Mathematica Slovaca

Jozef Tvarožek

S-cubes

Mathematica Slovaca, Vol. 36 (1986), No. 1, 55--68

Persistent URL: http://dml.cz/dmlcz/132866

Math. Slovaca 36, 1986, No. 1, 55—68

S-cubes

JOZEF TVAROŽEK

Introduction

Let I^n be the n-dimensional cube and J_i^n its i-th "double face". Let s_i: $\partial I^n \to \partial I^n$ be the symmetry of ∂I^n with respect to the hyperplane $x_i = 0$. A group of transformations of ∂I^n generated by the set $\{s_1, \ldots, s_n\}$ will be denoted by G. To each n-touple $(u^1, \ldots, u^n) \in G^n$ we assign a factorspace as follows: Let S be the binary relation on I^n defined via

$$xSy \Leftrightarrow x = y \text{ or there is an index } i \in \{1, 2, \ldots, n\}$$
$$\text{such that } x, y \in J_i^n \text{ and } x = u^i(y).$$

The space I^n/T, where T is the least equivalence relation on I^n containing S, will be denoted by $I^n/(u^1, \ldots, u^n)$ and called an *s-cube*.

The aim of this paper is:

1) To prove some basic properties of s-cubes (part 1).

2) To discuss some special types of s-cubes and the irreducibility of s-cubes (part 2).

3) To give a necessary and a sufficient condition for an s-cube to be a manifold (part 3).

Notation

$N_n = \{1, 2, \ldots, n\}$, $N_0 = \emptyset$

$M^{[r]} = \{x - r; x \in M\}$ where $M \subset N_n - N_r$ is a nonempty given set

$I^n = \{x \in R^n; |x_i| \leqq 1, i \in N_n\}$ an n-dimensional cube

$\partial I^n =$ the boundary of I^n

$S^n = \{x \in R^{n+1}; \sqrt{(x_1^2 + x_2^2 + \ldots + x_{n+1}^2)} = 1\}$ an n-dimensional sphere

$J_i^n = \{x \in I^n; |x_i| = 1\}$ (briefly J_i) the i-th "double-face" of the cube I^n

CX, S^kX a cone and a k-fold suspension over a topological space X

s_i: $\partial I^n \to \partial I^n$, $x \mapsto (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$ the symmetry of ∂I^n with respect to the hyperplane $x_i = 0$, $i \in N_n$

G the subgroup of the group of all transformations of ∂I^n generated by the set $\{s_i; i \in N_n\}$.

The group G is abelian, because $G \cong (Z_2)^n$. Each $u \in G$, $u \neq id$, is the product of mutually different transformations $s_{i_1}, \ldots, s_{i_k}$ and may be uniquely written in the form

$$s_{i_1 i_2 \ldots i_k} = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_k}, \text{ where } i_1 < i_2 < \ldots < i_k.$$

Since to every $u \in G$, $u = s_{i_1 \ldots i_k}$, there corresponds a unique subset $\{i_1, \ldots, i_k\} \in 2^{N_n}$, there is a bijective map

$$\tau: G \to 2^{N_n}, \ \tau(s_{i_1 \ldots i_k}) = \{i_1, \ldots, i_k\}, \ \tau(id) = \emptyset.$$

1. Basic properties of s-cubes

We start with an adapted definition of the s-cube since that given in the Introduction is not suitable for future proofs.

Definition 1.1. *Let $u^1, \ldots, u^n \in G$. An s-cube $I^n/(u^1, \ldots, u^n)$ is a factorspace I^n/T, where T is an equivalence relation on I^n defined as follows:*

xTy if $x = y$ or there are numbers $i_1, \ldots, i_k \in N_n$ such that $x, y \in \bigcap_{j=1}^{k} J_{i_j}$ and $x = u^{i_1} \circ u^{i_2} \circ \ldots \circ u^{i_k}(y)$.

To simplify the notation, any given s-cube $I^n/(u^1, \ldots, u^n)$ will be alternatively written in the form $I^n/(U_1, \ldots, U_n)$, where $U_i = \tau(u^i)$, $i \in N_n$.

Now we give the basic information about the general properties of s-cubes.

Proposition 1.2. *Every s-cube is a Hausdorff space.*

Proposition 1.3. *Let $I^n/(U_1, \ldots, U_n)$ be an s-cube, $f: N_n \to N_n$ a bijection and $F: I^n \to I^n$, $F(x) = (x_{f(1)}, \ldots, x_{f(n)})$. Then there is a map $\tilde{F}: I^n/(U_1, \ldots, U_n) \to I^n/(f(U_{f^{-1}(1)}), \ldots, f(U_{f^{-1}(n)}))$, $[x] \mapsto [F(x)]$, which is a homeomorphism.*

Lemma 1.4. *Let $k, r \in N_n$, $k \neq r$ and let $I^n/(u^1, \ldots, u^n)$ be such an s-cube that $u^r = s_k$. Then $I^n/(u^1, \ldots, u^n) \approx I^n/(v^1, \ldots, v^n)$, where $v^i = u^i$ for $i \neq r$ and $v^r = u^k$.*

Proof. Without loss of generality we can suppose (see Prop. 1.3.) that $r = 1$, $k = 2$. We find the homeomorphism $I^n/(s_2, u^2, \ldots, u^n) \approx I^n/(u^2, u^2, u^3, \ldots, u^n)$ first in the case of $n = 2$.

Let us denote $A = (-2, 0)$, $B = (2, 0)$, $S = (0, 0)$, $A_1 = (-1, -1)$, $A_2 = (1, -1)$, $A_3 = (1, 1)$, $A_4 = (-1, 1)$, $B_1 = (0, -1)$, $B_2 = (1, 0)$, $B_3 = (0, 1)$, $B_4 = (-1, 0)$, $S_i = \frac{1}{2}(A_i - S)$, $i \in N_4$. Now we define three PL-maps f_1, f_2, f_3:

f_1 maps the square $A_1A_2A_3A_4 \equiv I^2$ on the deltoid AB_1BB_3: it is the identity on the square $B_1B_2B_3B_4$, it is linear on the triangles $A_1B_1B_4$, $A_2B_2B_1$, $A_3B_3B_2$, $A_4B_4B_3$ and $f_1(A_1) = f_1(A_4) = A$, $f_1(A_2) = f_1(A_3) = B$.

f_2 maps the deltoid AB_1BB_3 on the square $B_1B_2B_3B_4$: it is the identity on the segment B_1B_3, it is linear on the trianles B_1B_3A, B_1B_3B and $f_2(A) = B_4$, $f_2(B) = B_2$.

f_3 maps the square $B_1B_2B_3B_4$ on the square $A_1A_2A_3A_4$: it is the identity on the segments B_1B_3, B_2B_4, it is linear on the triangles B_1SB_2, B_2SB_3, B_3SB_4, B_4SB_1 and $f_3(S_i)=A_i$, $i\in N_4$.

Now we define a map $F_2\colon I^2\to I^2$, $F_2=f_3\circ f_2\circ f_1$. The induced map $\tilde{F}_2\colon I^2/ /(s_2, u^2)\to I^2/(u^2, u^2)$, $[x]\mapsto[F_2(x)]$, is a homeomorphism. Thus the assertion is proved for $n=2$. This result can be extended to the general case via the cartesian product; after a tedious computation it is possible to show that the map $\tilde{F}_n\colon I^n/ /(u^1, \dots, u^n)\to I^n/(v^1, \dots, v^n)$, induced by the map $F_n=F_2\times(id)^{n-2}$, is the demanded homeomorphism $n\geqq 2$.

Proposition 1.5. *Let* $n, r\in N$, $1\leqq r<n$ and let $I^n/(U_1, \dots, U_n)$ *be an s-cube such that*

1) $U_i\subset N_r$ *for* $i\in N_r$,
2) $U_i\subset N_n-N_r$ *for* $i\in N_n-N_r$.

Then the map $h\colon I^n/(U_1, \dots, U_n)\to I^r/(U_1, \dots, U_r)\times I^{n-r}/(U_{r+1}^{[r]}, \dots, U_n^{[r]})$, $[x]\mapsto([(x_1, \dots, x_r)], [(x_{r+1}, \dots, x_n)])$, is a homeomorphism.

Proof. Denote s-cubes $I^n/(U_1, \dots, U_n)$, $I^r/(U_1, \dots, U_r)$, $I^{n-r}/(U_{r+1}^{[r]}, \dots, U_n^{[r]})$ by I^n/T, I^r/T_1, I^{n-r}/T_2, respectively. It is not difficult to show that $T=T_1\times T_2$. Since s-cubes are compact Hausdorff spaces, the map h is a homeomorphism.

Example 1.6. Applying Lemma 1.4 and Proposition 1.5 to the s-cube $X=I^8/ /(s_2, s_1, s_3, s_{34}, s_6, s_{56}, s_7, s_8)$ we get:

$$X\approx I^8/(s_1, s_1, s_3, s_{34}, s_{56}, s_{56}, s_7, s_8)\approx$$
$$\approx I^2/(s_1, s_1)\times I^2/(s_1, s_{12})\times I^2/(s_{12}, s_{12})\times I/(s_1)\times I/(s_1)\approx$$
$$\approx S^2\times Kb\times RP^2\times S^1\times S^1$$

where Kb is the Klein bottle and RP^2 is the real projective plane.

Remark 1.7. Proposition 1.5 enables to represent any finite product of s-cubes as an s-cube. In [2] and [3] it was shown that $I^n/(s_1, \dots, s_1)\approx S^n$, $I^n/(s_{12\dots n}, \dots, s_{12\dots n})\approx RP^n$ and $I^n/(s_{1\dots n-k}, \dots, s_{1\dots n-k})\approx S^kRP^{n-k}$. Making use of these results we get immediatelly that every finite product of spheres, real projective spaces and their suspensions can be represented as an s-cube.

2. Special types of s-cubes

In Example 1.6 we have seen an s-cube which was homeomorphic to a product of several s-cubes of lower dimensions. Such decompositions of x-cubes will now be introduced.

Let $U_1, \dots, U_n$ be given subsets of N_n. Define a binary relation $R(U_1, \dots, U_n)$ on N_n via

$$xRy\Leftrightarrow(x=y)\vee(x\in U_y)\vee(y\in U_x)\vee(\exists s\in N_n\colon x, y\in U_s)$$

The least transitive relation on N_n containing $R(U_1, \ldots, U_n)$ is an equivalence relation and will be denoted by $E(U_1, \ldots, U_n)$, briefly E.

Definition 2.1. *An s-cube $I^n/(U_1, \ldots, U_n)$ is said to be combinatorially irreducible (c-irreducible) if $N_n/E(U_1, \ldots, U_n)$ consists of exactly one equivalence class, otherwise X is said to be combinatorially reducible (c-reducible).*

Example 2.2. An s-cube $I^n/(s_1, s_2, \ldots, s_n)$ is c-reducible for $n>1$, s-cubes $I^n/(s_1, \ldots, s_1)$ and $I^n/(s_{12\ldots n}, \ldots, s_{12\ldots n})$ are c-irreducible.

Theorem 2.3. *Every c-reducible s-cube is homeomoprhic to a product of c-irreducible s-cubes.*

Proof. For a given c-reducible s-cube $I^n/(U_1, \ldots, U_n)$ denote $N_n/$ $/E(U_1, \ldots, U_n)=\{A_1, \ldots, A_q\}$, $c_i=\text{card } A_i$, $i \in N_q$, $q \geqq 2$. Let $h\colon N_n \to N_n$ be a bijection such that $A_1=\{h(1), \ldots, h(t_1)\}$, $A_2=\{h(t_1+1), \ldots, h(t_2)\}$, $\ldots$, $A_q=\{h(t_{q-1}+1, \ldots, h(t_q)\}$, where $t_i=c_1+\ldots+c_i$, $i \in N_q$. Using Proposition 1.3 for $f=h^{-1}$ we get the homeomorphism $I^n/(I_1, \ldots, U_n) \approx I^n/(h^{-1}(U_{h(1)}), \ldots, h^{-1}(U_{h(n)}))$. To complete the proof it is sufficient to apply $(q-1)$-times Proposition 1.5.

A c-irreducible s-cube need not to be irreducible. For example, an s-cube $X=I^3/(s_1, s_1, s_{123})$ is c-irreducible, but $X \approx I^2/(s_1, s_1) \times I/(s_1)$.

Definition 2.4. *An s-cube $I^n/(u^1, \ldots, u^n)$ is quasi-regular if there are not $i, j \in N_n$, $i \neq j$, such that $u^i=s_j$ and card $U_j>1$. An s-cube $I^n/(u^1, \ldots, u^n)$ is regular if for every $i, j \in N_n$ $u^i=s_j$ implies $u^j=s_j$. Regular s-cubes are called briefly r-cubes.*

Lemma 2.5. *Let $X=I^n/(u^1, \ldots, u^n)$ be an s-cube. Suppose that there are $i_1, \ldots, i_t \in N_n$ such that $u^{i_1}=s_{i_2}$, $u^{i_2}=s_{i_3}$, $\ldots$, $u^{i_{t-1}}=s_{i_t}$. Then $I^n/(u^1, \ldots, u^n) \approx I^n/(v^1, \ldots, v^n)$, where $v^{i_1}=v^{i_2}=\ldots=v^{i_t}=u^{i_t}$ and $v^i=u^i$ otherwise.*

Proof. By repeated application of Lemma 1.4 for $r=i_{t-1}, i_{t-2}, \ldots, i_1$ we get the homeomorphism f, $f\colon X \overset{\approx}{\to} X_1 \overset{\approx}{\to} \ldots \overset{\approx}{\to} X_{t-1}$, $X_j=I^n/(u^1_{(j)}, \ldots, u^n_{(j)})$, where $u^{i_k}_{(j)}=u^{i_t}$ for $k=t-j, t-j+1, \ldots, t-1$ and $u^i_{(j)}=u^i$ otherwise, $j=1, \ldots, t-1$. For $j=t-1$ we have $u^{i_1}_{(t-1)}=u^{i_2}_{(t-1)}=\ldots=u^{i_t}_{(t-1)}=u^{i_t}$, $u^i_{(t-1)}=u^i$ for $i \neq i_1, i_2, \ldots, i_t$.

Let $I^n/U_1, \ldots, U_n)$ be an s-cube. Let $\bar{U}_j=\{x \in N_n ; \exists i_1, \ldots, i_k \in N_n, k>1, i_1=x, i_k=j, u^{i_1}=s_{i_2}, u^{i_2}=s_{i_3}, \ldots, u^{i_{k-1}}=s_{i_k}\}$ for $j \in N_n$ such that card $U_j>1$ and $\bar{U}_j=\emptyset$ otherwise. It is not difficult to prove that for different $p, q \in N_n$ we have $\bar{U}_p \cap \bar{U}_q=\emptyset$. By a repeated application of Lemma 1.4 we obtain that $I^n/$ $/(U_1, \ldots, U_n) \approx I^n/(V_1, \ldots, V_n)$, where $V_i=U_j$ for $i \in \bar{U}_j$, $j \in N_n$ and $V_i=U_i$ otherwise. Further, the s-cube $I^n/(V_1, \ldots, V_n)$ is quasi-regular, because $u^i=s_j$ implies card $V_j=1$, $j \in N_n$. We have just proved

Proposition 2.6. *Every s-cube is homeomorphic to some quasi-regular s-cube.*

Let $X=I^n/(U_1, \ldots, U_n)$ be an s-cube, $P_U=\bar{U}_1 \cup \ldots \cup \bar{U}_n \cup \{j \in N_n ; \text{card } U_j>1\}$ and $M_U=N_n-P_U$. Let $\tilde{R}(U_1, \ldots, U_n)$ be a binary relation on M_U defined via

$$x \tilde{R}_U y \Leftrightarrow (x=y) \vee (x \in U_y) \vee (y \in U_x).$$

Suppose that $\tilde{E}(U_1, \ldots, U_n)$ (briefly $\tilde{E}_U$) is the least equivalence relation on M_U containing $\tilde{R}_U$ and $M_U/\tilde{E}_U = \{A_U^{(1)}, \ldots, A_U^{(r)}\}$.

Let $X = I^n/(U_1, \ldots, U_n)$, $Y = I^n/(V_1, \ldots, V_n)$ be the s-cubes defined in Lemma 2.5 and let $f\colon X \to Y$ be the homeomorphism constructed in the proof of Lemma 2.5. Making use of Lemma 1.4 it is not difficult to prove the following

Lemma 2.7. *If X is quasi-regular, then Y is quasi-regular. Further, $M_U = M_V$ and $M_U/\tilde{E}_U = M_V/\tilde{E}_V$.*

Lemma 2.8. *Let $s \in N_r$ and let $X = I^n/(u^1, \ldots, u^n)$ be a quasi-regular s-cube such that $M_U/\tilde{E}_U = \{A_U^{(1)}, \ldots, A_U^{(r)}\}$. Then X is homeomorphic to a quasi-regular s-cube $I^n/(v^1, \ldots, v^n)$ such that*

1) *$M_U = M_V$ and $M_U/\tilde{E}_U = M_V/\tilde{E}_V$*
2) *There is $k_s \in A_U^{(s)}$, for which $v^{k_s} = s_{k_s}$ and $v^i = u^i$* for *$i \notin A_U^{(s)}$.*

Proof. Suppose that there is not $k_s \in A_U^{(s)}$ for which $u^{k_s} = s_{k_s}$. Then there are $j_1, \ldots, j_t \in A_U^{(s)}$ such that $u^{j_1} = s_{j_2}$, $u^{j_2} = s_{j_3}$, $\ldots$, $u^{j_t} = s_{j_1}$, $2 \leqq t \leqq \text{card } A_U^{(s)}$. By Lemma 2.5 we get that $X \approx I^n/(v^1, \ldots, v^n)$, where $v^{j_1} = v^{j_2} = \ldots = v^{j_t} = s_{j_1}$ and $v^i = u^i$ otherwise. Then $k_s = j_1$ and $v^i = u^i$ for $i \in A_U^{(s)}$. Condition 1) follows from Lemma 2.7.

Corollary. The s-cube X is homeomorphic to a quasi-regular s-cube $I^n/$ $/(v^1, \ldots, v^n)$ such that

1) $M_U = M_V$ and $M_U/\tilde{E}_U = M_V/\tilde{E}_V$
2) For every $i \in N_r$ there is $k_i \in A_U^{(i)}$ such that $v^{k_i} = s_{k_i}$
3) $v^i = u^i$ for $i \in P_U$.

Le 2.9. *Let $I^n/(u^1, \ldots, u^n)$ be a quasi-regular s-cube, $M_U/\tilde{E}_U = \{A_U^{(1)}, \ldots, A_U^{(r)}\}$. Let there for some $s \in N_r$ $k_s \in A_U^{(s)}$ for which $u^{k_s} = s_{k_s}$. Then X is homeomorphic to a quasi-regular s-cube $I^n/(v^1, \ldots, v^n)$ such that*

1) *$M_U = M_V$ and $M_U/\tilde{E}_U = M_V/\tilde{E}_V$*
2) *$v^i = s_{k_s}$ for $i \in A_U^{(s)}$* and *$v^i = u^i$ otherwise.*

Proof. Let $j \in A_U^{(s)}$ be such an index that $u^j \neq s_{k_s}$. Since $j, k_s \in A_U^{(s)}$, $u^{k_s} = s_{k_s}$, there are $i_1, \ldots, i_t \in A_U^{(s)}$ such that $j = i_1$, $k_s = i_t$ and $u^{i_1} = s_{i_2}$, $u^{i_2} = s_{i_3}$, $\ldots$, $u^{i_{t-1}} = s_{i_t}$. By Lemma 2.5 we get that X is homeomorphic to an s-cube $I^n/(w^1, \ldots, w^n)$, where $w^{i_1} = w^{i_2} = \ldots = w^{i_t} = s_{k_s}$ and $w^i = u^i$ otherwise. With respect to Lemma 2.7 we have $M_U = M_W$, $M_U/\tilde{E}_U = M_W/\tilde{E}_W$ and the s-cube $I^n/(w^1, \ldots, w^n)$ is quasi-regular. In the case when $u^j = s_{k_s}$ for every $j \in A_U^{(s)}$ we finish. In the other case we continue in the outlined procedure until we get a quasi-regular s-cube $I^n/(v^1, \ldots, v^n)$ such that conditions 1), 2) are satisfied.

Corollary. Let $X = I^n/(u^1, \ldots, u^n)$ be a quasi-regular s-cube such that for every $s \in N_r$ there is $k_s \in A_U^{(s)}$ with the property $u^{k_s} = s_{k_s}$. Then there is a regular s-cube $I^n/(v^1, \ldots, v^n)$ homeomorphic to X such that

1) $v^i = s_{k_s}$ for $i \in A_U^{(s)}$, $s \in N_r$
2) $v^i = u^i$ for $i \in P_U$.

Proposition 2.10. *Every s-cube is homeomorphic to some r-cube.*

Proof. Let $X_U = I^n/(u^1, \ldots, u^n)$ be an s-cube. By Proposition 2.6 X_U is homeomorphic to a quasi-regular s-cube $X_V = I^n/(v^1, \ldots, v^n)$. Let $M_V/\tilde{E}_V = \{A_V^{(1)}, \ldots, A_V^{(r)}\}$. Then by Corollary of Lemma 2.8 X_V is homeomorphic to a quasi-regular s-cube $X_W = I^n/(w^1, \ldots, w^n)$, where $W_i = V_i$ for $i \in P_V$, $M_W = M_V$, $M_V/\tilde{E}_V = M_W/ /\tilde{E}_W$ and for every $i \in N_r$ there is $k_i \in A_V^i$ such that $u^{k_i} \approx s_{k_i}$. Further, by Corollary of Lemma 2.9, X_W is homeomorphic to a regular s-cube $X_Z = I^n/(z^1, \ldots, z^n)$, where $z^i = s_{k_j}$ for $i \in A_V^{(j)}$, $j \in N_r$, $z^i = v^i$ for $i \in P_V$.

Example 2.11. Making use of Lemma 1.4 we find an r-cube which is homeomorphic to the s-cube $X = I^5/(s_3, s_{123}, s_2, s_5, s_4)$.

$$X \approx I^5/(s_3, s_{123}, s_{123}, s_5, s_4) \approx I^5/(s_{123}, s_{123}, s_{123}, s_5, s_4) \approx$$
$$\approx I^5/(s_{123}, s_{123}, s_{123}, s_4, s_4) = Y$$

As we can see in Example 2.11, an s-cube is not homeomorphic to the unique r-cube in general, because $X \approx I^5/(s_{123}, s_{123}, s_{123}, s_5, s_5) \neq Y$.

Example 2.12. Let $X = I^n/(U_1, \ldots, U_n)$ be an s-cube with card $U_i = 1$ for $i \in N_n$. Then X is quasi-regular and $M_U = N_n$. Denote $N_n/\tilde{E}_U = \{A_U^{(1)}, \ldots, A_U^{(r)}\}$. By Corollary of Lemma 2.8 and by Corollary of Lemma 2.9 X is homeomorphic to a regular s-cube $Y = I^n/(v^1, \ldots, v^n)$, where $v^i = s_{k_j}$ for $i \in A_U^{(j)}$, $s_{k_j} \in A_U^{(j)}$, $j \in N_r$. Then in a way similar to that in the proof of Theorem 2.3, making use of Remark 1.7, we get the homeomorphism $Y \approx S^{c_1} \times \ldots \times S^{c_r}$, where $c_i = \text{card } A_U^{(i)}$, $i \in N_r$.

3. Are all r-cubes manifolds?

In dimensions 1 and 2 it is evident that r-cubes are not manifolds in general. As examples we mention r-cubes $I/(id)$, $I^2/(id, s_2) \approx S^1 \times I$ (these r-cubess are manifolds with a boundary). In a higher dimension it is sometimes difficult to decide whether a given r-cube is or is not a manifold. For example, an r-cube $I^3/(s_1, s_{12}, s_{123})$ is a manifold, but an r-cube $I^3/(s_1, s_{23}, s_{123})$ is neither a manifold nor a manifold with a boundary.

The solution of the problem whether a given r-cube is a manifold is in Theorem 3.18.

Definition 3.1. *An r-cube $X = I^n/(u^1, \ldots, u^n)$ has the property "M" if for each nonempty subset $P \subset N_n$ such that*

i) $\forall i, j \in P$: $i \neq j \Rightarrow u^i \neq u^j$ (1)

ii) $\forall i \in P$: card $U_i \neq 1$ (2)

we have

$$P \cap \tau\Big(\prod_{j \in P} u^j\Big) \neq \emptyset\,. \tag{3}$$

Example 3.2. r-cubes $I^3/(s_1, s_{12}, s_{123})$, $I^4/(s_2, s_2, s_4, s_4)$ have the property "M", r-cubes $I^3/(s_1, s_{12}, s_{12})$, $I^4/(s_{12}, s_{23}, s_{34}, s_{14})$ have not. Not every r-cube $I^n/(U_1, \ldots, U_n)$ with card $U_i = \emptyset$ for some $i \in N_n$ has the property "M".

Lemma 3.3. *Let $I^n/(U_1, \ldots, U_n)$ be an r-cube with the property "M" and let* card $U_k > 1$ *for some $k \in N_n$. Then $k \in U_k$.*

Proof. Suppose that $k \notin U_k$. Then for $P = \{k\}$ we have $P \cap \tau(u^k) = \{k\} \cap U_k = \emptyset$.

Definition 3.4. *An r-cube $I^n/(U_1, \ldots, U_n)$ is cube-fibreable (briefly c-fibreable) as there is a set Q, $\emptyset \subsetneqq Q \subsetneqq N_n$, such that*

$$\text{(i)}\quad Q \cap \Big(\bigcup_{k \in N_n - Q} U_j \Big) = \emptyset \tag{4}$$

$$\text{(ii)}\quad \textit{If } U_i = U_j \textit{ for some } i, j \in N_n, \textit{ then } i, j \in Q \textit{ or } i, j \in N_n - Q. \tag{5}$$

An r-cube which is not c-fibreable is called c-nonfibreable.

Example 3.5. An r-cube $I^3/(s_1, s_{12}, s_{123})$ is c-fibreable with $Q = \{3\}$ or $Q = \{2, 3\}$, an r-cube $I^2/(s_2, s_2)$ is c-nonfibreable.

Lemma 3.6. *Let $k \in N_n$ and let $I^n/(u^1, \ldots, u^n)$ be an r-cube with $k \in U_k$. Then $I^n/(u^1, \ldots, u^n) \approx I^n/(v^1, \ldots, v^n)$, where $v^i = u^i \circ u^k \circ s_k$ for such $i \in N_n$, $i \neq k$, that $k \in U_i$ and $v^i = u^i$ otherwise.*

Proof. First we define a map h_k: $I^n/(u^1, \ldots, u^n) \to I^n/(v^1, \ldots, v^n)$, $h_k([x]) = [(x_1, \ldots, x_{k-1}, x_k + 1, x_{k+1}, \ldots, x_n)]$ for $x_k \leqq 0$, $h_k([x]) = [(\tilde{x}_1, \ldots, \tilde{x}_{k-1}, x_k - 1, \tilde{x}_{k+1}, \ldots, \tilde{x}_n)]$ for $x_k \geqq 0$, where $\tilde{x}_j = x_j$ for $j \notin U_k$ and $\tilde{x}_j = -x_j$ for $j \in U_k$, $j \in N_n - \{k\}$. It is not difficult to show that h_k is well defined and continuous. The map g_k: $I^n/(v^1, \ldots, v^n) \to I^n/(u^1, \ldots, u^n)$, $g_k([x]) = [(x_1, \ldots, x_{k-1}, x_k - 1, x_{k+1}, \ldots, x_n)]$ for $x_k \geqq 0$, $g_k([x]) = [(\tilde{x}_1, \ldots, \tilde{x}_{k-1}, x_k + 1, \tilde{x}_{k+1}, \ldots, \tilde{x}_n)]$ for $x_k \leqq 0$, where $\tilde{x}_j = x_j$ for $j \notin V_k$, $x_j = -x_j$ for $j \in V_k$, $j \in N_n - \{k\}$, is also well defined, continuous and inverse to h_k. Hence both h_k and g_k are homeomorphisms.

Let $X = I^n/(u^1, \ldots, u^n)$, $Y = I^n/(v^1, \ldots, v^n)$ be the s-cubes from Lemma 3.6. Then the s-cube Y is not an r-cube in general. Let $K = \{i \in N_n;\ U_i = U_k,\ i \neq k\}$, $\alpha = \text{card } K$. Then for each $i \in K$ we have $v^i = s_k$ and $v^k = u^k$. Now it is easy to see that the s-cube is not an r-cube if and only if $\alpha \geqq 1$ and $u^k \neq s_k$. To obtain an r-cube from the s-cube Y it is sufficient to apply α-times Lemma 1.4. Therefore we can strengthen Lemma 3.6 into

Proposition 3.7. *Let $k \in N_n$ and let $X = I^n/(u^1, \ldots, u^n)$ be an r-cube with $k \in U_k$. Then the r-cube X is homeomorphic to an r-cube $Y = I^n/(w^1, \ldots, w^n)$, where $w^i = u^i \circ u^k \circ s_k$ for such $i \in N_n$ that $k \in U_i$, $U_i \neq U_k$ and $w^i = u^i$ otherwise.*

Proof. Let $K = \{i_1, \ldots, i_\alpha\}$, $\alpha \geqq 1$. According to Lemma 3.6 $I^n/(u^1, \ldots, u^n) \approx I^n/(v^1, \ldots, v^n)$, v^i, $i \in N_n$, are described in Lemma 3.6. Then using Lemma 1.4 successively for $r = i_1, \ldots, i_\alpha$ we get homeomorphisms $I^n/(v^1, \ldots, v^n) \overset{f_1}{\approx} I^n/(z^1_{(1)}, \ldots, z^n_{(1)}) \overset{f_2}{\approx} \ldots \overset{f_\alpha}{\approx} I^n/(z^1_{(\alpha)}, \ldots, z^n_{(\alpha)}) = I^n/(w^1, \ldots, w^n)$, where $z^i_{(j)} = v^k$ for $i = i_m$, $m \leqq j$ and $z^i_{(j)} = v^i$ otherwise. The map $\bar{h}_k = f_\alpha \circ f_{\alpha-1} \circ \ldots f_1 \circ h_k$ is the demanded homeomorphism.

Lemma 3.8. *Let $X=I^n/(u^1, \ldots, u^n)$, $Y=I^n/(w^1, \ldots, w^n)$ be r-cubes defined in Proposition 3.7. Then the r-cube X has the property "M" if and only if the r-cube Y has the property "M".*

Proof. Let the r-cube X not have the property "M". Then there is a nonempty set P, satisfying (1), (2), such that $P\cap\tau\left(\prod_{i\in P} u^i\right)=\emptyset$. We prove that the r-cube Y has not the property "M". We shall discuss two cases:

i) card $P\geqq 2$, ii) card $P=1$.

i) Let $P=\{i_1, \ldots, i_r\}$ and let s, $0\leqq s\leqq r$, be such a number that $k\in U_i$ for $i\leqq s$ and $k\notin U_i$ for $i>s$. It is clear that s is even. Suppose that $k\in P$ (the other case will be discussed later). We show that for $\tilde{P}=P-\{k\}$ we have $\tilde{P}\cap\tau\left(\prod_{i\in P} w^i\right)=\emptyset$. In fact,

$$\prod_{i\in\tilde{P}} w^i=\left(\prod_{i\in\tilde{P}} u^i\right)\circ(u^k\circ s_k)^{s-1}=\left(\prod_{i\in\tilde{P}} u^i\right)\circ(u^k\circ s_k)\circ(u^k\circ s_k)^s=$$

$$=s_k\circ\prod_{i\in P} u^i\ ,$$

becausee for every $u\in G$ we have $u^2=id$. Then $\tilde{P}\cap\tau\left(s_k\circ\prod_{i\in P} u^i\right)=\emptyset$, because $P\cap\tau\left(\prod_{i\in P} u^i\right)=\emptyset$.

In the case when $k\notin P$ we take $\tilde{P}=P$ for s even and $\tilde{P}=P\cup\{k\}$ for s odd.

ii) Let $P=\{p\}$. It is sufficient to take $\tilde{P}=P$ if $p\notin U_p$, $p\notin W_p$ and $\tilde{P}=P\cup\{k\}$ if $p\notin U_p$, $p\in W_p$.

Let now the r-cube $Y=I^n/(w^1, \ldots, w^n)$ not have the property "M". Taking $X=Y$ in Proposition 3.7 we get $Y\approx Z=I^n/(z^1, \ldots, z^n)$, where $z^i=u^i$ for $i\in N_n$. By the first part of the proof we obtain that the r-cube Z, $Z=X$, has not the property "M".

Lemma 3.9. *Let $X=I^n/(U_1, \ldots, U_n)$ be an r-cube without the property "M" such that for every $i\in N_n U_i\neq\emptyset$. Then there are an r-cube $Y=I^n/(V_1, \ldots, V_n)$, $X\approx Y$ and an integer $k\in N_n$ such that $k\notin V_k$ and* card *$V_k>1$.*

Proof. Suppose that $i\in U_i$ for every $i\in N_n$ such that card $U_i>1$. Since X has not the property "M", there is a nonempty set $P\subset N_n$ such that the conditions (1), (2) and $P\cap\tau\left(\prod_{j\in P} u^j\right)=\emptyset$ are satisfied. Without loss of generality we can suppose that $P=\{1, 2, \ldots, r\}$. Let $\tilde{K}_1=\{i\in N_n;\ 1\in U_i\}$, $K_1=\tilde{K}_1\cap\{2, \ldots, r\}$, card $K_1=\alpha_1$. Denote $X=I^n/(U_1^{(0)}, \ldots, U_n^{(0)})$. Then using Proposition 3.7 for $k=1$ we have $I^n/(U_1^{(0)}, \ldots, U_n^{(0)})\approx I^n/(U_1^{(1)}, \ldots, U_n^{(1)})$, where $u^i_{(1)}=u^i_{(0)}\circ u^1_{(0)}\circ s_1$ for $i\in\tilde{K}_1-\{1\}$ such that $U_i^{(0)}\neq U_1^{(0)}$ and $u^i_{(1)}=u^i_{(0)}$ otherwise. Let $P_k=P-\{1, \ldots, k\}$. Then for $k=1$ the following conditions are satisfied:

i) $\forall i, j \in P_k: i \neq j \Rightarrow U_i^{(k)} \neq U_j^{(k)}$

ii) $\forall i \in P_k$: card $U_i^{(k)} > 1$

iii) $P_k \cap \tau\left(\prod_{j \in P_k} u_{(k)}^j\right) = \emptyset$.

Conditions i), ii) are evident, we prove iii). Let $u_{(0)}^1 = s_1 \circ s_{i_1} \circ \dots \circ s_{i_m}$. Then

$$\prod_{j \in P_1} u_1^{(j)} = \left(\prod_{j \in P_1} u_{(0)}^1\right) \circ (u_{(0)}^1 \circ s_1)^{\alpha_1} =$$

$$= \left(\prod_{j \in P} u_{(0)}^j\right) \circ u_{(0)}^1 \circ (u_{(0)}^1 \circ s_1)^{\alpha_1} = \prod_{j \in P} u_{(0)}^j \circ s_1 ,$$

because α_1 is an odd integer and $u^2 = id$ for every $u \in G$. Since $P \cap \tau\left(\prod_{j \in P} u_{(0)}^j\right) = \emptyset$, we have

$$P_1 \cap \tau\left(\prod_{j \in P_1} u_{(1)}^j\right) = P_1 \cap \tau\left(\left(\prod_{j \in P} u_{(0)}^j\right) \circ s_1\right) = \emptyset .$$

There are two possibilities: 1) $2 \notin U_2^{(1)}$, 2) $2 \in U_2^{(1)}$.

In the case 1) the proof is finished. In the case 2) we continue in the outlined process until we get (by repeated application of Lemma 3.7) a number $k_0 \in \{3, \dots, r\}$ and an r-cube $I^n/(U_1^{(k_0-1)}, \dots, U_n^{(k_0-1)})$ such that $k_0 \notin U_{k_0}^{(k_0-1)}$. Now we outline the proof of the existence of such k_0. Let for $k = 3, \dots, r$ $k \in U_k^{(k-1)}$. It is possible to show that for $k = 2, \dots, r-1$ the conditions i), ii), iii) are satisfied. Then for $k = r-1$ we get from iii) that $\{r\} \cap \tau(u_{(r-1)}^r) = \{r\} \cap U_r^{(r-1)} = \emptyset$, a contradiction.

Lemma 3.10. *Let $X = I^n/(U_1, \dots, U_n)$ be an r-cube such that for some $k \in N_n$ card $U_k = m > 1$ and $k \notin U_k$. Then X is neither a manifold, nor a manifold with a boundary.*

Proof. Let $a \in \partial I^n$ be such a point that $a_k = 1$ and $a_j = 0$ for $j \in N_n - \{k\}$. Let $U = \{x \in I^n;\ d(x, a) \leqq \frac{1}{2}\}$ (d is the symbol of the Euclidean metric), $\pi_n: I^n \to I^n/ /(U_1, \dots, U_n)$ a projection, $V = \pi_n(U)$. V is a neighbourhood of the point $b = \pi_n(a)$, $V \approx C(I^{n-1}/(s_{12\dots m}, \dots, s_{12\dots m}))$, the point b corresponds to the top of the cone in this homeomorphism. Using [2] we get $V \approx C(S^{n-m-1}RP^m)$. Since V is contractible,

$$H_q(V, V - \{b\}) \cong \tilde{H}_{q-1}(V - \{b\}) \cong \tilde{H}_{q-1}(S^{n-m-1}RP^m) \cong \tilde{H}_{q-n+m}RP^m$$

for every $q \in N$ (the symbol $\tilde{H}$ denotes the reduced homology with Z coefficients). Then using [1], Proposition 3.2, page 59, it can be proved that there is not a neighbourhood of the point b homeomorphic to R^n or to $R_+^n (R_+^n = \{x \in R^n;\ x_n \geqq 0\})$.

Now we describe some simple properties of c-nonfibreable r-cubes. Let $X = I^n/ /(U_1, \dots, U_n)$ be a c-nonfibreable r-cube. By M_k we shall denote the set $\{i \in N_n;\ U_i = U_k\}$, $k \in N_n$.

Lemma 3.11. *If $U_k=\{k\}$ for some $k\in N_k$, then $X=I^n/(s_k, \ldots, s_k)$.*

Proof: Suppose that $M_k\neq N_n$. Then X is c-fibreable with $Q=N_n-M_k$.

Lemma 3.12. *If the r-cube X has the property "M" and* card $U_i>1$ *for some $i\in N_n$, then $M_i=N_n$ or there is $p_i\in U_i$ such that $p_i\notin M_i$ and $U_{p_i}\cap M_i=\emptyset$.*

Proof. Let $M_i\neq N_n$. If $U_i=M_i$, then X is c-fibreable with $Q=N_n-M_i$, a contradiction. Hence $M_i\subsetneqq U_i$. Let $p_i\in U_i-M_i$. We show that $U_{p_i}\cap M_i=\emptyset$. Let $j\in U_{p_i}\cap M_i$. Since card $U_k>1$ for every $k\in N_n$ (Lemma 3.11), we have $j\in U_{p_i}\cap M_i\subset U_{p_i}\cap U_j$, $p_i\in U_{p_i}\cap U_j$ and $\{j, p_i\}\cap\tau(u^j\circ u^{p_i})=\emptyset$. Hence the r-cube X has not the property "M", a contradiction.

Now we are going to describe c-nonfibreables r-cubes with the property "M".

Proposition 3.13. *Let $X=I^n/(U_1, \ldots, U_n)$ be a c-nonfibreable r-cube with the property "M". Then exactly one of the following conditions is satisfied:*

i) *There is $k\in N_n$ such that $X=I^n/(s_k, \ldots, s_k)$*

ii) $X=I^n/(s_{12\ldots n}, \ldots, s_{12\ldots n})$.

Proof. Suppose that $X\neq I^n/(s_k, \ldots, s_k)$, $k\in N_n$, $X\neq I^n/(s_{12\ldots n}, \ldots, s_{12\ldots n})$. As usually we denote $M_j=\{i\in N_n;\ U_i=U_j\}$, $j\in N_n$ and let $t=\text{card}\,\{M_j;\ j\in N_n\}$. According to Lemma 3.11 card $U_i>1$ for $i\in N_n$. Making use of Lemma 3.12 for $i=p_0=1$ we obtain an integer $p_1\in U_{p_0}$ such that $p_1\notin M_{p_0}$ and $U_{p_1}\cap M_{p_0}=\emptyset$. Let r be such an integer, $1\leqq r\leqq t-1$ that there are integers $p_0, p_1, \ldots, p_r$ for which the conditions

1) $p_i\in U_{p_i}-M_{p_i}$, $i=0, 1, \ldots, r-1$
2) $U_{p_j}\cap(M_{p_0}\cup M_{p_1}\cup\ldots\cup M_{p_{j-1}})=\emptyset$, $j=1, 2, \ldots, r$

are satisfiesd. We shall prove that there is $p_{r+1}\in U_{p_r}-M_{p_r}$ such that $U_{p_{r+1}}\cap(M_{p_0}\cup M_{p_1}\cup\ldots\cup M_{p_r})=\emptyset$. There are two possibilities:

i) For every $x\in U_{p_r}$ there is $x\in M_{p_0}\cup M_{p_1}\cup\ldots\cup M_{p_r}$. Then with regard to 2) the r-cube is c-fibreable with $Q=M_{p_r}$.

ii) There is $p_{r+1}\in U_{p_r}$ such that $p_{r+1}\notin M_{p_0}\cup M_{p_1}\cup\ldots\cup M_{p_r}$. We prove that the set $S=U_{p_{r+1}}\cap(M_{p_0}\cap M_{p_1}\cup\ldots\cup M_{p_r})$ is empty. Let $S\neq\emptyset$ and let $q\in\{0, 1, \ldots, r\}$ be the greatest index such that there is $s\in S$ with the property $s\in M_{p_q}$. By Lemma 3.12 we have $q<r$. Let now q_1, $q\leqq q_1\leqq r$ be the least index such that $p_{r+1}\in U_{p_{q,1}}$, let q_2, $q\leqq q_2<q_1$ be the least index such that $p_{q_1}\in U_{p_{q,2}}$, $\ldots$, let q_m, $q=q_m<q_{m-1}$ be the least index such that $p_{q_{m-1}}\in U_{p_{q,m}}$, $1\leqq m\leqq r+1$, where $p_{q,j}$ is used instead of p_{q_j}. Then

$$\{p_{r+1}, p_{q_1}, p_{q_2}, \ldots, p_{q_m}\}\cap\tau(u^{p_{r+1}}\circ u^{p_{q,1}}\circ\ldots\circ u^{p_{q,m}})=\emptyset$$

and the r-cube X has not the property "M", a contradiction. Hence there are integers $p_0, p_1, \ldots, p_{t-1}$ such that the conditions (1), 2) are satisfied for $i=0, 1, \ldots, t-2$, $j=1, 2, \ldots, t-1$. Then for $j=t-1$ we have

$$U_{p_{t-1}}\cap(M_{p_0}\cup M_{p_1}\cup\ldots\cup M_{p_{t-2}})=\emptyset$$

and $U_{p_{t-1}} = M_{p_{t-1}}$. We see that the r-cube X is c-fibreable with $Q = M_{p_{t-1}}$, a contradiction.

Let $X = I^n/(U_1, \ldots, U_n)$ be a c-fibreable r-cube with the property "M". Without loss of generality we can take $Q = N_r$ for some r, $1 \leqq r < n$. To the projection $p\colon I^n \to I^r$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_r)$ there is the induced map $\tilde{p}\colon X \to I^r/(U_1 \cap N_r, \ldots, U_r \cap N_r) = B_U$, $[x] \mapsto [p(x)]$, such that the Diagram 1 commutes. The r-cube

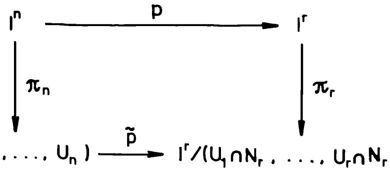

Diagram 1.

$B_U = I^r/(U_1 \cap N_r, \ldots, U_r \cap N_r)$ will be denoted in what follows briefly by $I^r/(\tilde{U}_1, \ldots, \tilde{U}_r)$ or by $I^r/(\tilde{u}^1, \ldots, \tilde{u}^r)$.

Lemma 3.14. *For every* $t \in I^r$ *we have* $\pi_n \circ p^{-1}(t) = \tilde{p}^{-1} \circ \pi_r(t)$.

Proof. Let $[x] \in \pi_n \circ p^{-1}(t)$. Then $\tilde{p}[x] = \tilde{p} \circ \pi_n(x) = = \pi_r \circ p(x) = \pi_r(t) = [t]$, hence $[x] \in \tilde{p}^{-1} \circ \pi_r(t)$. Let $[x] \in \tilde{p}^{-1} \circ \pi_r(t)$. We find such $z \in p^{-1}(t)$ that $[x] = [z]$ Since $\tilde{p}[x] = [p(x)] = [t]$, there are $i_1, \ldots, i_s \in N_r$ such that $p(x)$, $t \in \bigcap_{j=1}^{s} J^r_{i_j}$ and $p(x) = \tilde{u}^{i_1} \circ \ldots \circ \tilde{u}^{i_s}(t)$ $(t = (t_1, \ldots, t_r)$, $x = (x_1, \ldots, x_n))$. Let us define $z \in I^n$, $z = (t_1, \ldots, t_n)$ by $x = u^{i_1} \circ u^{i_2} \circ \ldots \circ u^{i_s}(z)$. Since $p(z) = t$ and x, $z \in \bigcap_{j=1}^{s} J^n_{i_j}$, we have $[x] = [z]$ and $z \in p^{-1}(t)$. Hence $[x] \in \pi_n \circ p^{-1}(t)$.

Now using the property "M" of the r-cube X we shall show that $\tilde{p}^{-1}[t] \approx I^{n-r}/ /(U^{[r]}_{r+1}, \ldots, U^{[r]}_n) = I^{n-r}/\Omega$ for each point $[t] \in B_U$. With regard to Lemma 3.14 is is sufficient to prove that $\pi_n(p^{-1}(t)) \approx I^{n-r}/\Omega$ for every $t \in I^r$. This fact is the direct colollary of the following

Lemma 3.15. *Let* x, $y \in p^{-1}(t)$, $t \in I^r$. *Then* $[x] = [y]$ *if and only if* $(x_{r+1}, \ldots, x_n)\Omega(y_{r+1}, \ldots, y_n)$.

Proof. Let $[x] = [y]$. Then there are $i_1, \ldots, i_k \in N_n$ such that x, $y \in \bigcap_{j=1}^{k} J^n_{i_j}$ and $y = u^{i_1} \circ \ldots \circ u^{i_k}(x)$. We can suppose that $u^{i_p} \neq u^{i_q}$ for $p \neq q$ (if $u^{i_p} = u^{i_q}$, the term $u^{i_p} \circ u^{i_q} = id$ can be omitted). Let $s \in N_n$ be such an integer that $i_j \leqq r$ for $j \leqq s$ and $i_j > r$ for $j > s$. Since $(x_1, \ldots, x_r) = (y_1, \ldots, y_r)$, we have

$$\tau(u^{i_1} \circ u^{i_2} \circ \ldots \circ u^{i_s}) \cap N_r = \emptyset\,. \tag{6}$$

Further, because X is c-fibreable with $Q = N_r$, (6) implies that $s = 0$ or there is

$j \in \{i_1, \ldots, i_s\}$ such that card $U_j > 1$. Let $S = \{j \in \{i_1, \ldots, i_s\};$ card $U_j > 1\}$. With regard to (6) we have

$$S \cap \tau\left(\prod_{j \in S} u^j\right) = \emptyset\,. \tag{7}$$

Since the r-cube X has the property "M" and the set S satisfies conditions (1), (2), we have with regard to (7) $S = \emptyset$ and $s = 0$. Hence $(x_{r+1}, \ldots, x_n)\Omega(y_{r+1}, \ldots, y_n)$.

The converse implication is trivial.

Corollary. $\tilde{p}^{-1}[t] \approx I^{n-r}/(U_{r+1}^{[r]}, \ldots, U_n^{[r]}) = F_U$ for every $t \in I^r$.

Lemma 3.16. *The r-cubes* $B_U = I^r/(\tilde{U}_1, \ldots, \tilde{U}_r)$, $F_U = I^{n-r}/(U_{r+1}^{[r]}, \ldots, U_n^{[r]})$ *have the property "M".*

Proof. We prove the assertion for B_U, the proof for F_U is similar. Let P, $\emptyset \subsetneqq P \subset N_r$ be such a set that the conditions (1), (2) are satisfied. Since the r-cube X has the property "M", $P \cap \tau\left(\prod_{j \in P} u^j\right) \neq \emptyset$. But then $P \cap \tau\left(\prod_{j \in P} \tilde{u}^j\right) \neq \emptyset$, because $\tilde{U}_j = U_j \cap N_r$ for $j \in N_r$.

The r-cube B_U can be embedded into X, an embedding i_U is given by $i_U[(t_1, \ldots, t_r)] = [(t_1, \ldots, t_r, 0, \ldots, 0)]$. Suppose now that the r-cube X is homeomorphic to an r-cube $Y = I^n/(V_1, \ldots, V_n)$ by Proposition 3.7 for some $k \leqq r$. Then the r-cube Y is c-fibreable with the same Q. Let us define a map $\bar{h}_k$: $I^r/(\tilde{U}_1, \ldots, \tilde{U}_r) \to I^r/(\tilde{V}_1, \ldots, \tilde{V}_r)$ by $\bar{h}_k = \tilde{p}_V \circ \tilde{h}_k \circ i_U$, where $\tilde{p}_V$: $I^n/(V_1, \ldots, V_n) \to I^r/(\tilde{V}_1, \ldots, \tilde{V}_r)$ is the induced map by p: $I^n \to I^r$. The map $\bar{h}_k$ is a homeomorphism and Diagram 2 commutes (i: $I^r \to I^n$, $(x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$). Further, the

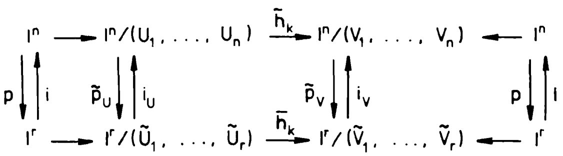

Diagram 2.

map $\bar{h}_k$ preserves fibres in such a way that the fibre over a point $[t] \in I^r/(\tilde{U}_1, \ldots, \tilde{U}_r)$ maps homeomorphically on the fibre over the point $\bar{h}_k[t] \in I^r/(\tilde{V}_1, \ldots, \tilde{V}_r)$

Lemma 3.17. *The fibration* $(X, \tilde{p}_U, B_U)$ *is locally trivial with the fibre* $F_U = I^{n-r}/(U_{r+1}^{[r]}, \ldots, U_n^{[r]})$.

Proof. We can suppose that there is an integer s, $0 \leqq s \leqq r$ such that card $U_i = 1$ for $i \leqq s$ and card $U_i > 1$ for $s < i \leqq r$. In the case when $s = r$, the fibration $(X, \tilde{p}_U, B_U)$ is trivial (Proposition 1.5). Now we give a local trivialization of the fibration $(X, \tilde{p}_U; B_U)$. Let $[a] \in B_U$.

1) If $a \notin \partial I^r$, then the set $A = \{[x] \in B_U;\, x \notin \partial I^r\}$ is a neighbourhood of $[a]$. We have $\tilde{p}_U^{-1}(A) \approx A \times F_U$ via $[(x_1, \ldots, x_n)] \mapsto ([x_1, \ldots, x_r)], [(x_{r+1}, \ldots, x_n)])$.

2) If $a \in \partial I^r$, then we shall discuss two cases:

I) $a_i \neq \pm 1$ for $i > s$. The set $A = \{[x] \in B_U;\ x_j \in \langle -1, 1\rangle$ for $j \in N_s$, $x_j \in (-1, 1)$ for $j \in N_r - N_s\}$ is a neighbourhood of $[a]$ and the map $f\colon \tilde{p}_U^{-1}(A) \to A \times F_U$, $[x] \mapsto ([x_1, \ldots, x_r)], [(x_{r+1}, \ldots, x_n)])$ is a homeomorphism.

II) $a_i = \pm 1$ for some $i > s$. Let $S = \{i \in N_r - N_s;\ a_i = \pm 1\} = \{i_1, \ldots, i_t\}$. Denote $I^n/(U_1, \ldots, U_n)$ by $I^n/(u^1_{(0)}, \ldots, u^n_{(0)})$. Then applying Proposition 3.7 for $k = i_1, \ldots, i_t$ we get the hemeomorphisms $\tilde{h}_{i_j}\colon I^n/(u^1_{(j-1)}, \ldots, u^n_{(j-1)}) \to I^n/(u^1_{(j)}, \ldots, u^n_{(j)})$, $j = 1, \ldots, t$, where $u^m_{(j)} = u^m_{(j-1)} \circ u^{i_j}_{(j-1)} \circ s_{i_j}$ for such m that $i_j \in U^{(j-1)}_m$, $U^{(j-1)}_{i_j} \neq U^{(j-1)}_m$ and $u^m_{(j)} = u^m_{(j-1)}$ otherwise. Let $\tilde{h} = \tilde{h}_{i_t} \circ \tilde{h}_{i_{t-1}} \circ \ldots \circ \tilde{h}_{i_1}$, $\tilde{h}(I^n/(U_1, \ldots, U_n)) = I^n/(V_1, \ldots, V_n)$, $\bar{h}[a] = [c]$, see Diagram 2, where $\tilde{h}$ is substituted for $\tilde{h}_k$($\bar{h}\colon B_U \to B_V$ is the map induced by $\tilde{h}$). Then $c_k = 0$ for $k \in S$, the set $C = \{[x] \in I^r/(\tilde{V}_1, \ldots, \tilde{V}_r);$ $x_j \in \langle -1, 1\rangle$ for $j \in N_s$, $x_j \in (-1, 1)$ for $j \in N_r - N_s\}$, is a neighbourhood of the point $[c] \in B_V$ a the map $f_C\colon \tilde{p}_V^{-1}(C) \to C \times I^{n-r}/(V^{[r]}_{r+1}, \ldots, V^{[r]}_n)$, $[(x_1, \ldots, x_n)] \mapsto ([(x_1, \ldots, x_r)], [(x_{r+1}, \ldots, x_n)])$ is a homeomorphism. Further, $I^{n-r}/(U^{[r]}_{r+1}, \ldots, U^{[r]}_n) = I^{n-r}/(V^{[r]}_{r+1}, \ldots, V^{[r]}_n)$, Let $A = \{[x] \in I^r/(\tilde{U}_1, \ldots, \tilde{U}_r)$, $x_j \in \langle -1, 1\rangle$ for $j \in N_s$, $x_j \in \langle -1, 0) \cup (0, 1\rangle$ for $j \in S$, $x_j \in (-1, 1)$ for $j \in N_r - N_s$, $j \notin S\}$. We see that A is a neighbourhood of the point $[a] \in B_U$ and the map $\bar{h}|_A\colon A \to C$ is a homeomorphism. The map $f_A = f_C \circ (\tilde{h}|\tilde{p}_U^{-1}(A))\colon \tilde{p}_U^{-1}(A) \to C \times I^{n-r}/(V^{[r]}_{r+1}, \ldots, V^{[r]}_n)$ is also a homeomorphism and the required local trivialization.

Theorem 3.18. *An r-cube $X = I^n/(U_1, \ldots, U_n)$ is a manifold if and only if it has the property "M".*

Proof. Let X not have the property "M". If $U_i = \emptyset$ for some $i \in N_n$, then X is not a manifold. If $U_i \neq \emptyset$ for all $i \in N_n$, then according to Lemma 3.9 and Lemma 3.10 X is neither a manifold nor a manifold with a boundary.

Let now X have the property "M"; there are two possibilities:

1) X is c-confibreable. Then by Proposition 3.13 and Remark 1.7 $X \approx S^n$ or $X \approx RP^n$.

2) X is c-fibreable. To prove that X is a manifold, it is sufficient to use Lemmas 3.16, 3.17, Proposition 3.13, Remark 1.7 and the induction.

REFERENCES

[1] DOLD, A.: Lectures on Algebraic Topology. Springer Verlag, Berlin—Heidelberg—New York 1972.

[2] DUDÁŠIKOVÁ, H.: Priestor $I^n/(s_{12\ldots k}, \ldots, s_{12\ldots k})$, $k > 1$, je suspenzia nad RP^k; In: Záverečná správa fakultnej výskumnej úlohy UK 364, Bratislava 1976.

[3] KULICH, I., TVAROŽEK, J.: Priestor $I^n/(s_{i_1}, \ldots, s_{i_n})$ je súčinom sfér; In: Záverečná správa FVÚ UK 364, Bratislava 1976.

Received May 12, 1981

Katedra matematiky
Elektrotechnickej fakulty SVŠT
Mlynská dolina, blok A
812 19 Bratislava

S-КУБЫ

Jozef Tvarožek

Резюме

В статье исследуются некоторые фактор-пространства n-мерного куба I^n, которые возникают отождествлением определенных точек на его границе. Возникающие пространства назвыны s-кубами.

В нервой части статьи установлены основные свойства s-кубов. Во второй части изучаются проблемы разложения s кубов. В третьей части найдено необходимое и достаточное условие для того, чтобы s-куб был многообразием.