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GRAPHS WHICH ARE EDGE-LOCALLY C_n

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ABSTRACT. A graph G is called edge-locally C_n (vertex-locally C_n) if the neighbourhood of each edge (vertex) of G is formed by a cycle of length n. The relationship between edge- and vertex-locally C_n graphs is clarified. This leads to a (geometrical) characterization of edge-locally C_n graphs for $n > 6$. A similar characterization for vertex-locally C_n graphs was given by Vince. Frončekproved that for n odd ($n > 3$) there is no edge-locally C_n graph. It is here proved that for all the remaining values of n such a graph exists. Moreover, a complete list of edge-locally C_n graphs for $n \leq 6$ is given. For every n even, $n > 6$, there are infinitely many such graphs.

0. Introduction

Let G be a graph and u one of its vertices. Denote by $G(u)$ the subgraph of G induced by the set of vertices adjacent to u. The graph G is called *vertex-locally* G_0 if $G(u) \cong G_0$ for all vertices u of G. A graph is called *locally homogeneous* if it is vertex-locally G_0 for some G_0. Much attention has been paid to two broad questions related to local homogeneity. The first one, posed by Zykov in 1963 [22], is the following: For which graphs G_0 does there exist a (finite) graph G that is vertex-locally G_0? (see for instance [1], [2], [3], [12]), while the second one reads as follows: For a given graph G_0 characterize all (finite) graphs that are vertex-locally G_0 (see [4], [9], [10], [19]). These questions seem to be difficult (see [3]). However, even partial results are of interest since the subject is related to algebraic topology and group theory (see [10], [17], [18], [19]). In particular, it was proved in [2] and also in [5] that for each cycle C_n of length $n \geq 3$ there is a vertex-locally C_n graph. Later, Ronan [17] showed that there are infinitely many vertex-locally C_n graphs for each $n \geq 6$. Finally, Vince [18] characterized graphs which are vertex-locally C_n in terms of groups. The main aim of this paper is to resolve edge versions of these results. All graphs

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considered will be undirected, connected, without loops or multiple edges and finite (except when explicitly stated otherwise).

1. Edge-locally C_n graphs

Let $e = uv$ be an edge of a graph G, and let $G(e)$ denote the subgraph of G induced by the vertices adjacent to u or v but different from u, v. Call a graph *edge-locally* G_0 if $G(e) \cong G_0$ for each edge e of G. A graph will be called *edge-locally homogeneous* if it is edge-locally G_0 for some G_0. The concept of edge-local homogeneity was introduced by Zelinka in [21], where he presented examples of edge-locally C_n graphs for $n = 3, 4, 6$ and 8. On the other hand, he proved that there is no such graph for $n = 5$. This result was improved by Fronček [7], who showed that there is no edge-locally C_n graph for n odd, $n \neq 3$. On the contrary, we prove (see Theorem 5) that for all even n such graphs exist. Denote by $L_{5,5}$ the graph obtained from the complete bipartite graph $K_{5,5}$ by deleting the edges of a perfect matching. Clearly, K_5, $K_{3,3}$ and $L_{5,5}$ are edge-locally C_3, C_4 and C_6, respectively. Another example of an edge-locally C_6 graph is the icosahedron, which is in addition vertex-locally C_5. Theorem 1 shows that any vertex-locally C_k graph without an induced C_4 is also edge-locally C_{2k-4}. Moreover, apart from K_5, each edge-locally C_n graph with a triangle ($n \geq 3$) may be obtained in this way.

THEOREM 1. *Let G be a connected graph and let G contains a triangle. Then G is edge-locally C_n ($n \geq 3$) if and only if either $n \geq 6$ is even and G is vertex-locally C_k without an induced C_4, where $k = (n+4)/2$, or $n = 3$ and $G \cong K_5$.*

Proof.

($\Longleftarrow$) Let G be a graph that is vertex-locally C_k for some $k \geq 4$, and which does not contain C_4 as an induced subgraph. Consider an arbitrary edge wx in G. Since $G(w)$ and $G(x)$ are both cycles of length k, their edges give rise to a Hamiltonian cycle C of length $2k-4$ in $G(wx)$. It remains to prove that C is induced. Suppose there is a chord uv in C. Since G is vertex-locally C_k, u and v cannot both be in either $G(w)$ or in $G(x)$. Thus we may assume that $u \in G(x) - G(w)$ and $v \in G(w) - G(x)$. Hence $(uvwx)$ is an induced cycle of length 4 in G, which is a contradiction.

($\Longrightarrow$) Let G be edge-locally C_n ($n \geq 3$). To finish the proof, we split the discussion into two separate claims.

Claim 1. Let $K_4 \subseteq G$, then $G \cong K_5$.

Proof. Denote by u, v, x, y the vertices of $K_4 \subseteq G$. Since G is edge-locally C_n and $n \geq 3$, we have $|V(G)| \geq 5$. Moreover, G is connected, so there is a vertex z in G distinct from u, v, x, y, but adjacent to at least one of them, say u. Since the triangle (vxy) belongs to $G(uz)$, we have $n = 3$. The vertices y, z and v belong to $G(ux)$, so z is adjacent with the vertices y and v, too. Analogously $z, x, y \in G(uv)$, and hence z is adjacent with x. Thus the subgraph of G induced by the vertices of $S = \{u, v, x, y, z\}$ is isomorphic to K_5. Suppose the subgraph of G induced by S is proper. By the connectivity of G, it follows that there is an edge tw in G joining a vertex $t \in S$ with a vertex w not in S. Consequently, $K_4 \subseteq G(tw)$, a contradiction. Thus $G \cong K_5$.

Claim 2. If $K_4 \nsubseteq G$, then G is vertex-locally C_k without an induced C_4, where $k = (n+4)/2$.

Proof. First we show that if uw is a common edge for two triangles (uwv) and (uwx), then $G(u)$ and $G(w)$ are cycles. Denote by C the cycle of length n in $G(uw)$. Set $x_1 = x$ and let $x_1, x_2, \dots, x_n$ be the vertices of C in this order. Clearly, $v = x_j$ for some $j \in \{1, 2, \dots, n\}$. Since $K_4 \nsubseteq G$, then $x_2 \neq x_j$ and $x_{j+1} \neq x_1$. Now we prove that each of the vertices $x_2, \dots, x_{j-1}$ is adjacent to exactly one vertex of uw. Suppose, on the contrary, that x_2 is adjacent to both u and w. Then the subgraph of G induced by the vertices x_1, x_2, u, w is isomorphic to K_4, a contradiction. Without loss of generality, we may assume x_2 is adjacent to u. Assume there is a vertex x_i, $3 \leq i \leq j-1$, adjacent to w. Choose x_i to be the first one in order. Then the vertices x_{i-1}, x_1, x_j and u belong to $G(x_i w)$. But u is adjacent to the vertices x_1, x_j, x_{i-1}, thus u is of degree at least 3 in $G(x_i w)$, a contradiction. Thus each vertex of $x_2, \dots, x_{j-1}$ is adjacent to u but not to w. Suppose x_{j+1} is adjacent to u. Then the vertices x_{j-2}, w, x_{j+1}, u belong to $G(x_{j-1}x_j)$, and u has degree at least 3 in $G(x_{j-1}x_j)$, a contradiction. Thus x_{j+1} is adjacent to w but not to u. Now, as above, we are able to prove that each of the vertices $x_{j+1}, \dots, x_{n-1}$ is adjacent to w but not to u. As a result $G(u) \cong (x_1, \dots, x_j, w)$ and $G(w) \cong (x_j, \dots, x_n, x_1, u)$. Now we show that for every vertex y in G, $G(y)$ is a cycle of the same length. By the assumption, there is a triangle (uvw) in G. Since $G(uv) \cong C_n$ there is a vertex x adjacent to w and also to u or v, say u. Hence uw is a common edge for triangles (uwv) and (uwx). Consequently, $G(u)$ and $G(w)$ are cycles. Since G is connected, there is a path $y_1, y_2, \dots, y_m$ joining the vertices u and y. We have proved that $G(y_1) = G(u)$ is a cycle. Suppose $G(y_i)$ is a cycle $(1 \leq i < m)$. Then $y_i y_{i+1}$ is a common edge for two triangles in G, whence $G(y_{i+1})$ is a cycle, too. Thus also $G(y_m) = G(y)$ is a cycle, and we are done. To complete the proof of Claim 2, we must show that each vertex in G has degree k. Since $G(x)$ is a cycle for each vertex x in G, x is incident with a triangle (xyz) in G. Using the fact $G(xy) \cong G(yz) \cong G(xz) \cong C_n$ we obtain

$$\begin{aligned}\deg(x) + \deg(y) &= n + 4\,,\\ \deg(y) + \deg(z) &= n + 4\,,\\ \deg(x) + \deg(z) &= n + 4\,.\end{aligned}$$

This system of linear equations has a unique solution $\deg(x) = \deg(y) = \deg(z) = (n+4)/2 = k$. Obviously, n is even and $n \geq 6$. Also G cannot have an induced C_4, since we would have a chord in any $G(uv)$ otherwise. □

Note that some ideas of the proof were extracted from Fronček's paper [7], namely, from a proof of the following statement.

PROPOSITION 2. *Let G be edge-locally C_n graph. Then either n is even or $n = 3$.*

From Theorem 1, we have the following characterization of edge-locally C_n graphs for small values of n.

COROLLARY 3. *Let G be a connected graph. Then*

(a) *G is edge-locally C_3 if and only if $G \cong K_5$,*
(b) *G is edge-locally C_4 if and only if $G \cong K_{3,3}$,*
(c) *G is edge-locally C_6 if and only if $G \cong L_{5,5}$ or G is the icosahedron.*

Proof. The statement follows immediately by the Theorem 1 and the well-known fact that for $k \leq 5$ there are exactly three vertex-locally C_k graphs, namely, the icosahedron, the octahedron and the tetrahedron. Of these three only the icosahedron is edge-locally C_n, namely for $n = 6$. Similarly, it is not difficult to check that the only edge-locally C_n graphs without triangles for $n \leq 6$ are $K_{3,3}$ and $L_{5,5}$. □

By Theorem 1, edge-locally C_n graphs containing triangles are with one exception just vertex-locally C_k graphs ($k = (n+4)/2$) without an induced C_4. It is not obvious that such a graph exists for each $k > 6$. However, as we prove in Theorem 6, there are infinitely many such graphs for each $k > 6$.

Remark. Corollary 3 may be viewed as a partial solution of the more general problem concerning the relationship between edge-local homogeneity and vertex-local homogeneity of a graph. This question is also related to methods of constructing edge-locally homogeneous graphs and vertex-locally homogeneous graphs. There is a well-known theorem (see [11]) establishing that each edge-transitive graph is either vertex-transitive or it is bipartite. These constructions of locally homogeneous graphs may be generalized in the way described in [18]. One can prove that all edge-locally homogeneous graphs obtained in this way are either vertex-locally homogeneous or bipartite. The mentioned facts suggest to us the following question.

Problem. Is it true that each edge-locally homogeneous graph is either vertex-locally homogeneous or is bipartite?

2. Vertex-locally C_k graphs

Here we summarize some known results on vertex-locally C_k graphs. In the rest of the paper, it is assumed that the reader is familiar with the terminology and basic results of topological graph theory. For undefined terms, the reader is referred to [8] or [20]. The concept of vertex-locally C_k graphs is closely related to the theory of 2-cell embeddings of graphs into closed surfaces. With each graph G we may associate a simplicial complex $K(G)$ in which the simplices are complete subgraphs and the incidence relation is subgraph inclusion. The following proposition was already known to Brown and Connelly [2] and shows that topological graph theory is useful in the study of vertex-locally C_k graphs.

PROPOSITION 4. *Graph G is vertex-locally C_k $(k \geq 3)$ if and only if $K(G)$ is a k-valent triangulation of a closed surface such that each triangle of G forms a boundary of a face in $K(G)$.*

The proof of Proposition 4 can be found in [16]. Using a technique of topological surgery Brown and Connelly proved in [2] that for each $k \geq 3$ there exists a vertex-locally C_k graph. By Theorem 1, each edge-locally C_n graph $(n > 3)$ containing a triangle is vertex-locally C_k as well, where $k = (n+4)/2$. However, there are vertex-locally C_k graphs which are not edge-locally C_n for they contain an induced C_4. Unfortunately, this is the case with all graphs constructed by Brown and Connelly in [2]. In Section 3, we shall give an alternative method for constructing vertex locally C_k graphs. Since these graphs will be free of induced 4-cycles, Proposition 2 and Corollary 3 give the following theorem.

THEOREM 5. *An edge-locally C_n graph exists if and only if $n = 3$ or n is even, $n > 4$.*

It is well-known that each closed surface of negative Euler characteristic admits a n-fold cover for each $n \geq 2$. Consequently, for an arbitrary vertex-locally C_k graph and for each $n \geq 2$ there exists an n-fold covering graph which is also vertex-locally C_k. Hence we have: For each $k \geq 6$ there are infinitely many vertex-locally C_k graphs. This fact was observed by Ronan in [17]. Arguing in the same way as Ronan we are able to prove the following theorem.

THEOREM 6. *For each even $n \geq 8$ there are infinitely many (connected!) edge-locally C_n graphs.*

Finally, Vince [18] gave the following geometrical characterization of vertex-locally C_k graphs. Let T_k $(k \geq 3)$ be the regular tessellations of a simply connected surface S into triangles with k triangles incident to each vertex. Such tessellations are discussed in Coxeter and Moser [6; p. 52]. The surface S is the sphere, the plane or the unit disk (hyperbolic plane) as $k < 6$, $k = 6$ or $k > 6$, respectively. In particular, these tessellations are exactly the tetrahedron, the octahedron and the icosahedron for $k = 3, 4, 5$. But for $k \geq 6$ the respective underlying graphs are infinite. Denote by Γ_k $(k \geq 3)$ the automorphism group of T_k. The group Γ_k is the well-known triangle group with presentation

$$\Gamma_k = \langle i, j, l;\ i^2 = j^2 = l^2 = (ij)^3 = (jl)^k = (li)^2 = 1 \rangle .$$

Call a subgroup B of Γ_k *properly discontinuous* if the vertices x and $\varphi(x)$ are at a distance of at least 4 for each $\varphi \in B$ and $x \in V(T_k)$. Vince proved the following theorem in [18].

THEOREM 7. *A (finite) graph G is vertex-locally C_k if and only if there exists a properly discontinuous subgroup $B \leq \Gamma_k$ of finite index such that $K(G) \cong T_k/B$.*

Theorem 7 shows that the question of existence of any vertex-locally C_k graph is equivalent to the question of existence of the corresponding subgroup of Γ_k. Now let us turn our attention to edge-locally C_n graphs for $n \geq 6$. Call a subgroup B of G *strongly discontinuous* if, for each $\varphi \in B$ and $x \in V(T_k)$, the vertices x and $\varphi(x)$ are at a distance of at least 5. Combining Theorems 7 and 1 we obtain:

THEOREM 8. *Let $k \geq 6$ and let G be a graph containing a triangle. The graph G is edge-locally C_{2k-4} if and only if there is a strongly discontinuous subgroup B of finite index such that $K(G) \cong T_k/B$.*

3. Existence of edge-locally C_n graphs

In this section, we prove Theorem 5. By Theorem 1 and Corollary 3, it is sufficient to prove the existence of vertex-locally C_k graphs without an induced C_4 for all $k \geq 6$. As already mentioned, examples of vertex-locally C_k graphs constructed by Brown and Connelly are not appropriate for us since they all contain an induced C_4. We shall present an alternative construction of k-valent triangulations which do not contain non-contractible cycles of length ≤ 4. The central idea is that of constructing regular maps from their planar quotients (see [13]). By a *map*, we mean a graph embedded cellularly into an orientable surface. Obviously, the k-valent triangulations we want to construct present a special family of maps. It is well-known that every map M can be described by two permutations l and r acting on the set of arcs $D = D(M)$ of M. The *arc-reversing involution* l sends an arc x to the oppositely directed arc x^{-1}, while the *rotation* r cyclically permutes arcs emanating from the same vertex. The vertices of the map correspond to the orbits of r, the edges correspond to the orbits of l, and the orbits of rl determine the boundaries of the faces of M. Thus any couple r, l of permutations, l being involutory, determines the map $M = (D; r, l)$ uniquely. The action of $\langle r, l\rangle$ is transitive, but in general, it is not regular. There is an associated map $M^{\#} = (D^{\#}; R, L)$ defined as follows: $D^{\#} = \langle r, l\rangle$, $R(x) = rx$, $L(x) = lx$ for every $x \in D^{\#}$. We call $M^{\#}$ the *generic* map of M. One can easily verify that right translations by elements of $D^{\#}$ form map automorphisms of $M^{\#}$. Thus $|\operatorname{Aut} M^{\#}| \geq |D^{\#}|$. On the other hand, no map can have more automorphisms than arcs. Therefore, the map automorphism group acts regularly on the arc-set of $M^{\#}$, and consequently, $M^{\#}$ is *regular*. It follows that if the original map M is regular, then M and $M^{\#}$ coincide; otherwise $M \neq M^{\#}$. The reader may find more information on regular maps and generic maps in [15] and [14].

In what follows, we shall consider graphs having three kinds of edges, *links* (possibly parallel), *loops* and *semiedges*. A link or a loop gives rise to two oppositely directed arcs, while there is just one arc associated with a semiedge. The arcs coming from semiedges are fixed by the arc-reversing involution. Now we are ready to describe the construction.

Construction.

Let us consider the following sequence H_k $(k = 1, 2, 3, \dots)$ of planar maps. The maps H_1, H_2, H_3 and H_4 are depicted on Fig. 1. For $k \geq 5$ the map H_k is formed from H_{k-3} by adding a semiedge and a loop in the way depicted on Fig. 1.

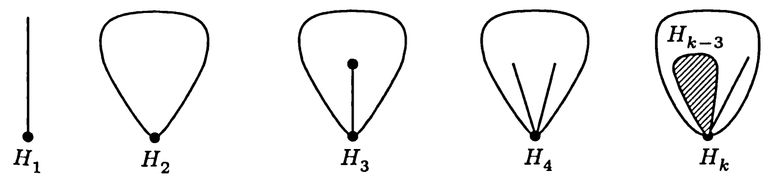

FIGURE 1.

Let r and l be the rotation and the arc-reversing involution associated with the map H_k. Then the orders of r and rl are k and 3 $(k \geq 3)$, respectively. Consequently, the generic map $H_k^{\#}$ is a k-valent triangulation. The problem is, however, to guarantee the contractibility of every cycle of length ≤ 4. To do this, a more sophisticated construction is needed.

First of all, we exclude the case $k = 6$ since it is easy to construct infinitely many 6-valent triangulations of the torus which are edge-locally C_8 (see Fig. 2).

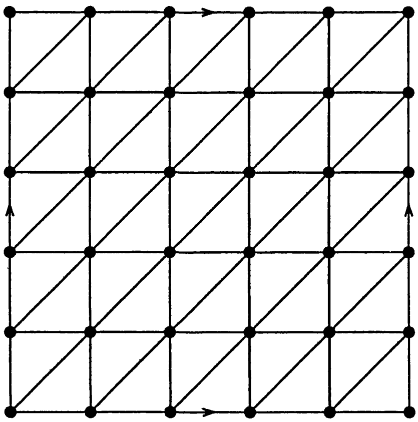

FIGURE 2.

Denote by τ_k the triangulation of a disc arising from the infinite k-valent triangular tessellation T_k of the plane by taking the part of T_k induced by the vertices at a distance ≤ 3 from a chosen vertex v. Let τ_k' and τ_k'' be two copies of τ_k, and let M_k be a spherical map obtained by glueing the boundary cycles of τ_k' and τ_k'' according to some isomorphism $\varphi\colon \tau_k' \to \tau_k''$. We call the cycle arising from the boundary cycles the *equator*, and the two images v' and v'' of the vertex v *poles*. Obviously, M_k is a spherical triangulation which is k-valent, with the exception of the vertices on the equator, where a sequence of $k-5$ consecutive vertices of valency 4 alternates with a single vertex of valency 6. We shall now modify the triangulation M_k locally such that a k-valent triangulation N_k is obtained. First we replace every edge on the equator by two parallel edges. Then we draw either the map H_{k-8} or H_{k-6} inside each digon, depending on whether the 'left' vertex in the digon has valency 8 or 6, respectively. The solution in the general case $k \geq 9$ is depicted on Fig. 3.

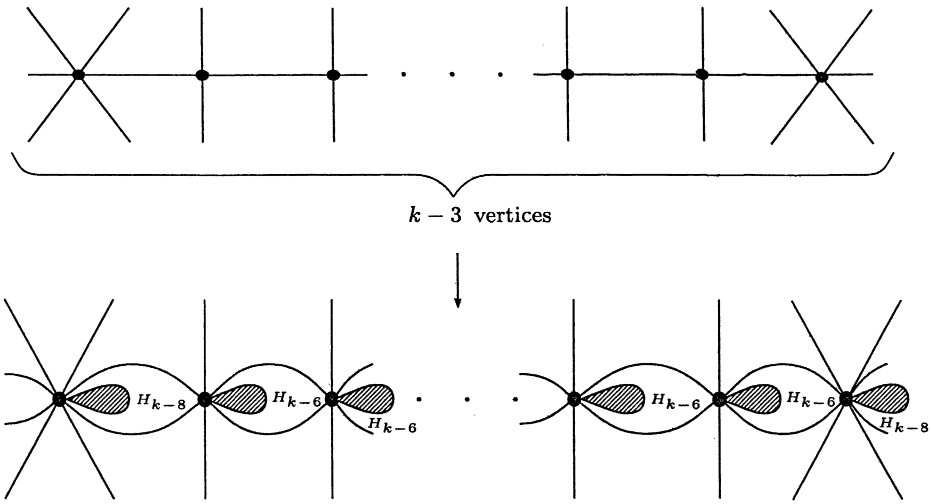

FIGURE 3.

The cases $k = 7$ and $k = 8$ have to be solved separately, see Fig. 4 and Fig. 5.

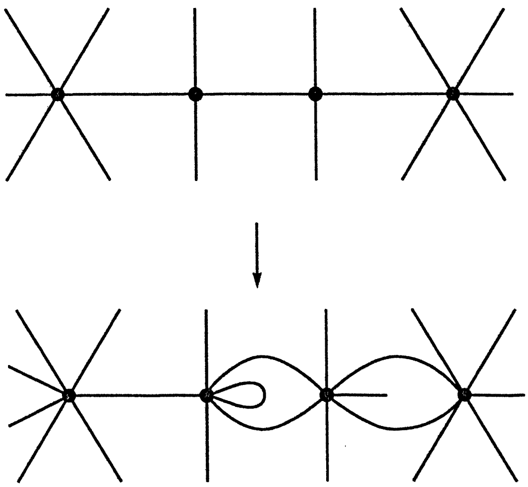

FIGURE 4.

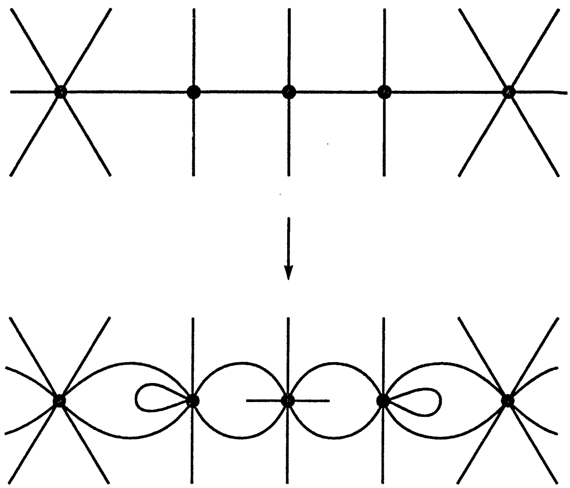

FIGURE 5.

We claim that the generic triangulation $N_k^{\#}$ is vertex- and edge-locally cyclic. The argument is geometrical. There is a natural (branched) covering projection $\pi\colon N_k^{\#} \to N_k$ (see [13], [15]). But the neighbourhood up to distance 2 of a preimage of the pole v' (v'') is mapped one to one. Therefore, the neighbourhood of every preimage $w \in \pi^{-1}(v')$ is formed by a cycle of length k, and the neighbourhood of a preimage of an edge incident with v' is formed by a cycle of length $2k-4$. Since the generic map $N_k^{\#}$ is vertex- and edge-transitive, the property extends to every vertex and edge of the triangulation $N_k^{\#}$.

Remark. Let us call the *edge-width* of a non-spherical triangulation T to be the length of a shortest non-contractible cycle in T. Clearly, the presented construction produces k-valent triangulations of edge-width at least two for any $k \geq 7$. A generalization of the above construction done in [13] allows us to construct k-valent triangulations of arbitrarily large edge-width for every $k \leq 7$.

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