Jerzy Kąkol

Some remarks on $*$-Baire-like and b-$*$-Baire-like spaces

Mathematica Slovaca, Vol. 36 (1986), No. 3, 233--239

Persistent URL: `http://dml.cz/dmlcz/132744`

Math. Slovaca 36, 1986, No. 3, 233—239

SOME REMARKS ON $*$-BAIRE-LIKE AND b-$*$-BAIRE-LIKE SPACES

JERZY KĄKOL

The paper presented studies the $*$-Baire-like and b-$*$-Baire-like spaces. An extension of the Banach-Mackey theorem is used to obtain the $*$-Baire-likeness and the b-$*$-Baire-likeness of spaces M lying between the product space $E = \Pi E_s$ and $\oplus E_s$, endowed with the product topology of E. This generalizes a similar theorem for barrelled and quasibarrelled spaces proved in [8]. Applications of $*$-Baire-like spaces to the open mapping theorem and to the theorem of bilinear mappings conclude the paper.

Introduction

In [5] we extended the Baire-like and b-Baire-like properties to arbitrary topological vector spaces (tvs). We distinguished two classes of tvs, containing strictly the Baire spaces and which are strictly included in the class of ultrabarrelled and quasiultrabarrelled spaces, respectively. Moreover, among locally convex spaces these new Baire properties imply the Baire-like and b-Baire-like properties, respectively. The new classes were called $*$-Baire-like and b-$*$-Baire-like, respectively. The main result of [5] was concerned with the closed graph theorem for $*$-Baire-like space. So it seems that the classes of $*$-Baire-like and b-$*$-Baire-like spaces are enough far-reading for applications.

Let E be a vector space over the field of the real or complex scalars. We shall write (E, t), or simply E, when E is provided with a vector topology t. All topologies on E will be Hausdorff. A sequence (U_i) of subsets of E is called an *almost string* [*a string*] if every U_i is balanced [and absorbing] and $U_{i+1} + U_{i+1} \subset U_i$ for all $i \in N$. A string (U_i) in a tvs E, is [1]

(a) *bornivorous* if every bounded subset of E is absorbed by every U_i;
(b) *closed* if every U_i is closed in E;
(c) *topological* if every U_i is a neighbourhood of zero in E.

A tvs is *ultrabarrelled* [*quasiultrabarrelled*] if every closed and bornivorous string is topological, [1], p. 31, p. 70.

Let E be a tvs. A double sequence $(K_i^n)_{i,\, n\in N}$ of balanced and closed subsets of E which satisfy conditions that

(a′) $K_{i+1}^n + K_{i+1}^n \subset K_i^n$ for all $i, n \in N$;

(b′) $K_i^n \subset K_i^{n+1}$ for all $i, n \in N$;

(c′) $\bigcup_{n=1}^{\infty} K_i^n$ is absorbing in E for all $i \in N$

is called a *sequence of almost strings.*

A sequence (K_i^n) of almost strings is:

bornivorous if for every $i \in N$ and every bounded subset B of E there exists $n \in N$ such that K_i^n absorbs B;

topological if for every $i \in N$ there exists $n \in N$ such that K_i^n is a neighbourhood of zero in E.

A tvs is [*b-∗-Baire-like*] *∗-Baire-like* if every [bornivorous] sequence of almost strings is topological. Every ultrabarrelled tvs whose completion is a Baire space must be ∗-Baire-like. Hence every metrizable ultrabarrelled tvs is ∗-Baire-like [5]. For the relationship of Baire spaces to ∗-Baire-like spaces we refer to [5].

Products of ∗-Baire-like and b-∗-Baire-like spaces

Let I be a set of indices and let E_s be a tvs for each $s \in I$. Let E_I be the (topological) product of the E_s and let H_I be its subspace formed by vectors whith at most countably many non zero components, endowed with the product topology. We denote by F_I the algebraic direct sum of the E_s 's endowed with the product topology. For any set $J \subset I$ we embed $E_J[F_J]$ into $E_I[F_I]$ in a natural way.

We shall need the following

Lemma 1. *Let M be a vector subspace of E_I and let (K_i^n) be a [bornivorous] sequence of almost strings in M. If $[F_I \subset M \subset E_I]$ $H_I \subset M \subset E_I$, then for every $i \in N$ there exist $n \in N$ and a finite subset $e_{i,n}$ of I such that $F_{I \setminus e_{i,n}} \subset K_i^n$.*

Proof. Let (K_i^n) be a [bornivorous] sequence of almost strings in M. Assume the assertion is false for some $i \in N$. Then, as in [1], p. 33, we construct a sequence (x^n) and a sequence (e_n) of subsets of I such that $x^n \notin nK_i^n$ and $x^n \in F_{I\setminus e_n}$, where $e_n = e_{n-1} \cup (s \in I\colon x_s^n \neq 0)$ for $n \geqslant 1$, $e_0 = \emptyset$. As it is easily seen the sequence (x^n) is bounded in F_I. Hence there exists $m \in N$ such that $(x^n) \subset K_i^m$. This contradicts the choice of x^n. On the other hand for every $n \in N$ the set $A_n = (k_s x_s^n\colon |k_s| \leqslant 1,\, s \in e_n)$ is absolutely convex and compact in E_{e_n}. Hence we can consider the compact set $A = \prod_n A_n$ in E_I (in fact in M). By the Baire theorem $A \subset K_i^n$ for some $n \in N$. This contradicts the choice of x^n.

Lemma 2. *Let M be a vector subspace of the product space (E_I, t) and let (K_i^n) be a [bornivorous] sequence of almost strings in M. If $[F_I \subset M \subset E_I]$ $H_I \subset M \subset E_I$, then for every $i \in N$ there exist $n \in N$ and a finite subset $e_{i,n}$ of I such that $M \cap E_{I \setminus e_{i,n}} \subset K_i^n$.*

Proof. By Lemma 1 for every $i \in N$ there exist $n \in N$ and a finite subset $e_{i,n}$ of I such that $F_{I \setminus e_{i,n}} \subset K_i^n$. Let $x = (x_s)$ belong to $M \cap E_{I \setminus e_{i,n}}$. Put $X = (k_s x_s : |k_s| \leq 1, s \in I \setminus e_{i,n})$. Then X is closed in $E_{I \setminus e_{i,n}}$. Moreover, we have

$$X = \overline{X \cap F_{I \setminus e_{i,n}}}^t.$$

Hence

$$x \in M \cap X = \overline{X \cap F_{I \setminus e_{i,n}} \cap M}^t \cap M \subset \overline{X \cap M}^t \cap M \cap K_i^n \cap M,$$

hence

$$x \in K_i^n.$$

Theorem 1. *Let M be a vector subspace of the product space E_I of $*$-Baire-like [b-$*$-Baire-like] spaces E_s, $s \in I$. If $H_I \subset M \subset E_I$ $[F_I \subset M \subset E_I]$, then M is a $*$-Baire-like [b-$*$-Baire-like] space.*

Proof. Let (K_i^n) be a [bornivorous] sequence of almost strings in M. For every $i \in N$ there exist $n \in N$ and a finite subset $e_{i+1,n}$ of I such that $M \cap E_{I \setminus e_{i+1,n}} \subset K_{i+1}^n$. As it is easily seen the finite product $E_{e_{i+1,n}} = E_{e_{i+1,n}} \cap M$ is a $*$-Baire-like [b-$*$-Baire-like] space. Hence $K_{i+1}^m \cap E_{e_{i+1,n}}$ is a neighbourhood of zero in $E_{e_{i+1,n}}$ for some $m \in N$. Since M is the topological direct sum of the spaces $E_{e_{i+1,n}}$ and $M \cap E_{I \setminus e_{i+1,n}}$, then

$$K_{i+1}^m \cap E_{e_{i+1,n}} + M \cap E_{I \setminus e_{i+1,n}} \subset K_{i+1}^m + K_{i+1}^n \subset K_i^r,$$

where $r = \max(n, m)$, is a neighbourhood of zero in M.

Corollary 1. *Let M be a vector subspace of the product space E_I of ultrabarrelled [quasiultrabarrelled] spaces E_s, $s \in I$. If $H_I \subset M \subset E_I$ $[F_I \subset M \subset E_I]$, then M is ultrabarrelled [quasiultrabarrelled].*

Proof. Let (U_i) be a closed and bornivorous string in M. It is enough to put $K_i^n := nU_i$ and apply the previous construction.

A similar result for barrelled and quasibarrelled spaces was obtained by Eberhardt in [3]. A tvs E is *ultrabornological* if every bornivorous string in E is topological, [1]. In [6] we have proved that no subspace M of E_I, card $I > \aleph_0$, such that $H_I \subset M \subset E_I$ and $0 < \dim (M/H_I) < \infty$, is ultrabornological.

The proof of the next result is based on a construction given in the proof of 3.1 Theorem of [8].

Proposition 1. *Let E_I be the product space of tvs E_s, $s \in I$. Let (K_i^n) be a sequence of almost strings in E_I. Then*

(a) *either for every $i \in N$ there exists $n \in N$ such that K_i^n is absorbing in E_I,*
(b) *or there exist $s \in I$ and $i \in N$ such that for every $n \in N$ the set K_i^n is not absorbing in E_s.*

Proof. Suppose that the assertion (a) is false for some $i \in N$. For the number $i+1 \in N$ there exist $n \in N$ and a finite subset $e_{i+1,n}$ of I such that $E_{I \setminus e_{i+1,n}} \subset K_{i+1}^n$. Let $r = \text{card}(e_{i+1,n})$ and let $P_{i+1+r} = (K_{i+1+r}^m: m \in N)$, $P_{i+1+r}^s = (K_{i+1+r}^m: K_{i+1+r}^m$ is not absorbing in $E_s)$.
We show that

$$(*) \qquad P_{i+1+r} = \bigcup_{s \in I} P_{i+1+r}^s .$$

If not, then some K_{i+1+r}^m is absorbing in every E_s. Then K_{i+1}^m is absorbing in $E_{e_{i+1,n}}$. Hence K_i^h is absorbing in E_I for some $h \in N$, a contradiction. Now assume that the assertion (b) is false. Then for every $s \in I$, $i \in N$, there exists $n \in N$ such that K_i^n is absorbing in E_s. Hence K_i^m is absorbing in E_s for all $m \geq n$. Let $p := i+1+r$ be the number defined before. By $(*)$ it is possible to define the (non-empty) set $I_0 := (s \in I: K_p^m$ is not absorbing in $E_s)$. For every $s \in I_0$, let $K_p^{n_s}$ be the largest set which is not absorbing in E_s. Hence for every $s \in I_0$ there exists $x_s \in E_s$ which is not absorbed by $K_p^{n_s}$. We define x in E_I by

$$x := \begin{cases} x_s & \text{for} \quad s \in I_0 \\ 0 & \text{for} \quad s \in I \setminus I_0. \end{cases}$$

If we put $X := (k_s x_s: |k_s| \leq 1, s \in I)$, then X is absolutely convex and compact in E_I, and hence it is absorbed by some K_p^m. By $(*)$ there exists $s \in I$ such that K_p^m is not absorbing in E_s. Hence $K_p^m \subset K_p^{n_s}$, which contradicts the choice of x_s.

The proof of the previous proposition can be easily modified to obtain the following.

Proposition 2. *Let E_I be the product space of tvs E_s, $s \in I$. Let (K_i^n) be a sequence of almost strings in E_I. Then*
(a) *either (K_i^n) is bornivorous in E_I,*
(b) *or there exist $s \in I$ and $i \in N$ such that for every $n \in N$ the set K_i^n does not absorb bounded subsets of E_s.*

Remark 1. A tvs E is boundedly summing [1] if for every bounded subset B of E there exists a scalar sequence (a_n), $a_n \neq 0$, such that $\bigcup_{n=1}^{\infty} \sum_{k=1}^{n} a_k B$ is bounded. Every locally convex, almost convex, locally pseudoconvex space is boundedly summing. It is known [5], Proposition 3.1, that every sequence of almost strings in a sequentially complete boundedly summing tvs is bornivorous. In view of Proposition 2 we derive that the last property is preserved under products. Note that an uncountable product of boundedly summing tvs need not be boundedly summing [1], p. 77. As a simple corollary we obtain that every closed string in a product of sequentially complete boundedly summing tvs is bornivorous.

In [5] we have proved the following theorems

($**$) *Let (E, t) be the inductive limit space of a family $(E_s, t_s\colon s \in I)$ of $*$-Baire-like spaces. Let (F, g) be the inductive limit space of an increasing sequence (F_n, g_n) of complete locally bounded spaces covering F and such that $g_{n+1}|F_n \leqslant g_n$ for all $n \in N$. If f is a closed (i.e. with the closed graph) linear map from (E, t) into (F, g), then f is continuous and $f(E) \subset F_n$ for some $n \in N$.*

($***$) *Let E be a $*$-Baire-like space and F its subspace of countable codimension. Then F is $*$-Baire-like.*

Remark 2. (a) There exist tvs M with the property that every closed linear map from M into any metrizable complete tvs is continuous but M is not $*$-Baire-like. This follows from the characterization of ultrabarrelled spaces by the closed graph theorem, [1], (2), p. 38, and from the fact that the class of $*$-Baire-like spaces is strictly included in the class of ultrabarrelled spaces [5]. (b) It is well known that if (E_n) is a sequence of subspaces of a Baire space E, covering E, then E_n must be a Baire space for some $n \in N$. In general, this property is false for $*$-Baire-like spaces. Indeed, let E be a metrizable non complete (LF)-space, generated by an increasing sequence (E_n) of Banach spaces. It is easy to show that there exists $p \in N$ such that, for $n \geqslant p$, E_n is not $*$-Baire-like, according to ($**$). Since every (LF)-space is ultrabarrelled we derive also that the class of $*$-Baire-like spaces is strictly larger than the class of Baire spaces.

Now we prove the following open mapping theorem

Theorem 2. *Let (E, t) be a $*$-Baire-like space and let (F, g) be the inductive limit space of an increasing sequence (F_n, g_n) of locally bounded complete tvs covering F and such that $g_{n+1}|F_n \leqslant g_n$ for all $n \in N$. If f is a closed linear map from (F, g) into (E, t) such that the codimension of $f(F)$ in E is countable, then f is open and $f(F)$ is closed in E.*

Proof. If $M := (x \in F\colon f(x) = 0)$, then M is closed in F. Moreover, the quotient space F/M is the inductive limit space of the sequence $(F_n/F_n \cap M, h_n)$ of complete locally bounded spaces, where h_n is the quotient topology. Let $\hat{f}$ be the associated injection from F/M onto $f(F)$. Then $\hat{f}$, and hence $\hat{f}^{-1}$, has the closed graph. By ($**$) and ($***$) $\hat{f}^{-1}$ is continuous which means that f is open. On the other hand $\hat{f}(F_n/F_n \cap M) = f(F)$ for some $n \in N$. Hence $\hat{f}$ has the closed graph in $(F_n/F_n \cap M, h_n) \times (f(F), t|f(F))$. In view of ($**$) $\hat{f}^{-1}$ is continuous from $f(F)$ onto $(F_n/F_n \cap M, h_n)$. Now we prove that $f(F)$ is closed in E. Let (x_s) be a net in $f(F)$ with $x_s \to x$. There exists a net $(\hat{x}_s)$ in $F_n/F_n \cap M$ such that $x_s = \hat{f}(\hat{x}_s)$. Since $\hat{f}^{-1}$ is closed and continuous, then $(\hat{x}_s)$ is a Cauchy net in the space $F_n/F_n \cap M$, and so $\hat{x}_s \to \hat{z} = \hat{f}^{-1}(x)$. Hence $x \in f(F)$.

Since every metrizable ultrabarrelled tvs is $*$-Baire-like, this theorem extends the Köthe theorem of [7], p. 111.

Corollary 2. *Let f be a continuous linear map from the inductive limit space F of an increasing sequence of complete locally bounded tvs into a tvs E. Let $f(F)$ be dense in E. Then either $f(F)=E$ or $f(F)$ is not $*$-Baire-like.*

The following result generalizes the Bourbaki theorem of [2], p. 43.

Proposition 3. *If E, G are metrizable tvs and F is a $*$-Baire-like space, then every separately equicontinuous set T of bilinear maps from $E\times F$ into G is equicontinuous.*

Proof. Let (U_n) and (W_i) be closed topological strings in E and G, respectively, which generate the topologies of E and G, respectively. For every i, $n\in N$ the sets

$$K_i^n := (y\in F\colon\ f(U_n, y)\subset W_i,\ f\in T)$$

are closed and balanced in F. As it is easily seen (K_i^n) is a sequence of almost strings in F. Hence for every $i\in N$ there exists $n\in N$ such that K_i^n is a neighbourhood of zero in F. Therefore $f(U_n, K_i^n)\subset W_i$, $f\in T$.

Corollary 3. *Let E be a metrizable algebra with separately continuous multiplication which as a vector space is ultrabarrelled. Then the multiplication is jointly continuous.*

Corollary 4. *If E, G are metrizable tvs and F is $*$-Baire-like, then every set T of separately continuous bilinear maps from $E\times F$ into G such that for every $(x, y)\in E\times F$ the set $(f(x, y)\colon\ f\in T)$ is bounded in G is equicontinuous.*

Proof. It is enough to apply the Banach-Steinhaus theorem for the ultrabarrelled space F, [1], (3), p. 39, and Proposition 3.

Let E, F, G be tvs. A bilinear map f from $E\times F$ into G is called *hypercontinuous* with respect to E (or *E-hypercontinuous*) if it is separately continuous and if for every neighbourhood of zero W in G and every bounded subset B of E there exists a neighbourhood V of zero in F such that $f(B, V)\subset W$, [4], Definition 2, p. 358. We define in an analogous way the bilinear maps f which are F-hypercontinuous. As it is easily seen every set T of separately equicontinuous bilinear maps from $E\times F$ into G is E-hyperequicontinuous, provided f is ultrabarrelled. In the same way as proposition 3 we get

Proposition 4. *If E, G are metrizable tvs and F is a b-$*$-Baire-like space, then every set T of F-hyperequicontinuous bilinear maps from $E\times F$ into G is equicontinuous.*

Since every metrizable tvs is b-$*$-Baire-like [5], we are able now to give.

Corollary 5. *Let E be a metrizable algebra. Then the multiplication in E is jointly continuous if and only if it is hypercontinuous.*

Proposition 5. *Every tvs is a closed subspace of some $*$-Baire-like space.*

Proof. Let E be a tvs. By [1], (5), p. 18, E is topologically isomorphic to a subspace of a product F of metrizable and complete tvs. F may be taken such that Theorem 1 is applicable Clearly F is $*$-Baire-like (Theorem 1). Let $(x_s)_{s\in I}$ be a Hamel basis of the complement subspace of E in F. For every $s\in I$ let $F_s := E+\text{lin}\,(x_r\colon\ r\neq s)$. By $(***)$ every F_s is $*$-Baire-like, and hence the product space F_I is $*$-Baire-like (Theorem 1). Therefore E is canonically embedded in the $*$-Bairelike space F_I. Observe that E is closed in F_I.

REFERENCES

[1] ADASCH, N.—ERNST, B.—KEIM, D.: Topological vector spaces. Lecture Notes in Math. 639, Berlin 1978.

[2] BOURBAKI, N.: Éléments de Mathématique. Topologie Générale. Chap. 9, Paris 1974.

[3] EBERHARDT, V.: Über einen Graphensatz fur Abbildungen mit normiertem Zielraum. Manuscripta Math., 12, 1974, 47—65.

[4] HORVATH, J.: Topological vector spaces and distributions. Addison-Wesley Publishing Company, 1966.

[5] KAKOL, J.: Topological linear spaces with some Baire-like properties. Functiones et Approximatio, 13, 1982, 109—116.

[6] KAKOL, J.: Some remarks on subspaces and products of ultrabornological spaces. Simon Stevin, 57, 1983, 83—97.

[7] KÖTHE, G.: Die Bildräume abgeschlossener Operatoren. J. reine engew. Math., 232, 1968, 110—111.

[8] TODD, A. R.: Covering of products of linear topological spaces. J. Austral. Math. Soc., 29, 1980, 281—290.

Received December 5, 1983

Institute of Mathematics
A. Mickiewicz University
ul. Matejki 48/49 Poznań
POLAND

НЕКОТОРЫЕ ЗАМЕЧАНИЯ ОБ $*$-БЭРОВСКО-ПОДОБНЫХ и b-$*$-БЭРОВСКОПОДОБНЫХ ПРОСТРАНСТВАХ

Jerzy Kąkol

Резюме

В настоящей работе предметом изучения являются монологические векторные пространства с некоторым бэровским свойством. Показаны тоже приложения этих пространств для получения теоремы об открытом отображении и теории билинейных отображениях.