, b) \in \phi \circ \theta \circ \phi$.

(ii) Put $f_1(x, y, z) := f(z, y, z, x)$ and $f_2(x, y, z) := f(x, x, y, z)$. Then f_1, f_2 satisfy the conditions in (i). □

Definition 5 $\mathcal{A}$ is called *n-modular* (for $n \geq 2$) if for every $\theta, \phi, \psi \in \mathrm{Con}(\mathcal{A})$ with $\theta \subseteq \psi$ we have

$$(\underbrace{\theta \circ \phi \circ \theta \circ \ldots}_{n \text{ factors}}) \cap \psi \subseteq \theta \vee (\phi \cap \psi).$$

We remark that congruence modularity is equivalent to the condition $\theta \subseteq \psi$ implies $(\theta \vee \phi) \cap \psi \subseteq \theta \vee (\phi \cap \psi)$. Thus our concept of n-modularity is weaker than congruence modularity. Obviously $(n+1)$-modularity implies n-modularity.

Theorem 4 *Every algebra $\mathcal{A}$ is 3-modular (and hence 2-modular).*

Proof Suppose $\theta, \phi, \psi \in \mathrm{Con}(\mathcal{A})$ with $\theta \subseteq \psi$ and $(c, d) \in (\theta \circ \phi \circ \theta) \cap \psi$. Then there exist $e, f \in A$ with $c\theta e\phi f\theta d$ and we obtain $e\psi c\psi d\psi f$ and hence $c\theta e(\phi \cap \psi)f\theta d$. □

Example 6 Let $\mathcal{A} = (A, s_1, s_2, s_3)$ be an algebra with 3 unary operations and $A = \{a, b, \ldots, g\}$ with

	a	b	c	d	e	f	g
s_1	c	d	e	e	e	e	d
s_2	e	e	e	f	g	g	f
s_3	d	c	b	a	a	b	c

Then $\mathrm{Con}(\mathcal{A}) \cong \mathrm{N}_5$ since $\mathrm{Con}(\mathcal{A})$ consists of the trivial congruences and

$$\begin{aligned} \theta &= \{a, b\}^2 \cup \{c, d\}^2 \cup \{e, f\}^2 \cup \{g\}^2, \\ \phi &= \{a\}^2 \cup \{b, c\}^2 \cup \{d, e\}^2 \cup \{f, g\}^2, \\ \psi &= \{a, b, g\}^2 \cup \{c, d\}^2 \cup \{e, f\}^2, \end{aligned}$$

with $\theta \subseteq \psi$. Hence $\mathrm{Con}(\mathcal{A})$ is not modular.

However, $\mathrm{Con}(\mathcal{A})$ is 4-modular: The only non-trivial case to be checked refers to the triple (θ, ϕ, ψ) and we have

$$(\theta \circ \phi \circ \theta \circ \phi) \cap \psi = \theta \subseteq \theta \vee (\phi \cap \psi).$$

We remark that $\theta \circ \phi \circ \theta \circ \phi$ is not a congruence since $(a, e) \in \theta \circ \phi \circ \theta \circ \phi$ while $(e, a) \notin \theta \circ \phi \circ \theta \circ \phi$.

Definition 6 $\mathcal{A}$ is called (a, b)-*modular* if for all $\theta, \phi, \psi \in \mathrm{Con}(\mathcal{A})$ with $\theta \subseteq \psi$ we have $(a, b) \in (\theta \vee \phi) \cap \psi$ implies $(a, b) \in \theta \vee (\phi \cap \psi)$.

Remark 2 Of course, if for all $\theta, \phi \in \mathrm{Con}(\mathcal{A})$ it is true that

$$(a, b) \in \theta \vee \phi \text{ implies } (a, b) \in \theta \circ \phi \circ \theta \tag{1}$$

or

$$(a, b) \in \theta \vee \phi \text{ implies } (a, b) \in \theta \circ \phi \tag{2}$$

then, by Theorem 4, $\mathcal{A}$ is (a, b)-modular. Hence it is a natural to search for algebras satisfying the implications (1) or (2). Obviously (2) implies (a, b)-permutability and (1) implies 3-(a, b)-permutability. We are going to find sufficient conditions for (1) or (2).

Proposition 1 *Let R be a reflexive and compatible relation on $\mathcal{A}$.*

(i) If there exists an R-compatible unary function $f : A \to A$ such that $f(a) = b$ and $f(b) = a$ then $(a, b) \in R$ implies $(a, b) \in R^{-1}$.

(ii) If there exist a function $f : A^3 \to A$ compatible with R with respect to the first and third component such that $f(a, x, x) = a$ and $f(x, x, b) = b$ for all $x \in A$ then $(a, b) \in R \circ R$ implies $(a, b) \in R$.

Proof (i) If $(a, b) \in R$ then $(b, a) = (f(a), f(b)) \in R$ due to the compatibility of f with R.

(ii) Let $(a, b) \in R \circ R$. Then $aRcRb$ for some $c \in A$ and thus $a = f(a, c, c)$ R $f(c, c, b) = b$. □

For a binary relation R on A put $[a]R = \{x \in A \mid xRa\}$.

Definition 7 $\mathcal{A}$ is *n-permutable at a* ($n > 1$) if for all $\theta, \phi \in \mathrm{Con}(\mathcal{A})$

$$[a](\theta \circ \phi \circ \ldots) = [a](\phi \circ \theta \circ \ldots)$$

(with n factors on both sides).

Theorem 5 *Let $\mathcal{A}$ be n-permutable at a. Then for all $\theta, \phi \in \mathrm{Con}(\mathcal{A})$ we have $(a, c) \in \theta \vee \phi$ if and only if $(a, c) \in \theta \circ \phi \circ \ldots$ (n factors).*

Proof Evidently, $(a, c) \in \theta \circ \phi \circ \ldots$ implies $(a, c) \in \theta \vee \phi$. Now, let $(a, c) \in \theta \vee \phi$. Then there exists an integer m such that $(a, b) \in \theta \circ \phi \circ \ldots$ (m factors). If $m \leq n$ we are done. We proof the assertion for $m = n + 1$ and n even, the general proof works with the same idea. There exists an element $d \in A$ such that

$$a(\underbrace{\theta \circ \phi \circ \ldots \circ \phi}_{n \text{ factors}})d\theta c.$$

Hence $d \in [a](\phi \circ \theta \circ \ldots \circ \theta)$ (n factors). Due to n-permutability at a we have $d \in [a](\theta \circ \phi \circ \ldots \circ \phi)$ (n factors), i.e.

$$a(\underbrace{\phi \circ \theta \circ \ldots \circ \theta}_{n \text{ factors}})d\theta c,$$

hence

$$a(\underbrace{\phi \circ \theta \circ \ldots \circ \theta}_{n \text{ factors}})c,$$

and again by n-permutability at a we arrive at $(a, c) \in \theta \circ \phi \circ \ldots \circ \phi$ (n factors). □

Corollary 3 *If $\mathcal{A}$ is 3-permutable at a then $\mathcal{A}$ is (a, c)-modular for all $c \in A$.*

Proof Let $\theta, \phi, \psi \in \mathrm{Con}(\mathcal{A})$ with $\theta \subseteq \psi$ and $(a, c) \in (\theta \vee \phi) \cap \psi$. Then due to 3-permutability at a by Theorem 5 we have $(a, c) \in \theta \circ \phi \circ \theta$, i.e. there are $d, e \in A$ with $a\theta d\phi e\theta c$. Consequently we obtain $d\psi a\psi c\psi e$ and $a\theta d(\phi \cap \psi)e\theta c$. Thus $(a, c) \in \theta \vee (\phi \cap \psi)$ and we are done. □

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