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Class Preserving Mappings of Equivalence Systems

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Abstract

By an equivalence system is meant a couple $\mathcal{A} = (A, \theta)$ where A is a non-void set and θ is an equivalence on A. A mapping h of an equivalence system $\mathcal{A}$ into $\mathcal{B}$ is called a class preserving mapping if $h([a]_\theta) = [h(a)]_{\theta'}$ for each $a \in A$. We will characterize class preserving mappings by means of permutability of θ with the equivalence Φ_h induced by h.

Key words: Equivalence relation, equivalence system, relational system, homomorphism, strong homomorphism, permuting equivalences.

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For the basic concepts, the reader is referred to [1],[2],[3]. Let R and S be binary relations on a non-void set A. As usually, their *relational product* will be denoted by $R \circ S$, i.e. $R \circ S = \{\langle a, b\rangle \in A^2; \exists c \in A \text{ with } \langle a, c\rangle\rangle \in R$ and $\langle c, b\rangle \in S\}$. We will say that R, S *permute* (or they are *permutable*) if $R \circ S = S \circ R$.

Lemma 1 *Let R, S be symmetric relations on A. Then $R \circ S \subseteq S \circ R$ is equivalent to $R \circ S = S \circ R$.*

Proof If $R \circ S \subseteq S \circ R$ then, due to symmetry,

$$S \circ R = S^{-1} \circ R^{-1} = (R \circ S)^{-1} \subseteq (S \circ R)^{-1} = R^{-1} \circ S^{-1} = R \circ S$$

thus S, R permute. The converse is trivial. □

By a *relational system* is meant a pair $\mathcal{A} = (A, R)$, where $A \neq \emptyset$ is a set and R is a binary relation on A. If R is an equivalence relation, $\mathcal{A} = (A, R)$ will be called an *equivalence system*.

We are going to introduce a quotient relational system as follows.

Definition 1 Let $\mathcal{A} = (A, R)$ be a relational system and Φ be an equivalence on A. Define a binary relation R/Φ on the factor set (i.e. a partition) A/Φ as follows: $\langle [a]_\Phi, [b]_\Phi \rangle \in R/\Phi$ iff there exist $x \in [a]_\Phi, y \in [b]_\Phi$ with $\langle x, y \rangle \in R$. Then $\mathcal{A}/\Phi = (A/\Phi, R/\Phi)$ will be called a *quotient relational system* of $\mathcal{A}$ by Φ.

Remark 1 It is evident that if R is reflexive or symmetric then R/Φ has the corresponding property.

Lemma 2 *Let $\mathcal{A} = (A, R)$ be a relational system and R be transitive. Let Φ be an equivalence on A and $\Phi \circ R \subseteq R \circ \Phi$. Then R/Φ is transitive, too.*

Proof Suppose $\langle [a]_\Phi, [b]_\Phi \rangle \in R/\Phi$ and $\langle [b]_\Phi, [c]_\Phi \rangle \in R/\Phi$. Then there exist $x, y, y', z \in A$ such that $x \in [a]_\Phi$, $y, y' \in [b]_\Phi$, $z \in [c]_\Phi$ and $\langle x, y \rangle \in R$, $\langle y', z \rangle \in R$. Hence $\langle x, z \rangle \in R \circ \Phi \circ R \subseteq R \circ R \circ \Phi \subseteq R \circ \Phi$. Thus there is $w \in A$ with $\langle x, w \rangle \in R$ and $\langle w, z \rangle \in \Phi$, i.e. $w \in [z]_\Phi = [c]_\Phi$. By the Definition, $\langle [a]_\Phi, [c]_\Phi \rangle \in R/\Phi$ proving transitivity of R/Φ. □

Let $\mathcal{A} = (A, R), \mathcal{B} = (B, S)$ be relational systems. A mapping $h : A \to B$ is called a *homomorphism* of $\mathcal{A}$ into $\mathcal{B}$ if $\langle a, b \rangle \in R$ implies $\langle h(a), h(b) \rangle \in S$.

A homomorphism h of $\mathcal{A}$ into $\mathcal{B}$ is called *strong* if for each $\langle x, y \rangle \in S$ there exist $a, b \in A$ such that $\langle a, b \rangle \in R$ and $h(a) = x, h(b) = y$. Let $\mathcal{A} = (A, \theta), \mathcal{B} = (B, \theta')$ be equivalence systems. A mapping $h : A \to B$ is called *class preserving* if $h([a]_\theta) = [h(a)]_{\theta'}$ for each $a \in A$.

Lemma 3 *Let $\mathcal{A} = (A, \theta)$, $\mathcal{B} = (B, \theta')$ be equivalence systems and $h : A \to B$ a surjective class preserving mapping. Then h is a strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$.*

Proof It is evident that $\langle a, b \rangle \in \theta$ implies $\langle h(a), h(b) \rangle \in \theta'$, i.e. it is a surjective homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Suppose $\langle c, d \rangle \in \theta'$. Then there is $a \in A$ with $h(a) = c$ and $d \in [c]_{\theta'} = [h(a)]_{\theta'}$. Hence, there exists $x \in [a]_\theta$ such that $h(x) = d$. Since $\langle a, x \rangle \in \theta$, h is a strong homomorphism. □

Example 1 The converse of Lemma 3 does not hold in general. Consider e.g. $\mathcal{A} = (A, \theta)$, $\mathcal{B} = (B, \theta')$ where $A = \{x_1, x_2, y_1, y_2, z_1, z_2\}$, $B = \{a, b, c\}$, $\theta' = B \times B$ and θ is determined by the partition $\{x_1, x_2\}, \{y_1, y_2\}, \{z_1, z_2\}$. Let $h : A \to B$ is defined as follows: $h(x_1) = h(y_1) = a$, $h(x_2) = h(z_1) = b$, $h(y_2) = h(z_2) = c$. Then h is a surjective strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$ but it is not a class preserving mapping; e.g. for x_1 we have

$$h([x_1]_\theta) = h(\{x_1, x_2\}) = \{a, b\} \neq \{a, b, c\} = [a]_{\theta'} = [h(x_1)]_{\theta'}.$$

Theorem 1 *Let $\mathcal{A} = (A, \theta)$, $\mathcal{B} = (B, \theta')$ be equivalence systems and $h : A \to B$ a surjective mapping. The following are equivalent:*

(a) h is a class preserving mapping;

(b) h is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$ and for each $x, y \in A$ with $\langle h(x), h(y)\rangle \in \theta'$ there exists $z \in A$ such that $\langle x, z\rangle \in \theta$ and $h(y) = h(z)$.

Proof (a) $\Rightarrow$ (b) by Lemma 3 and its proof. Prove (b) $\Rightarrow$ (a). Since h is a homomorphism, we easily get $h([a]_\theta) \subseteq [h(a)]_{\theta'}$. Suppose $c \in [h(a)]_{\theta'}$. Then $c = h(w)$ for some $w \in A$. By (b) there exists $z \in A$ such that $\langle a, z\rangle \in \theta$ and $h(z) = h(w) = c$. Since $z \in [a]_\theta$, we conclude $h([a]_\theta) = [h(a)]_{\theta'}$. □

Let $h : A \to B$ be a mapping. Denote by Φ_h the so-called *h-induced equivalence* on A, i.e.

$$\langle x, y\rangle \in \Phi_h \quad \text{if and only if} \quad h(x) = h(y).$$

Let Φ be an equivalence on A. Denote by h_Φ the so-called *natural mapping* $h_\Phi : A \to A/\Phi$ defined by $h_\Phi(a) = [a]_\Phi$.

Theorem 2 *Let $\mathcal{A} = (A, \theta)$ be an equivalence system and Φ be an equivalence on A. Suppose that θ, Φ permute. Then the natural mapping h_Φ is a class preserving mapping of $\mathcal{A}$ onto the quotient equivalence system $\mathcal{A}/\Phi = (A/\Phi, \theta/\Phi)$.*

Proof By Lemma 2 and the previous Remark, $\mathcal{A}/\Phi$ is clearly a quotient equivalence system. Of course, h_Φ is a surjective mapping. Suppose $\langle a, b\rangle \in \theta$. Then $\langle [a]_\Phi, [b]_\Phi\rangle \in \theta/\Phi$ thus h_Φ is a homomorphism of $\mathcal{A}$ onto $\mathcal{A}/\Phi$. Let $\langle [x]_\Phi, [y]_\Phi\rangle \in \theta/\Phi$. Then there exist $a \in [x]_\Phi, b \in [y]_\Phi$ such that $\langle a, b\rangle \in \theta$. Hence $\langle x, b\rangle \in \Phi \circ \theta = \theta \circ \Phi$, i.e. there exists $z \in A$ such that $\langle x, z\rangle \in \theta$ and $\langle z, b\rangle \in \Phi$, i.e. $h_\Phi(z) = h_\Phi(b)$. By (b) of Theorem 1, h_Φ is a class preserving mapping. □

Theorem 3 *Let $\mathcal{A} = (A, \theta), \mathcal{B} = (B, \theta')$ be equivalence systems and $h : A \to B$ a surjective strong homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Then h is a class preserving mapping if and only if θ and the h-induced equivalence Φ_h permute.*

Proof Let h be a class preserving mapping and suppose $\langle x, z\rangle \in \Phi_h \circ \theta$. Then there exists $y \in A$ with $\langle x, y\rangle \in \Phi_h$ and $\langle y, z\rangle \in \theta$. Thus $h(x) = h(y)$ and, as h is a homomorphism, $\langle h(x), h(z)\rangle \in \theta'$. By (b) of Theorem 1, there exists $u \in A$ with $\langle x, u\rangle \in \theta$ and $h(u) = h(z)$, i.e. $\langle u, z\rangle \in \Phi_h$. Hence $\langle x, z\rangle \in \theta \circ \Phi_h$ showing $\Phi_h \circ \theta \subseteq \theta \circ \Phi_h$. By Lemma 1, θ and Φ_h permute.

Conversely, let h be a surjective strong homomorphism and suppose $\theta \circ \Phi_h = \Phi_h \circ \theta$. Since h is a homomorphism we have $h([a]_\theta) \subseteq [h(a)]_{\theta'}$. Let $x \in [h(a)]_{\theta'}$. Then $\langle x, h(a)\rangle \in \theta'$. Since h is a strong homomorphism, there exist $b, c \in A$ such that $\langle b, c\rangle \in \theta$ and $h(b) = x, h(c) = h(a)$. Thus $\langle c, a\rangle \in \Phi_h$ and we have $\langle b, a\rangle \in \theta \circ \Phi_h = \Phi_h \circ \theta$. Hence, there exists $z \in A$ with $\langle b, z\rangle \in \Phi_h, \langle z, a\rangle \in \theta$. Thus $z \in [a]_\theta$ and $h(z) = h(b) = x$, i.e. h is a class preserving mapping. □

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