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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 43 (2004), No. 1, 119--131

Persistent URL: `http://dml.cz/dmlcz/132941`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **43** (2004) 119–131

Estimation in Connecting Measurements with Constraints of Type II *

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(Received February 12, 2004)

Abstract

This paper is a continuation of the paper [6]. It dealt with parameter estimation in connecting two–stage measurements with constraints of type I. Unlike the paper [6], the current paper is concerned with a model with additional constraints of type II binding parameters of both stages.

The article is devoted primarily to the computational aspects of algorithms published in [5] and its aim is to show the power of $\mathbf{H}^*$-optimum estimators.

The aim of the paper is to contribute to a numerical solution of the estimation problem in the two stage model, where constraints of type II occur in the second stage.

Key words: Two stage regression models, uncertainty of the type A and B, BLUE, $\mathbf{H}$–optimum estimators.

2000 Mathematics Subject Classification: 62J05

1 Introduction

In mathematical models of measurements "the connectedness syndrome" is very often encountered. This paper is concerned with a two–stage measurement with an additional condition of type II on parameters of both stages. The value $\hat{\boldsymbol{\Theta}}$

*Supported by the Council of Czech Government J14/98: 153 100011.

of the parameter may be known prior to the measurement, and may or may not be changed as a result of measurement in the second stage.

In relation to the uncertainty in the estimator $\hat{\boldsymbol{\Theta}}$ the notion of "the uncertainty of type B" is introduced, compared to "the uncertainty of type A", which is linked to the uncertainty in measurement in the second stage. In case these uncertainties are not neglected, certain difficulties arise.

During the search for statistical solutions of connecting measurement we define $\mathcal{U}_{\boldsymbol{\beta}}$ of unbiased estimators $\tilde{\boldsymbol{\beta}}$ of the parameters $\boldsymbol{\beta}$ in the regular model, where we respect errors in connecting points; and class $\widetilde{\mathcal{U}}_{\boldsymbol{\beta}}$ of unbiased estimators $\tilde{\boldsymbol{\beta}}$ of parameter $\boldsymbol{\beta}$ satisfying the constraints between parameters of the first and the second stage.

The estimators from the class $\mathcal{U}_{\boldsymbol{\beta}}$ need not fulfil the constraints between parameters of the first and the second stages. There does not exist any jointly efficient estimator in the class $\mathcal{U}_{\boldsymbol{\beta}}$. Therefore we study estimators from the class $\widetilde{\mathcal{U}}_{\boldsymbol{\beta}}$ which minimize a linear functional of the covariance matrix of the estimator $\tilde{\boldsymbol{\beta}}$.

2 Estimation in model of connecting measurements with constraints of type II

Definition 1 The two stage model of the second stage measurement is

$$\begin{pmatrix} \hat{\boldsymbol{\Theta}} \\ \mathbf{Y} - \mathbf{D}\hat{\boldsymbol{\Theta}} \end{pmatrix} \sim_n \left(\begin{pmatrix} \boldsymbol{\Theta} \\ \mathbf{X}\boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & -\boldsymbol{\Sigma}_{1,1}\mathbf{D}' \\ -\mathbf{D}\boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{2,2} + \mathbf{D}\boldsymbol{\Sigma}_{1,1}\mathbf{D}' \end{pmatrix} \right),$$

The parametric space of the two stage model with constraints of the type II is

$$\underline{\boldsymbol{\Theta}} = \{(\boldsymbol{\Theta}', \boldsymbol{\beta}') : \mathbf{B}^*\boldsymbol{\beta} + \mathbf{C}^*\boldsymbol{\Theta} + \mathbf{G}\boldsymbol{\gamma} + \mathbf{a} = \mathbf{0}\}$$

where $\mathbf{B}^*, \mathbf{C}^*, \mathbf{G}$ are given matrices with dimensions $q \times k_2, q \times k_1, q \times k_3$ and $\mathbf{a}$ is given q-dimensional vector, such that $\mathcal{M}(\mathbf{C}^*) \subset \mathcal{M}(\mathbf{B}^*)$, and $r(\mathbf{B}^*) = q < k_2$.

The vector $\boldsymbol{\Theta}$ is the parametr of the first stage (connecting stage).

The vector $\boldsymbol{\beta}$ is the parametr of the second stage (connected stage).

The estimator $\hat{\boldsymbol{\Theta}}$ of the parameter $\boldsymbol{\Theta}$ is given from the first stage.

$\mathbf{D}$ is the incidence matrix, which identify parameters of connecting network, that were used in the course of measurement in the second stage,

$\mathbf{X}$ is known matrix of the connecting network,

$\boldsymbol{\Theta}$ and $\boldsymbol{\beta}$ are effective values of the parameter from the first and second stage,

$\boldsymbol{\Sigma}_{1,1}$ is the covariance matrix of the estimator $\hat{\boldsymbol{\Theta}}$, $\boldsymbol{\Sigma}_{2,2}$ is the covariance matrix of the observation vector $\mathbf{Y}$.

The notation $\boldsymbol{\xi} \sim_n (\boldsymbol{\mu}, \boldsymbol{\Sigma}_{2,2})$ means, that the n-dimensional vector parameter $\boldsymbol{\xi}$ has the mean value equal to $\boldsymbol{\mu}$ and its covariance is $\boldsymbol{\Sigma}_{2,2}$.

From the first stage the unbiased estimator $\hat{\mathbf{\Theta}}$ and its covariance matrix $\mathbf{\Sigma}_{1,1}$ are at our disposal only.

The aim is to determine an estimator of the parameter $\boldsymbol{\beta}$ on the basis of random vector $\mathbf{Y} - \mathbf{D}\hat{\mathbf{\Theta}}$, where $\mathbf{Y}$ is the observation vector of the second stage and on the basis of the estimator $\hat{\mathbf{\Theta}}$.

Lemma 1 *If* $\mathbf{\Theta}$ *in the model from Definition 1 is known, then the BLUE of the parameter* $(\boldsymbol{\beta}', \boldsymbol{\gamma}')'$ *is*

$$\begin{aligned}\hat{\hat{\boldsymbol{\beta}}} = \Bigg(\mathbf{I} - \left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)'\Bigg\{\left[\mathbf{B}^*\left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1} \\ - \left[\mathbf{B}^*\left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\Big\{\mathbf{G}'\Big[\mathbf{B}^*\left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1} \\ \times (\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}'\left[\mathbf{B}^*\left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\Bigg\}\mathbf{B}^*\Bigg) \\ \times \left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y} - \mathbf{D}\mathbf{\Theta}) \\ - \left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)'\Bigg\{\left[\mathbf{B}^*\left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1} \\ - \left[\mathbf{B}^*\left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\Big\{\mathbf{G}'\Big[\mathbf{B}^*\left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)' \\ + \mathbf{G}\mathbf{G}'\Big]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}'\left[\mathbf{B}^*\left(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}\right)^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\Bigg\}(\mathbf{a}^* + \mathbf{C}^*\mathbf{\Theta}),\end{aligned}$$

and

$$\begin{aligned}\hat{\hat{\boldsymbol{\gamma}}} = -\Big\{\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}' \\ \times \left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{Y} + \mathbf{a}^* + \mathbf{C}^*\mathbf{\Theta}\right].\end{aligned}$$

Their covariance matrices and cross covariance matrix are

$$\begin{aligned}\mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}}) &= (\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1} - (\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)'\big[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1} \\ &\quad \times (\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\big]^{-1}\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1} + (\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1} \\ &\quad \times (\mathbf{B}^*)'\left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1} \\ &\quad \times \mathbf{G}\Big\{\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\Big\}^{-1}\mathbf{G}' \\ &\quad \times \left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}, \\ \mathrm{cov}(\hat{\hat{\boldsymbol{\beta}}}, \hat{\hat{\boldsymbol{\gamma}}}) &= -(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)'\left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1} \\ &\quad \times \mathbf{G}\Big\{\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\Big\}^{-1}, \\ \mathrm{Var}(\hat{\hat{\boldsymbol{\gamma}}}) &= \Big\{\mathbf{G}'\left[\mathbf{B}^*(\mathbf{X}'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\right]^{-1}\mathbf{G}\Big\}^{-1} - \mathbf{I}.\end{aligned}$$

Proof [5], section 3.

Definition 2 The estimator from Lemma 1 obtained under the condition $\boldsymbol{\Sigma}_{1,1} = \mathbf{0}$ ($\Rightarrow \mathrm{Var}(\boldsymbol{\Theta}) = \mathbf{0}$) is called the standard estimator if in this estimator the vector $\boldsymbol{\Theta}$ is substituted by $\hat{\boldsymbol{\Theta}}$.

Remark 1 If $\boldsymbol{\Theta}$ in Lemma 1 is substituted by $\hat{\boldsymbol{\Theta}}$, the standard estimator is obtained. Its covariance matrix is given by the following relationships.

$$\begin{aligned}
\mathrm{Var}(\hat{\boldsymbol{\beta}}) &= \mathrm{Var}[\mathbf{N}_1(\mathbf{Y} - \mathbf{D}\hat{\boldsymbol{\Theta}})] + \mathrm{Var}[\mathbf{N}_2(\mathbf{C}^*\hat{\boldsymbol{\Theta}} + \mathbf{a})] \\
&+ \mathrm{cov}[\mathbf{N}_1(\mathbf{Y} - \mathbf{D}\hat{\boldsymbol{\Theta}}), \mathbf{N}_2(\mathbf{C}^*\hat{\boldsymbol{\Theta}} + \mathbf{a})] + \mathrm{cov}[\mathbf{N}_2(\mathbf{C}^*\hat{\boldsymbol{\Theta}} + \mathbf{a}), \mathbf{N}_1(\mathbf{Y} - \mathbf{D}\hat{\boldsymbol{\Theta}})] \\
&= \mathbf{N}_1(\boldsymbol{\Sigma}_{2,2} + \mathbf{D}\boldsymbol{\Sigma}_{1,1}\mathbf{D}')\mathbf{N}_1' + \mathbf{N}_2\mathbf{C}^*\boldsymbol{\Sigma}_{1,1}(\mathbf{C}^*)'\mathbf{N}_2' \\
&- \mathbf{N}_1\mathbf{D}\boldsymbol{\Sigma}_{1,1}(\mathbf{C}^*)'\mathbf{N}_2' - \mathbf{N}_2\mathbf{C}^*\boldsymbol{\Sigma}_{1,1}\mathbf{D}'\mathbf{N}_1',
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{N}_1 &= \Big(\mathbf{I} - (\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)'\Big\{\Big[\mathbf{B}^*(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1} \\
&- \Big[\mathbf{B}^*(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1}\mathbf{G}\mathbf{G}'\Big[\mathbf{B}^*(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1}\mathbf{G}' \\
&\times \Big[\mathbf{B}^*(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1}(\mathbf{B}^*)'(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}, \\
\mathbf{N}_2 &= -(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)'\Big\{\Big[\mathbf{B}^*(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1} \\
&- \Big[\mathbf{B}^*(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1}\mathbf{G} \\
&\times \{\mathbf{G}'\Big[\mathbf{B}^*(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1}\mathbf{G}\}^{-1}\mathbf{G}' \\
&\times \Big[\mathbf{B}^*(\mathbf{X}'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X})^{-1}(\mathbf{B}^*)' + \mathbf{G}\mathbf{G}'\Big]^{-1}\Big\}.
\end{aligned}$$

Theorem 1 *In the model*

$$\mathbf{Y} - \mathbf{D}\hat{\boldsymbol{\Theta}} \sim_n (\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}_{2,2} + \mathbf{D}\boldsymbol{\Sigma}_{1,1}\mathbf{D}'), \qquad \mathbf{a}^* + \mathbf{C}^*\boldsymbol{\Theta} + \mathbf{B}^*\boldsymbol{\beta} + \mathbf{G}\boldsymbol{\gamma} = \mathbf{0},$$

the class of all unbiased linear estimators of $\binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}}$ *based on the vectors* $\hat{\boldsymbol{\Theta}}$ *and* $\mathbf{Y} - \mathbf{D}\hat{\boldsymbol{\Theta}}$ *is*

$$\mathcal{U}_{\beta,\gamma} = \left\{ \begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{K}_1, \ \mathbf{K}_2 \\ \mathbf{K}_3, \ \mathbf{K}_4 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\Theta}} \\ \mathbf{Y} - \mathbf{D}\hat{\boldsymbol{\Theta}} \end{pmatrix} \right\},$$

where

$$\begin{pmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -\mathbf{G}^- \end{pmatrix}\mathbf{a}^* + \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_3 \end{pmatrix}(\mathbf{I} - \mathbf{G}\mathbf{G}^-)\mathbf{a}^*,$$

$$\begin{pmatrix} \mathbf{K}_1, \ \mathbf{K}_2 \\ \mathbf{K}_3, \ \mathbf{K}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{0}, & \mathbf{X}^- \\ -\mathbf{G}^-\mathbf{C}^*, & -\mathbf{G}^-\mathbf{B}^*\mathbf{X}^- \end{pmatrix}$$
$$+ \begin{pmatrix} \mathbf{Z}_1, \ \mathbf{Z}_2 \\ \mathbf{Z}_3, \ \mathbf{Z}_4 \end{pmatrix} \begin{pmatrix} (\mathbf{I}-\mathbf{G}\mathbf{G}^-)\mathbf{C}^*, & (\mathbf{I}-\mathbf{G}\mathbf{G}^-)\mathbf{B}^*\mathbf{X}^- \\ \mathbf{0}, & \mathbf{I}-\mathbf{X}\mathbf{X}^- \end{pmatrix},$$

where $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4$ are arbitrary matrices with suitable dimensions.

The covariance matrix of the estimator $\binom{\tilde{\boldsymbol{\beta}}}{\tilde{\gamma}} \in \mathcal{U}_{\beta,\gamma}$ is

$$\operatorname{Var}\begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_1, \ \mathbf{K}_2 \\ \mathbf{K}_3, \ \mathbf{K}_4 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & -\boldsymbol{\Sigma}_{1,1}\mathbf{D}' \\ -\mathbf{D}\boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{2,2}+\mathbf{D}\boldsymbol{\Sigma}_{1,1}\mathbf{D}' \end{pmatrix} \begin{pmatrix} \mathbf{K}_1, \ \mathbf{K}_2 \\ \mathbf{K}_3, \ \mathbf{K}_4 \end{pmatrix}'.$$

Proof [5], section 3.

Lemma 2 *Let in Lemma 1 $\boldsymbol{\Theta}$ be substituted by $\hat{\boldsymbol{\Theta}}$. Then such estimator (it is usually used in practice) belong to the class $\mathcal{U}_{\beta,\gamma}$.*

Proof [5], section 3.

Theorem 2 *The class $\tilde{\mathcal{U}_{\beta,\gamma}}$ of all linear unbiased estimators which in addition satisfy the constraints*

$$\mathbf{a}^* + \mathbf{C}^*\hat{\boldsymbol{\Theta}} + \mathbf{B}^*\tilde{\boldsymbol{\beta}} + \mathbf{G}\tilde{\gamma} = \mathbf{0},$$

is given by such a choice of the matrices $\mathbf{Z}_1, \ldots, \mathbf{Z}_4$, in Theorem 1, which satisfy the following equation

$$\begin{pmatrix} \mathbf{Z}_1, \ \mathbf{Z}_2 \\ \mathbf{Z}_3, \ \mathbf{Z}_4 \end{pmatrix} = (\mathbf{B}^*, \mathbf{G})^-[-(\mathbf{I}-\mathbf{G}\mathbf{G}^-), \mathbf{0}] \begin{pmatrix} \mathbf{I}-\mathbf{G}\mathbf{G}^-, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I}-\mathbf{X}\mathbf{X}^- \end{pmatrix}$$
$$+ \begin{pmatrix} \mathbf{W}_1, \ \mathbf{W}_2 \\ \mathbf{W}_3, \ \mathbf{W}_4 \end{pmatrix} - (\mathbf{B}^*, \mathbf{G})^-(\mathbf{B}^*, \mathbf{G}) \begin{pmatrix} \mathbf{W}_1, \ \mathbf{W}_2 \\ \mathbf{W}_3, \ \mathbf{W}_4 \end{pmatrix} \begin{pmatrix} \mathbf{I}-\mathbf{G}\mathbf{G}^-, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I}-\mathbf{X}\mathbf{X}^- \end{pmatrix},$$

where the matrices $\mathbf{W}_1, \ldots, \mathbf{W}_4$ are arbitrary.

Proof [5], section 3.

3 H*-optimum estimator for constraints II

Definition 3 Let $\mathbf{H}^*$ be a given $(k+l)\times(k+l)$ positive semidefinite matrix. The estimator $\binom{\tilde{\boldsymbol{\beta}}}{\tilde{\gamma}}$ from $\tilde{\mathcal{U}}_{\beta,\gamma}$ is said to be $\mathbf{H}^*$-optimum if it minimizes the value

$$\operatorname{Tr}\left[\mathbf{H}^* \operatorname{Var}\begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\gamma} \end{pmatrix}\right], \begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\gamma} \end{pmatrix} \in \tilde{\mathcal{U}}_{\beta,\gamma}.$$

Theorem 3 *An estimator $\binom{\tilde{\boldsymbol{\beta}}}{\tilde{\gamma}}$ is $\mathbf{H}^*$-optimum if the matrices $\mathbf{W}_1, \mathbf{W}_2$, $\mathbf{W}_3, \mathbf{W}_4$ (Theorem 2) are solution of the equation*

$$\left\{\mathbf{I} - (\mathbf{B}^*, \mathbf{G})'\left[(\mathbf{B}^*, \mathbf{G})^-\right]'\right\}\mathbf{H}^*\left[\mathbf{I} - (\mathbf{B}^*, \mathbf{G})^-(\mathbf{B}^*, \mathbf{G})\right]\mathbf{WSTS}' =$$
$$= -\left\{\mathbf{I} - (\mathbf{B}^*, \mathbf{G})'\left[(\mathbf{B}^*, \mathbf{G})^-\right]'\right\}\mathbf{H}^*(\mathbf{RTS}' + \mathbf{ASTS}'),$$

where

$$\mathbf{A} = (\mathbf{B}^*, \mathbf{G})^-\left[-(\mathbf{I}-\mathbf{G}\mathbf{G}^-, \mathbf{0}\right]\begin{pmatrix} \mathbf{I}-\mathbf{G}\mathbf{G}^-, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I}-\mathbf{X}\mathbf{X}^- \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_1, & \mathbf{W}_2 \\ \mathbf{W}_3, & \mathbf{W}_4 \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} \mathbf{0}, & \mathbf{X}^- \\ -\mathbf{G}^-\mathbf{C}^*, & -\mathbf{G}^-\mathbf{B}^*\mathbf{X}^- \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} (\mathbf{I}-\mathbf{G}\mathbf{G}^-)\mathbf{C}^*, & (\mathbf{I}-\mathbf{G}\mathbf{G}^-)\mathbf{B}^*\mathbf{X}^- \\ \mathbf{0}, & \mathbf{I}-\mathbf{X}\mathbf{X}^- \end{pmatrix},$$

$$\mathbf{T} = \mathrm{Var}\begin{pmatrix} \hat{\mathbf{\Theta}} \\ \mathbf{Y}-\mathbf{D}\hat{\mathbf{\Theta}} \end{pmatrix} = \begin{pmatrix} \mathbf{\Sigma}_{1,1}, & -\mathbf{\Sigma}_{1,1}\mathbf{D}' \\ -\mathbf{D}\mathbf{\Sigma}_{1,1}, & \mathbf{\Sigma}_{2,2}+\mathbf{D}\mathbf{\Sigma}_{1,1}\mathbf{D}' \end{pmatrix}.$$

Proof [5], section 4.

Remark 2 Since the matrices $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4$ of the $\mathbf{H}^*$-optimum estimator are functions of the matrix $\mathbf{H}^*$, the joint efficient estimator does not exist in the class $\tilde{\mathcal{U}}_{\beta,\gamma}$.

4 Numerical studies—constraints type II

In this part we will concentrate on a numerical calculation of the estimator of parameters. In all following examples we need to construct a condition expressing a relation between parameters of the first and the second stages. From this condition we can always construct a vector function $\mathbf{g}$ of parameter β and Θ where $\mathbf{g}(\beta, \Theta, \gamma) = \mathbf{0}$. We apply the Taylor expansion at point (β_0, Θ_0) to this function. So for estimators of parameters we get the condition

$$\mathbf{g}(\boldsymbol{\beta}, \mathbf{\Theta}, \gamma) = \mathbf{g}(\beta_0, \Theta_0, \gamma_0) + \mathbf{C}\delta\theta + \mathbf{B}\delta\beta + \mathbf{G}\delta\gamma = \mathbf{0}.$$

Example 1 Let us consider the point A_1 from the first stage with the plane coordinates $(\Theta_1,\ \Theta_2)$, that were measured as $(\hat{\Theta}_1, \hat{\Theta}_2) = (59999.91, 41339.81)$. The accuracy of measurement was given by the dispersion $\omega_1^2 = 0.04^2$.

In the second stage we will assume the same dispersion $\omega_1^2 = 0.04^2$ for the measured coordinates $(y_1,\ y_2), \ldots, (y_7, y_8) = (54999.95,\ 40000.04,\ 49999.94,\ 41339.70,\ 54999.89,\ 60000.01,\ 65000.05,\ 49999.88)$ of the points $P_i = (\beta_{2i-1}, \beta_{2i})$ for $i = 1, 2, 3, 4$.

The aim is to make the estimator of the coordinates of the points P_1, P_2, P_3 and P_4 more accurate under the constraint that all these points together with the point A_1 are located on a circle, with a radius γ_3 and a center $[\gamma_1, \gamma_2]$ unknown.

Our constraints are

$$\begin{aligned}
(\theta_1-\gamma_1)^2 + (\theta_2-\gamma_2)^2 - \gamma_3^2 &= 0\,, \\
(\beta_1-\gamma_1)^2 + (\beta_2-\gamma_2)^2 - \gamma_3^2 &= 0\,, \\
(\beta_3-\gamma_1)^2 + (\beta_4-\gamma_2)^2 - \gamma_3^2 &= 0\,, \\
(\beta_5-\gamma_1)^2 + (\beta_6-\gamma_2)^2 - \gamma_3^2 &= 0\,, \\
(\beta_7-\gamma_1)^2 + (\beta_8-\gamma_2)^2 - \gamma_3^2 &= 0\,.
\end{aligned}$$

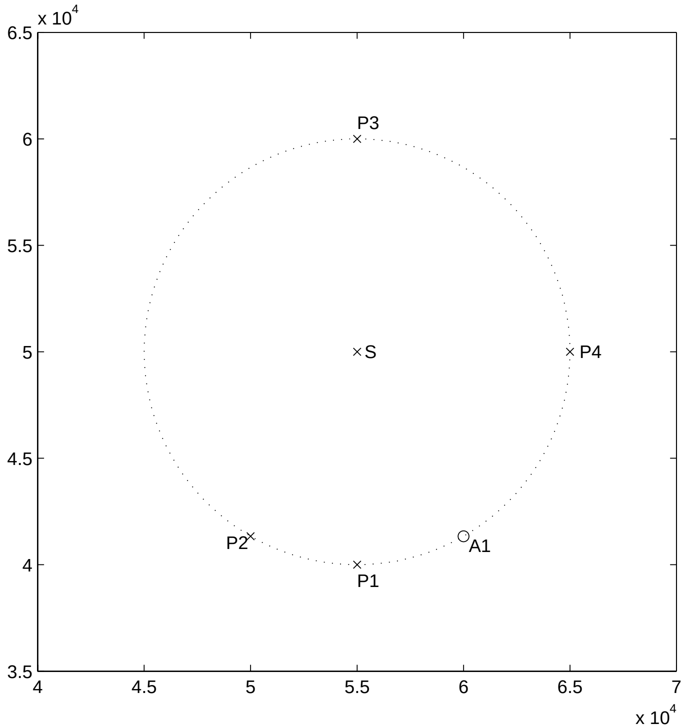

Figure 1: Situation (in S-JTSK)

In our linearized model we will determine numerically the estimator and the covariance matrix according to Lemma 1:

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} 54999.95 \\ 40000.14 \\ 50000.00 \\ 41339.80 \\ 54999.89 \\ 60000.11 \\ 64999.83 \\ 49999.88 \end{pmatrix} \text{ and } \hat{\boldsymbol{\gamma}} = \begin{pmatrix} 54999.84 \\ 50000.01 \\ 9999.98 \end{pmatrix}.$$

After that we will numerically determine $\mathbf{H}^*$–optimum estimator from Theorem 2 and 3 for the matrix

$$\mathbf{H}_1{}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ are } \tilde{\boldsymbol{\beta}} = \begin{pmatrix} 54999.95 \\ 40000.04 \\ 49999.91 \\ 41339.65 \\ 54999.89 \\ 60000.08 \\ 64999.69 \\ 49999.88 \end{pmatrix} \text{ and } \tilde{\boldsymbol{\gamma}} = \begin{pmatrix} 54999.67 \\ 50000.06 \\ 10000.02 \end{pmatrix}.$$

By chosen matrix $\mathbf{H}_1{}^*$ *minimizing data errors in the process estimation of the vector* $\tilde{\boldsymbol{\beta}}$ *we got better estimator of the parameter* β *in comparison with the standard estimator* $\hat{\boldsymbol{\beta}}$*. It follows from the fact that for the chosen matrix* $\mathbf{H}^*$ *is* $\mathrm{Tr}(\mathbf{H}^*\,\mathrm{Var}(\tilde{\boldsymbol{\beta}})) = 2.17 \cdot 10^{-3} < 2.66 \cdot 10^{-3} = \mathrm{Tr}(\mathbf{H}^*\,\mathrm{Var}(\hat{\boldsymbol{\beta}}))$.

Let us study the proportion accuracy of the standard estimator $\hat{\boldsymbol{\beta}}$ and the $\mathbf{H}_i{}^*$-optimum estimator $\tilde{\boldsymbol{\beta}}$ for $i = 2, 3, 4$. We will not determine the estimators from now, but we will only study the trace of the covariance matrix $\mathrm{Tr}(\mathbf{H}\,\mathrm{Var}(\tilde{\boldsymbol{\beta}}))$ for comparing it with the above mentioned $\mathrm{Tr}(\mathbf{H}\,\mathrm{Var}(\hat{\boldsymbol{\beta}}))$.

We get $\mathrm{Tr}(\mathbf{H}_2{}^*\,\mathrm{Var}(\tilde{\boldsymbol{\beta}})) = 2.59 \cdot 10^{-3} < \mathrm{Tr}(\mathbf{H}_2{}^*\,\mathrm{Var}(\hat{\boldsymbol{\beta}})) = 2.66 \cdot 10^{-3}$ for matrix

$$\mathbf{H}_2{}^* = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

we get $\mathrm{Tr}(\mathbf{H}_3{}^*\,\mathrm{Var}(\tilde{\boldsymbol{\beta}})) = 3.09 \cdot 10^{-3} < \mathrm{Tr}(\mathbf{H}_3{}^*\,\mathrm{Var}(\hat{\boldsymbol{\beta}})) = 3.21 \cdot 10^{-3}$ for matrix

$$\mathbf{H}_3{}^* = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

and we get $\mathrm{Tr}(\mathbf{H}_4{}^*\,\mathrm{Var}(\tilde{\boldsymbol{\beta}})) = 2.72 \cdot 10^{-3} < \mathrm{Tr}(\mathbf{H}_4{}^*\,\mathrm{Var}(\hat{\boldsymbol{\beta}})) = 3.49 \cdot 10^{-3}$ for matrix

$$\mathbf{H}_4{}^* = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

It is evident that $\mathrm{Tr}(\mathbf{H}_i{}^*\,\mathrm{Var}(\tilde{\boldsymbol{\beta}})) < \mathrm{Tr}(\mathbf{H}_i{}^*\,\mathrm{Var}(\hat{\boldsymbol{\beta}}))$ for $i = 1, \dots 4$.

Now let us study the proportion of this values for different covariance matrices $\boldsymbol{\Sigma}_{1,1}$ and $\boldsymbol{\Sigma}_{2,2}$. In other numerical calculations we choose the matrix $\boldsymbol{\Sigma}_{1,1}$ as the fixed one and we change the matrix $\boldsymbol{\Sigma}_{2,2}$ by the multiplication by the number k.

The proportions in dependence on k are shown in the following table and graph.

The proportion $\mathrm{Tr}(\mathbf{H}_i{}^* \,\mathrm{Var}(\tilde{\boldsymbol{\beta}}))$ and $\mathrm{Tr}(\mathbf{H}_i{}^* \,\mathrm{Var}(\hat{\boldsymbol{\beta}}))$				
k	$i = 1, \mathbf{H}_1{}^*$	$i = 2, \mathbf{H}_2{}^*$	$i = 3, \mathbf{H}_3{}^*$	$i = 4, \mathbf{H}_4{}^*$
100	99.99 %	100.00 %	100.00 %	99.99%
64	99.98 %	100.00 %	100.00 %	99.98%
50	99.97 %	100.00 %	100.00 %	99.97%
25	99.90 %	99.99 %	99.98 %	99.87%
16	99.77 %	99.97 %	99.96 %	99.70%
9	99.33 %	99.92 %	99.89 %	99.12%
5	98.14 %	99.78 %	99.67 %	97.57%
4	97.30 %	99.67 %	99.52 %	96.50%
3	95.74 %	99.47 %	99.22 %	94.54%
2	92.29 %	98.99 %	98.52 %	90.27%
1	81.73 %	97.24 %	96.00 %	77.81%
1/2	65.01 %	93.41 %	90.64 %	59.52%
1/4	45.48 %	86.18 %	81.00 %	39.92%
1/10	23.72 %	69.58 %	61.02 %	19.92%
1/16	16.02 %	58.29 %	48.90 %	13.26%
1/25	10.77 %	46.87 %	37.66 %	8.83%
1/50	5.64 %	30.34 %	22.98 %	4.58%
1/64	4.45 %	25.34 %	18.86 %	3.60%
1/100	2.89 %	17.79 %	12.91 %	2.33%
1/400	0.74 %	5.11 %	3.55 %	0.59%

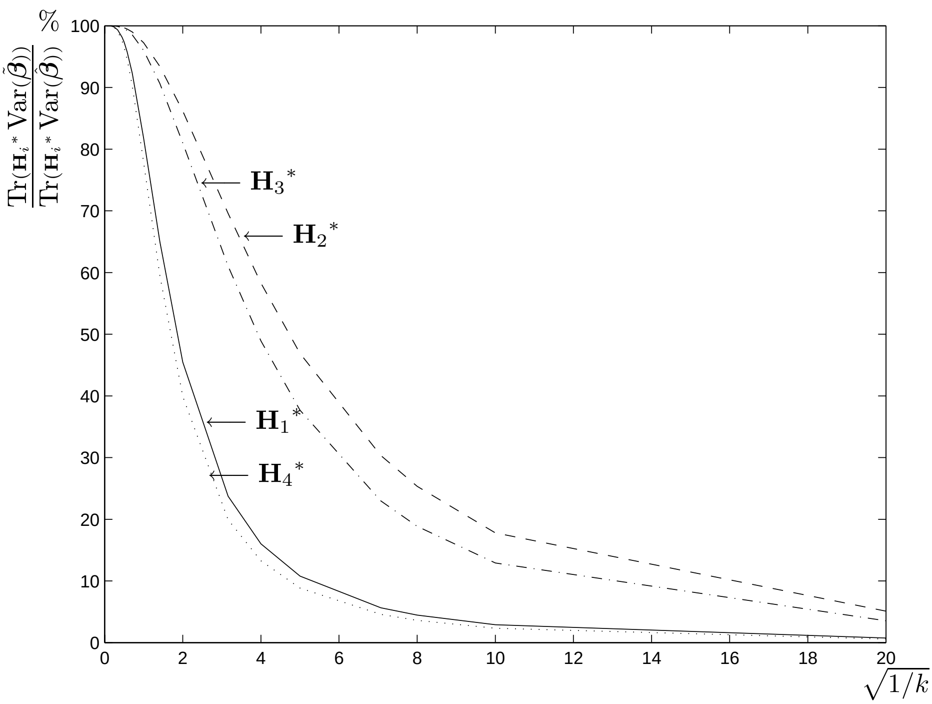

Figure 2: The proportion $\mathrm{Tr}(\mathbf{H}_i{}^* \,\mathrm{Var}(\tilde{\boldsymbol{\beta}}))$ and $\mathrm{Tr}(\mathbf{H}_i{}^* \,\mathrm{Var}(\hat{\boldsymbol{\beta}}))$

Example 2 We have at our disposal the coordinates of the points A_1, A_2, A_3, A_4, A_5 that are given from the first stage – from the connecting measurement.

All the angles $\angle y_1 = (A_2 A_1 P_1)$, $\angle y_2 = (A_1 P_1 P_2)$, $\angle y_3 = (P_1 P_2 P_3)$, $\angle y_4 = (P_2 P_3 P_4)$, $\angle y_5 = (P_3 P_4 A_5)$, $\angle y_6 = (P_4 A_5 A_4)$, $\angle y_7 = (P_1 A_3 P_2)$ and distances $y_8 = A_1 P_1$, $y_9 = P_1 P_2$, $y_{10} = P_2 P_3$, $y_{11} = P_3 P_4$, $y_{12} = P_4 A_5$ were measured in the second stage—in the connecting stage.

The aim is to find an estimator for the plane coordinates $(\beta_1, \beta_2), \ldots, (\beta_7, \beta_8)$ of the points P_1, P_2, P_3 and P_4 from the second stage, in such a way so as the distance between the points P_1 and P_3 would be determined as accurately as possible.

Values of plane-coordinates and distances will be given in meters, values of angles will be given in radians.

The accuracy of measurement is given by the dispersion or covariance matrices. We suppose that the points from the first stage are determined with the dispersion 0.06^2 m. Measurement of angles in the second stage was performed with the standard deviation $\omega_a = 10/206265$. Measurement of distances in the second stage was performed with the standard deviation $\omega_d = 0.005$ m.

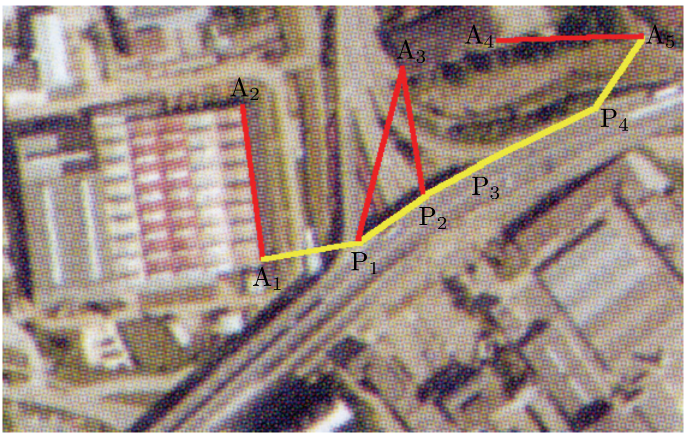

Figure 3: The aerial photograph of the Tovární Street, Olomouc

We carry out numerical studies in this example for the plane coordinates of points A_i

	Y	X	
A_1	543330,15	1121488,64	corner of the assembly hall
A_2	544347,49	1121390,53	corner of the assembly hall
A_3	544246,27	1121374,30	corner of the assembly hall
A_4	544187,59	1121350,71	corner of the assembly hall
A_5	544101,01	1121357,58	plastic point

and for measured values from the second stage

$y_1 = 1.6091000$, $y_2 = 2.7466880$, $y_3 = 3.2469781$, $y_4 = 3.2134906$,
$y_5 = 2.5395759$, $y_6 = 1.1120582$, $y_7 = 4.4991793$,
$y_8 = 56.515$, $y_9 = 50.889$, $y_{10} = 43.064$, $y_{11} = 80.486$, $y_{12} = 41.524$.

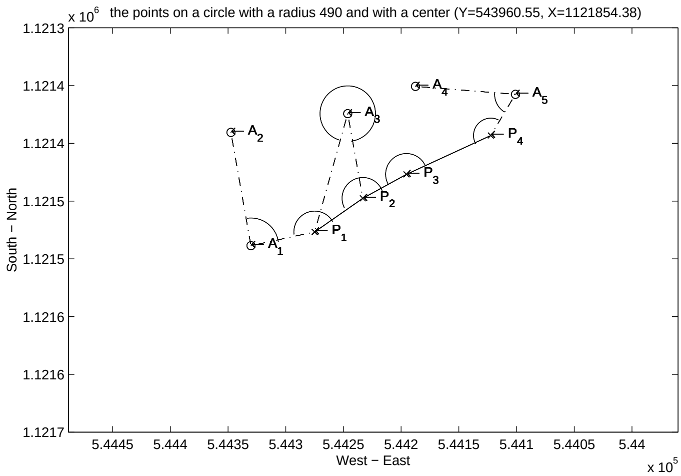

Figure 4: Situation (S-JTSK)

Now we make nonlinear model of our example $\mathbf{Y} = f(\boldsymbol{\Theta}, \boldsymbol{\beta}) + \varepsilon$.

$$\mathbf{Y} = \begin{pmatrix} f_1(\Theta,\beta) \\ f_2(\Theta,\beta) \\ f_3(\Theta,\beta) \\ f_4(\Theta,\beta) \\ f_5(\Theta,\beta) \\ f_6(\Theta,\beta) \\ f_7(\Theta,\beta) \\ f_8(\Theta,\beta) \\ f_9(\Theta,\beta) \\ f_{10}(\Theta,\beta) \\ f_{11}(\Theta,\beta) \\ f_{12}(\Theta,\beta) \end{pmatrix} = \begin{bmatrix} \arctan(\frac{\beta_2-\theta_2}{\beta_1-\Theta_1}) - \arctan(\frac{\Theta_4-\Theta_2}{\Theta_3-\Theta_1}) \\ \pi - \arctan(\frac{\beta_4-\beta_2}{\beta_3-\beta_1}) - \arctan(\frac{\Theta_2-\beta_2}{\Theta_1-\beta_1}) \\ \pi - \arctan(\frac{\beta_6-\beta_4}{\beta_5-\beta_3}) + \arctan(\frac{\beta_2-\beta_4}{\beta_1-\beta_3}) \\ \pi - \arctan(\frac{\beta_8-\beta_6}{\beta_7-\beta_5}) + \arctan(\frac{\beta_4-\beta_6}{\beta_3-\beta_5}) \\ \pi - \arctan(\frac{\Theta_{10}-\beta_8}{\Theta_9-\beta_7}) + \arctan(\frac{\beta_6-\beta_8}{\beta_5-\beta_7}) \\ -\arctan(\frac{\Theta_8-\theta_{10}}{\Theta_7-\Theta_9}) + \arctan(\frac{\beta_8-\Theta_{10}}{\beta_7-\Theta_9}) \\ \pi - \arctan(\frac{\beta_2-\Theta_6}{\beta_1-\Theta_5}) + \arctan(\frac{\beta_4-\Theta_6}{\beta_3-\Theta_5}) \\ \sqrt{{\beta_1}^2 - 2\,\beta_1\,\Theta_1 + {\Theta_1}^2 + {\beta_2}^2 - 2\,\beta_2\,\Theta_2 + {\Theta_2}^2} \\ \sqrt{{\beta_3}^2 - 2\,\beta_3\,\beta_1 + {\beta_1}^2 + {\beta_4}^2 - 2\,\beta_4\,\beta_2 + {\beta_2}^2} \\ \sqrt{{\beta_5}^2 - 2\,\beta_5\,\beta_3 + {\beta_3}^2 + {\beta_6}^2 - 2\,\beta_6\,\beta_4 + {\beta_4}^2} \\ \sqrt{{\beta_7}^2 - 2\,\beta_7\,\beta_5 + {\beta_5}^2 + {\beta_8}^2 - 2\,\beta_8\,\beta_6 + {\beta_6}^2} \\ \sqrt{{\beta_7}^2 - 2\,\beta_7\,\Theta_9 + {\Theta_9}^2 + {\beta_8}^2 - 2\,\beta_8\,\Theta_{10} + {\Theta_{10}}^2} \end{bmatrix}$$

Points from the second stage are situated on a circle—additional constraints are

$$\begin{aligned} g_1(\Theta,\beta,\beta) &= (\beta_1-\gamma_1)^2 + (\beta_2-\gamma_2)^2 - \gamma_3^2 \;=\; 0\,, \\ g_2(\Theta,\beta,\beta) &= (\beta_3-\gamma_1)^2 + (\beta_4-\gamma_2)^2 - \gamma_3^2 \;=\; 0\,, \\ g_3(\Theta,\beta,\beta) &= (\beta_5-\gamma_1)^2 + (\beta_6-\gamma_2)^2 - \gamma_3^2 \;=\; 0\,, \\ g_4(\Theta,\beta,\beta) &= (\beta_7-\gamma_1)^2 + (\beta_8-\gamma_2)^2 - \gamma_3^2 \;=\; 0\,. \end{aligned}$$

Now we need to use the Taylor expansion $\mathbf{Y} = f_0 + \mathbf{B}\delta\beta + \mathbf{D}\delta\Theta + \mathbf{G}\delta\gamma = \mathbf{0}$, where the matrices $\mathbf{B} = \frac{\partial f_i(\Theta^0,\beta^0)}{\partial\beta'}$, $\mathbf{D} = \frac{\partial f_i(\Theta^0,\beta^0)}{\partial\Theta'}$, $\mathbf{G} = \frac{\partial g_j(\Theta^0,\beta^0,\gamma^0)}{\partial\gamma'}$ and $f_0 = \mathbf{Y}(\Theta^0,\beta^0)$.

In the linearized model, we calculate the estimator $\hat{\boldsymbol{\beta}}$ and the distance estimator $\hat{y}_{10}$.

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} 544274.921 \\ 1121476.680 \\ 544233.140 \\ 1121447.619 \\ 544195.417 \\ 1121426.860 \\ 544122.290 \\ 1121393.236 \end{pmatrix}, \qquad \hat{y}_{10} = 43.0569.$$

Next by the same procedure as in the preceding example we calculate, by Lemma 2.2 , the $\mathbf{H}^*$-optimum estimator $\tilde{\boldsymbol{\beta}}$. The matrix $\mathbf{H}^*$ of the type

$$\mathbf{H}^* = \mathbf{p}\mathbf{p}', \quad \mathbf{p}' = \frac{\partial\sqrt{(\beta_5-\beta_3)^2 + (\beta_6-\beta_4)^2}}{\partial\beta'}$$

is chosen in such a way, so that the resulting estimator would be optimal for determining the distance between the points P_2 and P_3. We arrive at the estimator

$$\tilde{\boldsymbol{\beta}} = \begin{pmatrix} 544274.916 \\ 1121476.678 \\ 544233.150 \\ 1121447.604 \\ 544195.416 \\ 1121426.854 \\ 544122.286 \\ 1121393.239 \end{pmatrix}, \qquad \widetilde{y}_{10} = 43.0637.$$

Let us conclude with the comparison of the resulting estimators of distance between P_2 and P_3. The mesurement distance y_{10} was 43.064 m, the distance determined by the standard estimator $\hat{y}_{10}$ was 43.0569 m and by the $\mathbf{H}^*$-optimum estimator $\widetilde{y}_{10} = 43.0637$ m (the difference between estimators is 6.8 mm).

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