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TANGENCY RELATIONS FOR SETS IN SOME CLASSES IN GENERALIZED METRIC SPACES

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ABSTRACT. In this paper the compatibility and the equivalence problem of the tangency relations of sets of the classes $\tilde{M}_{p,k}$ and $A^*_{p,k}$ having the *Darboux* property in generalized metric space (E,l) is considered. Some sufficient conditions for the compatibility and the equivalence of the tangency relations are given here.

Introduction

Let (E,l) be a generalized metric space. E denotes here an arbitrary non-empty set and l is a non-negative real function defined on the *Cartesian* square $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let k be any, but fixed positive real number and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r\to 0^+]{} 0 \qquad \text{and} \qquad b(r) \xrightarrow[r\to 0^+]{} 0\,. \tag{1}$$

The *tangency relation* $T_l(a,b,k,p)$ of sets of the family E_0 in generalized metric space (E,l) is defined as follows (see [10]):

$$T_l(a,b,k,p) = \Big\{(A,B):\ A,B \in E_0\,,\ \text{the pair } (A,B) \text{ is } (a,b)\text{-clustered at the point } p \text{ of the space } (E,l) \text{ and } \frac{1}{r^k} l\Big(A \cap S_l(p,r)_{a(r)},\, B \cap S_l(p,r)_{b(r)}\Big) \xrightarrow[r\to 0^+]{} 0\Big\}\,. \tag{2}$$

If $(A,B) \in T_l(a,b,k,p)$, then we say that the set $A \in E_0$ is (a,b)*-tangent* (or briefly: is *tangent*) *of order* k to the set $B \in E_0$ at the point p of the space (E,l).

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The pair (A,B) of sets of the family E_0 is called (a,b)*-clustered* at the point p of the space (E,l) if 0 is the cluster point of the set of all numbers $r > 0$ such that

$$A \cap S_l(p,r)_{a(r)} \neq \emptyset \qquad \text{and} \qquad B \cap S_l(p,r)_{b(r)} \neq \emptyset . \tag{3}$$

The sets $S_l(p,r)_{a(r)}$ and $S_l(p,r)_{b(r)}$ denote here so-called $a(r)$-, $b(r)$-neighbourhoods of the sphere $S_l(p,r)$ with the centre at the point $p \in E$ and the radius $r > 0$ in the space (E,l).

Two tangency relations of sets $T_{l_1}(a_1,b_1,k,p)$, $T_{l_2}(a_2,b_2,k,p)$ are said to be *compatible* (see [4]), if $(A,B) \in T_{l_1}(a_1,b_1,k,p)$ if and only if $(A,B) \in T_{l_2}(a_2,b_2,k,p)$ for $(A,B) \in E_0$.

Let ρ be an arbitrary metric of the set E. We shall denote by $d_\rho A$ the diameter of the set $A \in E_0$, and by $\rho(A,B)$ the distance of sets $A,B \in E_0$ in the metric space (E,ρ).

Let f be any subadditive increasing real function defined in a certain right-hand side neighbourhood of 0, such that $f(0) = 0$. By $\overline{F}_f$ we denote the class of all functions l fulfilling the conditions:

1^0 $l\colon E_0 \times E_0 \to \langle 0, \infty)$,

2^0 $f\big(\rho(A,B)\big) \le l(A,B) \le f\big(d_\rho(A \cup B)\big)$ for $A,B \in E_0$.

It is easy to notice that every function $l \in \overline{F}_f$ generates on the set E the metric l_0 defined by the formula:

$$l_0(x,y) = l\big(\{x\},\{y\}\big) = f\big(\rho(x,y)\big) \qquad \text{for} \quad x,y \in E . \tag{4}$$

In [9], the problem of the compatibility for the tangency relations of sets in the classes $A^*_{p,k}$ and $\tilde{M}_{p,k}$ having the *Darboux* property at the point $p \in E$ for the functions l belonging to the class $F_f \subset \overline{F}_f$, where f is moreover a continuous function, was examined.

We say (see [7]) that the set $A \in E_0$ has the *Darboux* property at the point p of the space (E,l) and we shall write this as: $A \in D_p(E,l)$, if there exists a number $\tau > 0$ such that the set $A \cap S_l(p,r) \neq \emptyset$ for $r \in (0,\tau)$.

In this paper we shall consider the problem of the compatibility and the equivalence for the tangency relations of sets in the classes $\tilde{M}_{p,k}$ and $A^*_{p,k}$ having the *Darboux* property at the point p of the generalized metric spaces (E,l) for $l \in \overline{F}_f$. Some theorems (sufficient conditions) concerning the compatibility and the equivalence of the tangency relations will be given here.

1. On the tangency of sets in the classes $\tilde{M}_{p,k}$

Let ρ be a metric of the set E and A any set of the family E_0 of subsets of the set E. Let A' denote the set of all cluster points of the set $A \in E_0$ and

$$\rho(x, A) = \inf\{\rho(x,y) : \ y \in A\} \qquad \text{for} \quad x \in E. \tag{5}$$

The classes of sets $\tilde{M}_{p,k}$ mentioned in Introduction are defined as follows (see [5]):

$$\begin{aligned}\tilde{M}_{p,k} = \Big\{A \in E_0 : \ & p \in A' \text{ and there exists } \mu > 0 \text{ such that} \\ & \text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ & \text{for every pair of points } (x,y) \in [A, p; \mu, k] \\ & \quad \text{if } \rho(p,x) < \delta \text{ and } \tfrac{\rho(x,A)}{\rho^k(p,x)} < \delta, \text{ then } \tfrac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon\Big\},\end{aligned} \tag{6}$$

where

$$[A, p; \mu, k] = \{(x,y) : \ x \in E, \ y \in A \text{ and } \mu\rho(x,A) < \rho^k(p,x) = \rho^k(p,y)\}. \tag{7}$$

EXAMPLE 1. Let $E = \mathbb{R}^2$ be the two-dimensional Euclidean space. Let $A \subset E$ be a set of the form

$$A = \{(x,y) : \ x \geq 0, \ 0 \leq y \leq x^{k+1} \text{ and } k \geq 1\}. \tag{8}$$

We shall prove that A defined by the formula (8) is the set of the class $\tilde{M}_{p,k}$, where $p = (0,0)$ and $k \geq 1$. For this purpose let us denote

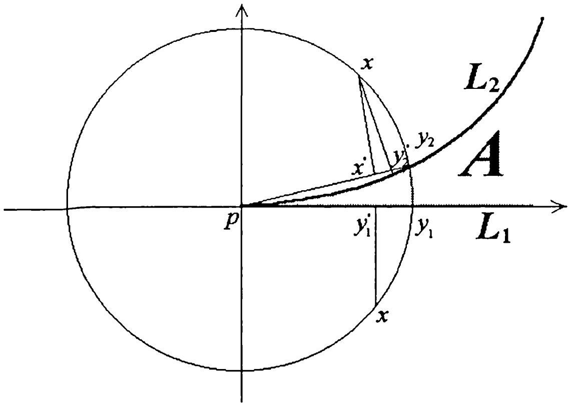
$$L_1 = \{(t,0) : \ t \geq 0\}, \qquad L_2 = \{(t, t^{k+1}) : \ t \geq 0\}. \tag{9}$$

Let y_1, y_2 be the points of the set A such that for $r > 0$

$$y_1 \in L_1 \cap S_\rho(p,r), \qquad y_2 \in L_2 \cap S_\rho(p,r). \tag{10}$$

If we denote $y_2 = (t, t^{k+1})$, then

$$r = \rho(p, y_2) = \sqrt{t^2 + t^{2k+2}} = t\sqrt{1 + t^{2k}}. \tag{11}$$



Hence it follows that $y_1 = \left(t\sqrt{1+t^{2k}}, 0\right)$. From (11) it results also that $r \to 0+$ if and only if $t \to 0+$. If we denote by $d_\rho A$ the diameter of the set A in the metric space (E, ρ), then (see [8])

$$\begin{aligned}
&\frac{1}{r^{2k}} d_\rho^2\big(A \cap S_\rho(p,r)\big) \\
&= \frac{1}{r^{2k}} \rho^2(y_1, y_2) \\
&= \frac{\left(t\sqrt{1+t^{2k}} - t\right)^2 + t^{2k+2}}{t^{2k}(1+t^{2k})^k} \\
&= \frac{1}{(1+t^{2k})^k}\left(\frac{\left(\sqrt{1+t^{2k}}-1\right)^2}{t^{2k-2}} + t^2\right) \xrightarrow[t\to 0+]{} \frac{\left(\sqrt{1+t^{2k}}-1\right)^2}{t^{2k-2}} \\
&= \frac{t^{4k}}{t^{2k-2}\left(\sqrt{1+t^{2k}}+1\right)^2} = \frac{t^{2k+2}}{\left(\sqrt{1+t^{2k}}+1\right)^2} \xrightarrow[t\to 0+]{} 0,
\end{aligned}$$

which means that

$$\frac{1}{r^k} d_\rho\big(A \cap S_\rho(p,r)\big) \xrightarrow[r\to 0+]{} 0. \tag{12}$$

Hence for an arbitrary $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$\frac{1}{r^k} d_\rho\big(A \cap S_\rho(p,r)\big) < \frac{\varepsilon}{2} \qquad \text{for} \quad 0 < r < \delta_1. \tag{13}$$

Now we shall prove that for an arbitrary $\varepsilon > 0$ there exists $\delta_2 > 0$ such that for every pair of points $(x, y_1) \in [L_1, p; \mu, k]$

$$\frac{\rho(x, y_1)}{\rho^k(p,x)} < \frac{\varepsilon}{2}, \tag{14}$$

when

$$r = \rho(p,x) < \delta_2 \qquad \text{and} \qquad \frac{\rho(x,L_1)}{\rho^k(p,x)} < \delta_2 \,. \tag{15}$$

Let y_1' be the projection of the point $x \in (E \setminus A)$ at the arc L_1, i.e. the point of the arc L_1 such that $\rho(x,y_1') = \rho(x,L_1)$.

Since $x = \left(t, \pm\sqrt{r^2-t^2}\right)$, for $0 \le t < r$ we have

$$\rho(y_1,y_1') = r - t = \sqrt{(r-t)^2} \le \sqrt{(r+t)(r-t)} = \sqrt{r^2-t^2} = \rho(x,y_1') \,. \tag{16}$$

Let $\mu = 2$, $\delta_2 = \min\left(\frac{1}{2}, \frac{\varepsilon}{4}\right)$. Hence, from (15), (16) and from the triangle inequality we obtain

$$\frac{\rho(x,y_1)}{\rho^k(p,x)} \le \frac{\rho(x,y_1') + \rho(y_1',y_1)}{\rho^k(p,x)} \le \frac{2\rho(x,L_1)}{\rho^k(p,x)} < \frac{\varepsilon}{2} \,,$$

which gives inequality (14).

Finally we prove that for an arbitrary $\varepsilon > 0$ there exists $\delta_3 > 0$ such that for every pair of points $(x,y_2) \in [L_2,p;\mu,k]$

$$\frac{\rho(x,y_2)}{\rho^k(p,x)} < \frac{\varepsilon}{2} \,, \tag{17}$$

when

$$r = \rho(p,x) < \delta_3 \qquad \text{and} \qquad \frac{\rho(x,L_2)}{\rho^k(p,x)} < \delta_3 \,. \tag{18}$$

If x' is the projection of the point $x \notin A$ on the segment $\overline{py_2}$, then from (16) it follows that for $0 \le t < r$

$$\rho(y_2,x') \le \rho(x,x') \,. \tag{19}$$

Let y_2' be the projection of the point $x \in (E \setminus A)$ at the arc L_2, i.e. the point of the arc L_2 such that $\rho(x,y_2') = \rho(x,L_2)$.

Since

$$\rho(x,x') < \rho(x,y_2') \,, \tag{20}$$

from (19) and from the triangle inequality we obtain

$$\rho(y_2,y_2') \le \rho(y_2,x') + \rho(x',x) + \rho(x,y_2') < 3\rho(x,y_2') = 3\rho(x,L_2) \,. \tag{21}$$

Let us put $\mu = 4$, $\delta_3 = \min\left(\frac{1}{4}, \frac{\varepsilon}{8}\right)$. From here and from (21) we have

$$\frac{\rho(x,y_2)}{\rho^k(p,x)} \le \frac{\rho(x,y_2') + \rho(y_2',y_2)}{\rho^k(p,x)} < \frac{4\rho(x,L_2)}{\rho^k(p,x)} < \frac{\varepsilon}{2} \,,$$

which as a consequence gives (17).

Let $\mu = 4$, $\delta = \min(\delta_1,\delta_2,\delta_3)$ and let (x,y) be any pair of points belonging to the class $[A,p;\mu,k]$. In this example $\rho(x,A) = \rho(x,L_1)$ or $\rho(x,A) = \rho(x,L_2)$, when $x \notin A$.

Let us suppose that $\rho(x, A) = \rho(x, L_1)$. Hence, from the triangle inequality and from (13), (14) it follows that for $(x, y) \in [A, p; \mu, k]$ if

$$r = \rho(p, x) < \delta \qquad \text{and} \qquad \frac{\rho(x, A)}{\rho^k(p, x)} < \delta, \tag{22}$$

then

$$\frac{\rho(x, y)}{\rho^k(p, x)} \leq \frac{\rho(x, y_1)}{\rho^k(p, x)} + \frac{\rho(y_1, y)}{\rho^k(p, x)} \leq \frac{\rho(x, y_1)}{\rho^k(p, x)} + \frac{1}{r^k} d_\rho\big(A \cap S_\rho(p, r)\big) < \varepsilon. \tag{23}$$

Similarly, if $\rho(x, A) = \rho(x, L_2)$, then using (13) and (17) for $(x, y) \in [A, p; \mu, k]$ we get that

$$\frac{\rho(x, y)}{\rho^k(p, x)} \leq \frac{\rho(x, y_2)}{\rho^k(p, x)} + \frac{1}{r^k} d_\rho\big(A \cap S_\rho(p, r)\big) < \varepsilon. \tag{24}$$

If $x \in E$ is a point of the set $A \subset E$, then from (13) it follows immediately that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every pair of points $(x, y) \in [A, p; \mu, k]$ (for an arbitrary fixed number $\mu > 0$)

$$\frac{\rho(x, y)}{\rho^k(p, x)} < \frac{1}{r^k} d_\rho\big(A \cap S_\rho(p, r)\big) < \varepsilon, \tag{25}$$

when

$$r = \rho(p, x) < \delta \qquad \text{and} \qquad \frac{\rho(x, A)}{\rho^k(p, x)} < \delta. \tag{26}$$

Hence, from (23) and (24) it follows that the set A belongs to the class $\tilde{M}_{p,k}$ defined by (6).

From the definition of the set A it follows evidently that $A \in D_p(E, \rho)$.

Let l be an arbitrary function of the class $\overline{F}_f$. From (4) and from the properties of the function f it follows

$$\begin{aligned} f(d_\rho A) &= f\big(\sup\{\rho(x, y) :\ x, y \in A\}\big) = \sup\big\{f\big(\rho(x, y)\big) :\ x, y \in A\big\} \\ &= \sup\big\{l_0(x, y) :\ x, y \in A\big\} = d_l A, \end{aligned}$$

therefore

$$f(d_\rho A) = d_l A \qquad \text{for} \quad A \in E_0. \tag{27}$$

Let a_i, b_i $(i = 1, 2)$ be any non-negative real functions defined in a certain right-hand side neighbourhood of 0 and fulfilling the condition

$$a_i(r) \xrightarrow[r \to 0^+]{} 0 \qquad \text{and} \qquad b_i(r) \xrightarrow[r \to 0^+]{} 0. \tag{28}$$

Let us denote

$$\breve{a} = \max(a_1, a_2), \qquad \breve{b} = \max(b_1, b_2). \tag{29}$$

THEOREM 1. *If* $l \in \overline{F}_f$ *and*

$$\frac{a_i(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \alpha_i\,, \qquad \frac{b_i(r)}{r^{k+1}} \xrightarrow[r\to 0+]{} \beta_i\,, \tag{30}$$

where $\alpha_i, \beta_i < \infty$ *for* $i = 1, 2$, *then the tangency relations* $T_l(a_1, b_1, k, p)$ *and* $T_l(a_2, b_2, k, p)$ *are compatible in the classes of sets* $\tilde{M}_{p,k} \cap D_p(E, l)$.

P r o o f. Let us assume that $(A, B) \in T_l(a_1, b_1, k, p)$ for $A, B \in \tilde{M}_{p,k} \cap D_p(E, l)$. Then we have

$$\frac{1}{r^k}\, l\Big(A \cap S_l(p,r)_{a_1(r)},\, B \cap S_l(p,r)_{b_1(r)}\Big) \xrightarrow[r\to 0+]{} 0\,. \tag{31}$$

From the inequality

$$d_\rho(A \cup B) \le d_\rho A + d_\rho B + \rho(A, B) \qquad \text{for} \quad A, B \in E_0\,, \tag{32}$$

from (29), from the properties of the function f and from the fact that $l \in \overline{F}_f$ we obtain

$$\begin{aligned}
&\Big|\frac{1}{r^k}\, l\Big(A \cap S_l(p,r)_{a_2(r)},\, B \cap S_l(p,r)_{b_2(r)}\Big) \\
&\qquad - \frac{1}{r^k}\, l\Big(A \cap S_l(p,r)_{a_1(r)},\, B \cap S_l(p,r)_{b_1(r)}\Big)\Big| \\
&\le \frac{1}{r^k} f\Big(d_\rho\Big(\big(A \cap S_l(p,r)_{a_2(r)}\big) \cup \big(B \cap S_l(p,r)_{b_2(r)}\big)\Big)\Big) \\
&\qquad - \frac{1}{r^k} f\Big(\rho\big(A \cap S_l(p,r)_{a_1(r)},\, B \cap S_l(p,r)_{b_1(r)}\big)\Big) \\
&\le \frac{1}{r^k} f\Big(d_\rho\Big(\big(A \cap S_l(p,r)_{\breve{a}(r)}\big) \cup \big(B \cap S_l(p,r)_{\breve{b}(r)}\big)\Big)\Big) \\
&\qquad - \frac{1}{r^k} f\Big(\rho\big(A \cap S_l(p,r)_{\breve{a}(r)},\, B \cap S_l(p,r)_{\breve{b}(r)}\big)\Big) \\
&\le \frac{1}{r^k} f\Big(d_\rho\big(A \cap S_l(p,r)_{\breve{a}(r)}\big) + d_\rho\big(B \cap S_l(p,r)_{\breve{b}(r)}\big) \\
&\qquad\qquad + \rho\big(A \cap S_l(p,r)_{\breve{a}(r)},\, B \cap S_l(p,r)_{\breve{b}(r)}\big)\Big) \\
&\qquad - \frac{1}{r^k} f\Big(\rho\big(A \cap S_l(p,r)_{\breve{a}(r)},\, B \cap S_l(p,r)_{\breve{b}(r)}\big)\Big) \\
&\le \frac{1}{r^k} f\Big(d_\rho\big(A \cap S_l(p,r)_{\breve{a}(r)}\big)\Big) + \frac{1}{r^k} f\Big(d_\rho\big(B \cap S_l(p,r)_{\breve{b}(r)}\big)\Big)\,.
\end{aligned} \tag{33}$$

From (29), (30) and from [5; Lemma 1.1] it follows

$$\frac{1}{r^k}\, d_l\big(A \cap S_l(p,r)_{\breve{a}(r)}\big) \xrightarrow[r\to 0+]{} 0\,. \tag{34}$$

Hence (27) implies

$$\frac{1}{r^k} f\big(d_\rho(A \cap S_l(p,r)_{\check{a}(r)})\big) \xrightarrow[r\to 0^+]{} 0 \,. \tag{35}$$

Analogously

$$\frac{1}{r^k} f\big(d_\rho(B \cap S_l(p,r)_{\check{b}(r)})\big) \xrightarrow[r\to 0^+]{} 0 \,. \tag{36}$$

From (31), (35), (36) and from the inequality (33) we have

$$\frac{1}{r^k} l\Big(A \cap S_l(p,r)_{a_2(r)},\, B \cap S_l(p,r)_{b_2(r)}\Big) \xrightarrow[r\to 0^+]{} 0 \,. \tag{37}$$

From the fact that the sets $A, B \in D_p(E,l)$ it follows that the pair of sets (A,B) is (a,b)-clustered at the point p of the space (E,l). Hence from (37) we obtain $(A,B) \in T_l(a_2, b_2, k, p)$.

If $(A,B) \in T_l(a_2, b_2, k, p)$, then identically we prove that $(A,B) \in T_l(a_1, b_1, k, p)$. Therefore the tangency relations $T_l(a_1, b_1, k, p)$ and $T_l(a_2, b_2, k, p)$ are compatible in the classes of sets $\tilde{M}_{p,k} \cap D_p(E,l)$. □

Using this theorem we shall prove:

THEOREM 2. *If $l \in \overline{F}_f$,*

$$\frac{a(r)}{r^{k+1}} \xrightarrow[r\to 0^+]{} \alpha \qquad \text{and} \qquad \frac{b(r)}{r^{k+1}} \xrightarrow[r\to 0^+]{} \beta \,, \tag{38}$$

where $\alpha, \beta < \infty$, then the tangency relation $T_l(a,b,k,p)$ is an equivalence in the classes of sets $\tilde{M}_{p,k} \cap D_p(E,l)$.

P r o o f . From [5; Lemma 1.1] and from the assumptions of this theorem it follows that for $A \in \tilde{M}_{p,k} \cap D_p(E,l)$

$$\frac{1}{r^k} d_l\big(A \cap S_l(p,r)_{a(r)}\big) \xrightarrow[r\to 0^+]{} 0 \,, \tag{39}$$

and

$$\frac{1}{r^k} d_l\big(A \cap S_l(p,r)_{b(r)}\big) \xrightarrow[r\to 0^+]{} 0 \,. \tag{40}$$

Since

$$\rho\Big(A \cap S_l(p,r)_{a(r)},\, A \cap S_l(p,r)_{b(r)}\Big) = 0 \qquad \text{for} \qquad A \in E_0 \,, \tag{41}$$

then from here, from (27) and (32), from the properties of the function f, and from the fact that $l \in \overline{F}_f$ we have

$$\begin{aligned}
0 &\le l\Big(A \cap S_l(p,r)_{a(r)},\, A \cap S_l(p,r)_{b(r)}\Big) \\
&\le f\Big(d_\rho\Big(\big(A \cap S_l(p,r)_{a(r)}\big) \cup \big(A \cap S_l(p,r)_{b(r)}\big)\Big)\Big) \\
&\le f\Big(d_\rho\big(A \cap S_l(p,r)_{a(r)}\big)\Big) + f\Big(d_\rho\big(A \cap S_l(p,r)_{b(r)}\big)\Big) \\
&= d_l\big(A \cap S_l(p,r)_{a(r)}\big) + d_l\big(A \cap S_l(p,r)_{b(r)}\big) \,.
\end{aligned}$$

Hence (39) and (40) imply

$$\frac{1}{r^k} l\Big(A \cap S_l(p,r)_{a(r)}, A \cap S_l(p,r)_{b(r)}\Big) \xrightarrow[r\to 0^+]{} 0. \tag{42}$$

Since $A \in D_p(E,l)$, the pair of sets (A,A) is (a,b)-clustered at the point p of the space (E,l). Hence from (42) it follows $(A,A) \in T_l(a,b,k,p)$, which means that the tangency relation $T_l(a,b,k,p)$ is reflexive in the classes of sets $\tilde{M}_{p,k} \cap D_p(E,l)$.

Let us assume now that $(A,B) \in T_l(a,b,k,p)$ for $A,B \in \tilde{M}_{p,k} \cap D_p(E,l)$. From here and from Theorem 1 it follows that $(A,B) \in T_l(b,a,k,p)$. Hence

$$\frac{1}{r^k} l\Big(A \cap S_l(p,r)_{b(r)}, B \cap S_l(p,r)_{a(r)}\Big) \xrightarrow[r\to 0^+]{} 0. \tag{43}$$

From (27), from the inequality (32), and from the fact that $l \in \overline{F}_f$ we get

$$\begin{aligned}
0 &\le l\Big(B \cap S_l(p,r)_{a(r)}, A \cap S_l(p,r)_{b(r)}\Big)\\
&\le f\Big(d_\rho\Big((B \cap S_l(p,r)_{a(r)}) \cup (A \cap S_l(p,r)_{b(r)})\Big)\Big)\\
&\le f\Big(d_\rho(A \cap S_l(p,r)_{b(r)})\Big) + f\Big(d_\rho(B \cap S_l(p,r)_{a(r)})\Big)\\
&\quad + f\Big(\rho(A \cap S_l(p,r)_{b(r)}, B \cap S_l(p,r)_{a(r)})\Big)\\
&\le d_l(A \cap S_l(p,r)_{b(r)}) + d_l(B \cap S_l(p,r)_{a(r)})\\
&\quad + l\Big(A \cap S_l(p,r)_{b(r)}, B \cap S_l(p,r)_{a(r)}\Big).
\end{aligned}$$

From here, from (43) and from [5; Lemma 1.1] one derives

$$\frac{1}{r^k} l\Big(B \cap S_l(p,r)_{a(r)}, A \cap S_l(p,r)_{b(r)}\Big) \xrightarrow[r\to 0^+]{} 0. \tag{44}$$

Since by our assumption $A,B \in D_p(E,l)$, the pair of sets (B,A) is (a,b)-clustered at the point p of the space (E,l). Hence (44) gives $(B,A) \in T_l(a,b,k,p)$, which means that $T_l(a,b,k,p)$ is a symmetric relation in the classes of sets $\tilde{M}_{p,k} \cap D_p(E,l)$.

Finally we assume that $(A,B) \in T_l(a,b,k,p)$ and $(B,C) \in T_l(a,b,k,p)$ for the sets $A,B,C \in \tilde{M}_{p,k} \cap D_p(E,l)$. Hence

$$\frac{1}{r^k} l\Big(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}\Big) \xrightarrow[r\to 0^+]{} 0, \tag{45}$$

and

$$\frac{1}{r^k} l\Big(B \cap S_l(p,r)_{a(r)}, C \cap S_l(p,r)_{b(r)}\Big) \xrightarrow[r\to 0^+]{} 0. \tag{46}$$

Then Theorem 1 yields

$$\frac{1}{r^k} l\Big(B \cap S_l(p,r)_{b(r)},\, C \cap S_l(p,r)_{b(r)}\Big) \xrightarrow[r\to 0+]{} 0\,. \tag{47}$$

From (27), (41) and (32), from the properties of the function f, and from the fact that $l \in \overline{F}_f$ we obtain

$$\begin{aligned}
0 &\le l\Big(A \cap S_l(p,r)_{a(r)},\, C \cap S_l(p,r)_{b(r)}\Big)\\
&\le f\Big(d_\rho\Big((A \cap S_l(p,r)_{a(r)}) \cup (C \cap S_l(p,r)_{b(r)})\Big)\Big)\\
&\le f\Big(d_\rho\Big(((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}))\\
&\qquad\qquad \cup ((B \cap S_l(p,r)_{b(r)}) \cup (C \cap S_l(p,r)_{b(r)}))\Big)\Big)\\
&\le f\Big(d_\rho\Big((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)})\Big)\\
&\qquad\qquad + d_\rho\Big((B \cap S_l(p,r)_{b(r)}) \cup (C \cap S_l(p,r)_{b(r)})\Big)\Big)\\
&\le f\Big(d_\rho(A \cap S_l(p,r)_{a(r)})\Big) + f\Big(d_\rho(B \cap S_l(p,r)_{b(r)})\Big)\\
&\quad + f\Big(\rho(A \cap S_l(p,r)_{a(r)},\, B \cap S_l(p,r)_{b(r)})\Big)\\
&\quad + f\Big(d_\rho(B \cap S_l(p,r)_{b(r)})\Big) + f\Big(d_\rho(C \cap S_l(p,r)_{b(r)})\Big)\\
&\quad + f\Big(\rho(B \cap S_l(p,r)_{b(r)},\, C \cap S_l(p,r)_{b(r)})\Big)\\
&\le d_l(A \cap S_l(p,r)_{a(r)}) + 2d_l(B \cap S_l(p,r)_{b(r)}) + d_l(C \cap S_l(p,r)_{b(r)})\\
&\quad + l(A \cap S_l(p,r)_{a(r)},\, B \cap S_l(p,r)_{b(r)}) + l(B \cap S_l(p,r)_{b(r)},\, C \cap S_l(p,r)_{b(r)})\,.
\end{aligned}$$

From here, from [5; Lemma 1.1] and from (45), (47) it follows that

$$\frac{1}{r^k} l\Big(A \cap S_l(p,r)_{a(r)},\, C \cap S_l(p,r)_{b(r)}\Big) \xrightarrow[r\to 0+]{} 0\,. \tag{48}$$

From the fact $A, C \in D_p(E,l)$ we obtain that the pair of sets (A,C) is (a,b)-clustered at the point p of the space (E,l). Hence and from (48) it follows that $(A,C) \in T_l(a,b,k,p)$, in other words, the tangency relation $T_l(a,b,k,p)$ is transitive in the classes of sets $\tilde{M}_{p,k} \cap D_p(E,l)$.

From the above considerations one sees that $T_l(a,b,k,p)$ is an equivalence relation in the classes of sets $\tilde{M}_{p,k} \cap D_p(E,l)$. This ends the proof. □

2. Some remarks on the tangency of sets in the classes $A^*_{p,k}$

Similarly as in Section 1, let ρ be any metric of the set E. By A' we denote the set of all cluster points of the set A of the family E_0 of all non-empty subsets of the set E.

Let us assume by the definition that for an arbitrary but fixed number $k > 0$ (see [4]):

$$A^*_{p,k} = \Big\{A \in E_0 : \ p \in A' \text{ and there exists a number } \lambda > 0 \text{ such that} \limsup_{[A,p;k]\ni(x,y)\to(p,p)} \tfrac{\rho(x,y)-\lambda\rho(x,A)}{\rho^k(p,x)} \le 0\Big\}, \tag{49}$$

where

$$[A,p;k] = \{(x,y) : \ x \in E\,, \ \ y \in A \text{ and } \rho(x,A) < \rho^k(p,x) = \rho^k(p,y)\}\,. \tag{50}$$

Analogously as Theorem 1 we can prove:

THEOREM 3. *If $l \in \overline{F}_f$ and for $i = 1,2$*

$$\frac{a_i(r)}{r^k} \xrightarrow[r\to 0+]{} 0\,, \qquad \frac{b_i(r)}{r^k} \xrightarrow[r\to 0+]{} 0\,, \tag{51}$$

then for arbitrary sets of the classes $A^*_{p,k} \cap D_p(E,l)$ the tangency relations $T_l(a_1,b_1,k,p)$ and $T_l(a_2,b_2,k,p)$ are compatible.

The proof of this theorem is based on the inequality (33) and on the following lemma (see [4]):

LEMMA 1. *If*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0+]{} 0\,, \tag{52}$$

then for an arbitrary set $A \in A^*_{p,k} \cap D_p(E,l)$

$$\frac{1}{r^k} d_l\big(A \cap S_l(p,r)_{a(r)}\big) \xrightarrow[r\to 0+]{} 0\,. \tag{53}$$

It turns out that if the functions a, b fulfil the condition

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0+]{} 0\,, \qquad \frac{b(r)}{r^k} \xrightarrow[r\to 0+]{} 0\,, \tag{54}$$

then for an arbitrary function $l \in \overline{F}_f$ Theorem 2 will be true in the classes of sets $A^*_{p,k} \cap D_p(E,l)$.

Analogously as in case of Theorem 2, using Theorem 3 and Lemma 1 we can prove the following theorem:

THEOREM 4. *If the functions* a, b *fulfil the condition* (54), *then for an arbitrary function* $l \in \overline{F}_f$ *the tangency relation* $T_l(a, b, k, p)$ *is an equivalence in the classes of sets* $A^*_{p,k} \cap D_p(E, l)$.

This theorem is also fulfilled under somewhat weakened assumptions concerning the functions a, b, which follows from the fact (see [5; Theorem 1.1]) that the classes of sets $A^*_{p,k}$ are contained in the classes $\tilde{M}_{p,k}$ for any $k > 0$ and $p \in E$.

If we put $f = \mathrm{id}$, where id denotes the identity function defined in a right-hand side neighbourhood of 0, then the class $\overline{F}_{\mathrm{id}}$ of the functions l is equal to the class F^*_ρ considered in some papers of mine mentioned in References below. From here it results that all theorems about the problem of the compatibility and the equivalence for the tangency relations of sets for the functions of the class F^*_ρ given in these papers follow from the theorems of the present paper.

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